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A TREATISE ON PHYSICS

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A TREATISE ON PHYSICS

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VOL. I

DYNAMICS AND PROPERTIES OF MATTER

LONDON
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PREFACE.

My object in the present work is to provide a treatise on physics which may serve for those who, beginning at the elements of the subject, wish to have in one book an account of theoretical and experimental physics which may be sufficient for most practical purposes of scientific and technical education. The undue prominence in our educational system given to examinations, and the over estimation in which the mere passing of examinational tests has been held, has led to a system of division of a great subject like physics into isolated compartments; but I think it desirable that, from the beginning, those who wish to make a serious study of the subject should have in one book all they require for applications of physical science to the arts and industries. Accordingly I have conceived that no mechanical or electrical engineer would desire to be content with, even at the first, such a view of the subject as can be obtained by the student whose mathematical attainments are the proverbial "Two Books of Euclid and Algebra to Quadratic Equations," and therefore I have not scrupled to employ the processes of the Differential and Integral Calculus where the discussion required such analysis. With much that has been written regarding the teaching of mathematics to engineering students I entirely agree, but it is absolutely necessary that all concerned should realise what is undoubtedly the fact—that no good progress can be made in the study of engineering or natural philosophy without mathematics, or without persevering and anxious effort on the part of the student, and that even to apprehend the fundamental notions of dynamics,

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and the principles of physics generally, the same processes of thought are required as are involved in those branches of mathematics which are used in the discussion of physical theories.

Chapters and Sections of the book which are really advanced are marked with asterisks: these portions may be omitted on a first reading. There have, however, been introduced in many places explanations of mathematical conceptions and processes which may render unnecessary reference to special treatises, and economise the time and energy of the student. Thus I hope that the chapters on General Dynamics, Fluid Motion, Hydrostatics, and Elasticity may be useful to students of engineering, and help them to be able to render a reason for their faith in the statements of the indispensable "pocket-books" and of the treatises on Applied Mechanics and Strength of Materials.

The reader will notice that several subjects usually treated under Properties of Matter are not dealt with—for example, Diffusion and Osmosis, and Viscosity. These, however, are better treated in connection with the Theory of Gases, and are therefore with various other topics held over for discussion in vol. ii. One dynamical subject is omitted, of which some use might have been made from time to time—the Principle of Dimensions or Dynamical Similarity. Much information with regard to very recondite problems, the full solution of which is hardly possible, can be obtained by this method. It will be explained and exemplified in connection with such applications in later chapters.

In many parts of the book I have been under obligations to other writers, and in many places, though I fear not in all, I have acknowledged my indebtedness. I would mention especially Professor Love's "Treatise on Elasticity," Lord Rayleigh's Papers, now happily available in a collected form to the physical student, Professor Greenhill's "Treatise on Hydrostatics," and Thomson and Tait's "Natural Philosophy." Some of my former students in Wales have rendered me willing help in reading proofs, and most of the sheets have been read by my colleague in the Natural Philosophy

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Department here, Dr. Walter Stewart, to whom I am under special obligations. I have to thank Messrs. Ballantyne, Hanson & Co.'s staff, readers and compositors alike, for their careful attention to the work of setting and correction, and their anxiety to make the book thoroughly satisfactory in respect of typography.

It would be too much to hope that all the errors and obscurities in the text have been detected: no work of the kind can be freed from such blemishes except by the care of a large number of competent readers. I shall be very grateful for any suggestions for the improvement of the work from those who do me the honour to read it.

ANDREW GRAY.

THE UNIVERSITY, GLASGOW.
January 30, 1901.

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ERRATA.

P. 15, line 20 from top for k read h.

18, in equation (13) *for* cos BCA *read* cos BAC.

27, line 10 from foot *after* varying *insert* velocity.

95, in Fig. 80 *for reference letter* C' *substitute* D.

,, line 18 from foot *after* the two great circles *insert* IA, IA', or IB, IB'.

133, line 9 from top *for* particles *read* particle.

197, line 9 from foot the reference should be *Measurements and Instruments* (chap. xvii.)

226, last paragraph of § 263. It has not been found possible to afford space in chap. xii. for a discussion of *Precession and Nutation.*

271, line 3 from foot *for* resulant *read* resultant.

324, line 5 from bottom delete the words within parentheses.

491, in headline *for* Astrononical *read* Astronomical.

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NOTES AND ADDITIONAL ERRATA.

P. 135, line 16 from foot, *after* applied forces *insert* supposed transferred to the centroid in the same way.

189, line 19 from top, for kmm/r^2 read kmm'/r^2.

198. The discussion of friction-wheels here given is not intended to be a full mathematical treatment of the problem. The exact ratio in which the activity spent against friction is diminished by the wheels involves the ratio in which they increase the weight supported, the angle $\tan^{-1}p$ (where p is the coefficient of friction for the rubbing surfaces) and the angle between a vertical and the line drawn from the centre of the supported wheel to the centre of a side-wheel. But for any practical case in which the added weight is negligible, and the wheels overlap sufficiently, the ratio r/R given in the text is almost exact.

200, line 22 from top, *after* body *insert* producing rolling.

„ line 15 from foot, for $mk^2\dot{\theta}$ read $mk^2\ddot{\theta}$.

222, in the first of equations (58) *for* $A\theta$ *read* $A\ddot{\theta}$.

267, line 15 from foot, *for* a vertical component *read* vertical components.

„ line 14 from foot, *delete* with a slight modification due to the rotation of the wheels.

[The reference, which is incomplete, was intended to be to a reaction analogous to that indicated in (73) or (74) p. 231.]

307, equation (45) for $e^{s/R}$ read $e^{-s/R}$.

316, equation (2) for $q=$ read $\dot{q}=$.

406, line 16 from top, *after* those *insert* pressures.

589, lines 10 and 8 from foot, for $dr.rd\theta$ read $dr.d\theta$.

624, line 20 from top, *for* abscissal *read* abscissæ.

651, line 12 from top, *for* is *read* x.

„ line 13 from top, *after* vertically *insert* from that point.

„ line 15 from top for $(1+p^2)_{\frac{1}{2}}$ read $(1+p^2)^{\frac{1}{2}}$.

653, in equation (21) x is to be understood as measured to the right from a vertical tangent to the drop.

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DYNAMICS, PROPERTIES OF MATTER.

INTRODUCTION.

THE present treatise will deal with Physics as far as possible from the point of view from which natural phenomena are regarded by modern science. We have here to treat of only the relations of matter and energy; to endeavour to investigate the laws which regulate the transference, transformation and distribution of energy; and to trace as far as possible that order which regulates physical phenomena, and which when fully perceived will form their true explanation. It is a mistake to suppose, as is frequently done, that the principles which suffice for the deduction of physical results are the conservation of matter and the conservation of energy. These, important as they are, cannot afford guidance as to the play of energy in matter without the aid of certain dynamical principles which regulate in that regard the relations between different material systems or different parts of the same system.

Without the aid of the theorems of dynamics, which are far from being summed up in the doctrine of the conservation of energy, we are unable to make any satisfactory advance in the study of physical theories. To regard the phenomena only, to catalogue them, and note their agreements and disagreements, is a most important part of scientific investigation. But the classificatory stage of the science has long ceased to be the only one; we have ascended to the philosophical stage, that on which an attempt is unceasingly made to unite phenomena in the bond of physical theory, and to test, rectify, and strengthen the scheme of natural philosophy by using its theories for the prediction of hitherto unknown results, which can then be subjected to the ordeal of observation or experiment.

It will be necessary first, then, to give some account of dynamical principles, and this we shall attempt to do in an elementary manner, keeping always physical applications in view. It has been too much the habit, especially since the divorce of what has been called "mixed mathematics" from physics in degree and diploma courses has been countenanced by several of our universities and examining bodies, to treat dynamical principles as mere pegs on which to hang collections of mathematical exercises, practically destitute of physical interest, and without example in the everyday experience and observation of the student.

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Dynamics is undoubtedly a subject which cannot be studied without the use of mathematics; but it is just as certainly a science capable of being illustrated by everyday sights and events. The motion of a football kicked into the air, the flight of a tennis ball, the fall of a stone, the very different manner of fall of a rain-drop, the ascent of smoke, the stability of a bicycle and the mode in which an expert rider guides its motion, the spinning of a top, and a thousand other familiar things which meet the gaze of every observer whose eyes are open to what is taking place around him, are all capable of more or less elementary explication, and ought to be kept in view in a satisfactory discussion of dynamical principles.

In the portion which deals with dynamics, as elsewhere in the book, we may use the language of cause and effect; but this, it is to be carefully observed, is done only for the sake of convenience of expression. The action of one body may be spoken of as the cause of the acceleration of another, an electromotive force in a voltaic circuit as the cause of a current in that circuit; but it is not to be inferred that what is spoken of as the cause has necessarily an existence apart from the effect, or that the former precedes the latter in time. What we really deal with are two aspects of a phenomenon which co-exist, and are related in a way which it is our object to discover and quantitatively express.

We shall include in the dynamical discussion an account of vibrational motion, or, as it is frequently called, simple harmonic motion, and of the motion of a system of particles, and of a material system the relative configuration of the parts of which undergoes no change (what is somewhat improperly termed a *rigid* system), for these theories are required in many physical problems which will subsequently present themselves for solution. But, as far possible as, each part of the subject will be treated from the point of view of a definite case of motion easily apprehended by the student, and no effort will be spared to make the discussion clear and accurate without having undue recourse to mathematical analysis. To some the treatment may appear unduly prolix; but it is to be remembered that hardly any subject presents more difficulties and sources of misunderstanding, or conduces more to clearness of thought and the cultivation of a logical habit of mind when its principles are correctly apprehended and applied by the student. Hardly any, therefore, stands more in need of such a presentment as may guard the student against error arising from misconceptions or preconceived ideas when grappling with the further difficulties of experimental physics. A few pages of formal statements of definitions and principles, followed by a few more of algebra, unmarred by letterpress, may be made to contain the essentials of elementary dynamics; but such a work, except as a collection of notes for the use of a teacher who has carefully thought out and solved the difficulties of the subject for himself, and who is therefore able to supply the necessary commentary, is of no educational value whatever.

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CHAPTER I.

MEASUREMENT OF LENGTH AND TIME.

1. **Physical Units.**—All physical quantities are measured in units chosen arbitrarily or otherwise, each of which is a quantity of the same kind as that which it measures. Thus, the numerical measure of the length of a straight line is the number of times it contains a chosen unit of length; that is, if the unit is realised in the form of a measuring-rod, for example a yard rule, the numerical value of the length is obtained by placing the rule in successive end to end positions until the length is exhausted, and reckoning the number of times this is done. Of course this number may be an exact whole number, or a proper or improper fraction. If the length contains the unit n times, where n is a whole number, and does not contain it $n+1$ times, the fraction remaining after the n units have been taken is to be determined by applying a smaller unit, which is contained in the same way by the larger an exact number of times. Thus, a distance may be 37 yards 6 inches; that is, it contains the distance 1 yard thirty-seven times, and besides a distance which contains six times a unit defined by the fact that the yard contains it exactly thirty-six times. If the inch were not contained an exact whole number of times in the distance to which it is applied, that is if the distance were less than an inch or contained the inch less than a whole number $n+1$ times and more than n times, a unit which is a submultiple of the inch would have to be applied to determine the fraction.

2. **Fundamental Units.**—The fundamental quantities in dynamics are length, time, and mass, and on the units of these are founded the units of other quantities, as momentum, force, energy, which are therefore called *derived units*. We shall here consider the measurement of length or distance and time, leaving mass to be dealt with later.

3. **Unit of Length or Distance. The Imperial Yard.**—First, then, let us consider more fully the unit of length. In this country the standard unit of length is what is called the Imperial Yard. It is defined by Act of Parliament 18 & 19 Vict. c. 72, July 30, 1853, as follows: "The straight line or distance between the centres of the transverse lines in the two gold plugs in the bronze bar deposited in the Office of the Exchequer shall be the genuine standard of length

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at 62° F., and if lost it shall be replaced by means of its copies." The copies referred to are specified, and are those preserved at the Royal Mint, the Royal Society of London, the Royal Observatory at Greenwich, and the New Palace at Westminster. Their lengths have been very exactly compared with that of the standard, so that in the event of the destruction of the latter the copies would enable it to be replaced with the utmost exactness.

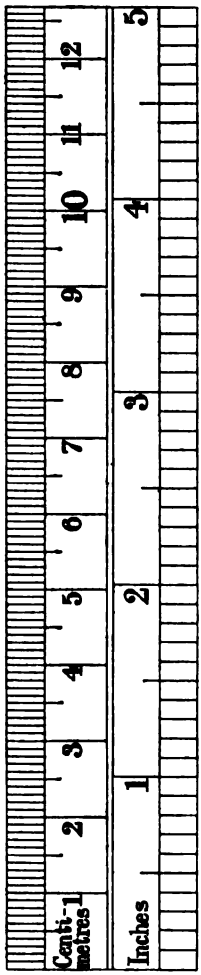

FIG. 1.

The length of a pendulum which beats seconds has been determined in terms of the length of the standard for a number of exactly defined places, so that, even if the copies were also destroyed, pendulum observations at these places would suffice for the recovery of the standard.

The yard yields the subordinate units—the foot, which is $\frac{1}{3}$ of the yard, and the inch, which is $\frac{1}{36}$ of the yard; as explained above. Larger units of length are the pole, $5\frac{1}{2}$ yards; the furlong, which is 40 poles, or 220 yards; and the mile, which is 1760 yards. The length given in the lower scale along the page in Fig. 1 is 5 inches, and each inch is divided into tenths. In many measuring-rules in use inches are divided to eighths.

4. **Unit of Length. The Metre.** — In France, after the great Revolution, a scientific committee was appointed to devise a completely new system of weights and measures. It was decided by a decree of the Republic, passed in 1795 (*Loi du* 18 *germinal, an iii*), that the new unit of length should be based on the earth's dimensions, and it was defined as one ten-millionth part of the distance from the Equator to the North Pole along the meridian passing through Paris.

This unit was realised by means of a measurement of the length, in terms of a certain measuring rod, of this quadrant of the earth's circumference, which was carried out mainly by Delambre and Méchain, with the assistance of Borda. A base line was laid down by means of this rod, and from that, by trigonometrical observations, the length of a certain arc of the meridian, that between a certain point at Dunkirk and another point at Barcelona, was found. The ratio of this arc to the whole quadrant being known, the length of the latter in terms of the length of the measuring rod became known, and hence also the ratio of the length

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of the measuring rod to that of one ten-millionth part of the quadrant. Then it was possible to make a rod which should have at a certain temperature the same length as the required unit.

This was done by Borda, who made a rod of platinum which at the temperature 0° of the Centigrade scale was intended to be of the required length. From various causes the utmost exactitude was not achieved; but Borda's rod has nevertheless remained the standard. The French standard of length is called the *mètre*,* and is defined as the distance between the extremities of Borda's rod at temperature 0° C., and not at all with reference to the earth's dimensions, though approximately the relation stated above is fulfilled.

5. **The Metric System of Measures.**—The system of measures founded on this unit of length is called "the metric system." It has come into general use on the Continent, and has been legalised in this country. For convenience in every respect it is infinitely superior to the British system of measures. It is truly astonishing that the latter should have been tolerated so long.

In the metric system the decimal mode of reckoning is adopted throughout. The metre is divided into ten equal parts, each called a decimetre; the decimetre into ten equal parts, each therefore a hundredth of a metre, and called a centimetre; the centimetre is divided into ten equal parts called millimetres; then further subdivisions are tenths of a millimetre, and so on. A length of ten metres is called a decametre, of one hundred metres a hectometre, and one thousand metres a kilometre. The kilometre is the unit adopted on the Continent for the measurement of distances between places.

A length of one and a half decimetres subdivided to centimetres and millimetres is given on the upper scale of the rule shown in Fig. 1.

For most scientific purposes the centimetre has been adopted as the fundamental unit of length, and will be used in what follows for the expression of results far more frequently than any other unit. We therefore give here a table showing the values of the various British units of length in terms of the centimetre. Their values in terms of the decimetre can be obtained by shifting the decimal point one place towards the left, in terms of the metre by shifting the decimal point two places to the left, and so on.

1 inch	=	2·5400	centimetres
1 foot	=	30·4797	,,
1 yard	=	91·4392	,,
1 mile	=	160933·	,,
1 sea mile	=	185230·	,,

1 mm.	=	·03937	inch	=	·0032809 feet
1 cm.	=	·3937	,,	=	·032809 ,,
1 metre	=	39·371	,,	=	3·2809 ,,

* Since the metric system has come into use in this country, the words *metre, centimetre*, &c., without accents, have become naturalised English words, and are so pronounced.

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6. **Units of Area and Volume.**—On the unit of length alone are founded two derived units, the unit of area and the unit of volume. The unit of area is the area of a square the side of which is the unit of length, the unit of volume is the volume of a cube the edge of which is the unit of length. Thus in Great Britain small areas are measured in square yards, square feet or square inches, while larger areas are measured in acres (a unit of 4840 square yards) or square miles. In the metric system we have square metres, square decimetres, square centimetres, &c.

Again, we have in volume measurements cubic yards, cubic feet, cubic inches, &c., and cubic metres, cubic decimetres or litres, cubic centimetres. A cubic metre contains 1000 litres, or 1,000,000 cubic centimetres, or 1,000,000,000 cubic millimetres.

The following tables contain the principal units of area and volume and their relations:

1 sq. inch	=	6·451	sq. cm.
1 sq. foot	=	928·997	,,
1 sq. yard	=	0·8361	,,
1 sq. cm.	=	·155	sq. inch
1 ,,	=	10·764	sq. foot
1 ,,	=	1·196	sq. yard
1 cub. inch	=	16·386	cub. cm.
1 cub. foot	=	28·315	litres
1 cub. yard	=	·7645	cub. metres
1 cub. cm.	=	·06103	cub. in.
1 litre	=	61·027	cub. in.
1 ,,	=	·03532	cub. foot

7. **Change of Units, Dimensions.**—For the conversion of areas and volumes from one system of units to another the process which is illustrated by the following examples, and which will be developed later in connection with other units, will be found convenient.

Let it be required to express an area of N square feet in square centimetres. Then expressing the unit of area by the symbol $(\text{foot})^2$ we may write the area with the unit indicated by an equation thus:

$$\text{area} = N\,(\text{foot})^2$$

now

$$1 \text{ foot} = 30{\cdot}48 \text{ cm.}$$

therefore

$$(\text{foot})^2 = 30{\cdot}48^2\,(\text{cm.})^2$$

where $(\text{cm.})^2$ expresses the area of a square of 1 cm. side, and we have

$$\text{area} = N \times 30{\cdot}48^2\,(\text{cm.})^2.$$

The multiplier $30{\cdot}48^2$ is an example of what has been called the *change-ratio* required for the change of units. It is the number of times the *old* unit of area contains the *new*.

Again, let a volume expressed in cubic yards have a numerical value N, and it be required to express the same volume in cubic

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centimetres. We express the first unit of volume by $(\text{yard})^3$ the second by $(\text{cm.})^3$. Thus we write

$$\text{volume} = N \times (\text{yard})^3.$$

But $$(\text{yard})^3 = 3^3 \times 30{\cdot}48^3\ (\text{cm.})^3,$$

and therefore

$$\text{volume} = N \times 3^3 \times 30{\cdot}48^3\ (\text{cm.})^3.$$

The change-ratio in this case is $3^3 \times 30{\cdot}48^3$.

In general in physical mathematics the units are specified beforehand, and the calculation is conducted without introducing the units into the equations, which are therefore relations between numbers. The introduction of the units into the equations is useful, though not essential, when change of units is to be made, and in some other circumstances. The whole subject of units and the dimensional formulæ of physical quantities will be discussed later.

8. **Measurement of Time.**—For civil and scientific purposes time is measured by the rotation of the earth, and equal intervals of time are defined as those in which the earth turns through equal angles about its axis. The dynamical definition of equal intervals of time must be deferred, but it may be here stated that there is reason to believe that the civil reckoning requires a slight correction; that in fact, taking the dynamical definition as the basis of reckoning, we are almost forced to the conclusion that the rotational motion of the earth is very gradually becoming slower (see § 14). This question will be discussed later; at present we take the rotating earth as timekeeper.

9. **Sidereal Day.**—The period of the earth's rotation is called a *sidereal day*, and is so called because it is the interval between two successive passages of a fixed star across the meridian plane of any place. The meridian plane of any point on the earth's surface is a plane drawn through the point and the earth's axis. It therefore turns with the earth, and the sun or a star is said to be on the meridian at the instant when the plane of the meridian passes through the sun's centre or through the star. To an observer looking along the axis of a telescope in the plane of the meridian, the heavenly body seems to move across the field of view in which a certain line marks the meridian, and thus the body is said to cross the meridian. Of course in each rotation of the earth a star crosses the meridian of any place twice, but does so in the two cases from opposite sides of the plane. Thus, there is only one crossing of a fixed star, from a given side to the other, for each complete turn of the earth on its axis, and the interval between two such crossings measures a sidereal day.

10. **Sidereal Time and Solar Time.**—It will be seen from Fig. 2 that, in consequence of the earth's orbital motion round the sun, the interval between two such successive crossings of the meridian by the sun is longer than that between two successive passages of the meridian by a fixed star. The orbit of the earth, which is really an ellipse of small eccentricity, may for simplicity be taken

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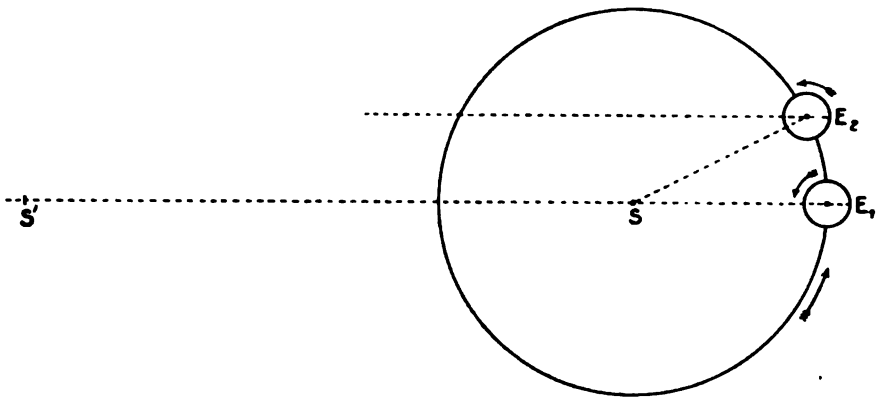
as a circle. This circle is rather less than 93,000,000 miles in radius, and is described in 365 days, 6 hours, 9 minutes, 9 seconds, the length of the so-called *sidereal year*, that is, the interval between two successive appearances of the sun in the same relative position among the stars. Thus, the earth moves with an orbital velocity of about 18·5 miles per second, and traverses while making a single turn about its axis a distance of very nearly 1,600,000 miles. Fig. 2 shows two positions of the earth, E_1, E_2, at the beginning and end of a rotation; the circle with S, the sun, as centre represents the orbit, and the arrows show the directions of the axial and orbital motions as they would appear to an observer in space looking from beyond the North Pole. The dimensions of the diagram are, of course, entirely out of proportion, the diameter of the earth is only about 8000 miles,

FIG. 2.



the distance SE, of the earth from the sun is 11,600 times as much. The diagram also is drawn as if the inclination of the earth's axis to the plane of the orbit, which is in reality 66° 32′ 48″, were 90°.

The two positions E_1, E_2 are 1,600,000 miles apart. When the earth is in the first position let the sun S and a fixed star S' be on the meridian. After the earth has made exactly one turn the meridian is parallel to its former position, and no longer passes through the centre of the sun, nor, strictly speaking, through that of the star. From the position shown at E_2 the meridian must turn through a further angle equal to E_1SE_2 to be coincident with S, and through a smaller angle $E_1S'E_2$ to contain S'.

The distance of the nearest fixed star is, however, so great, that the latter interval is too small to be observed, and thus the meridian, as nearly as observation can detect, passes in the position E_2, again through the star S'. Of course the distance $S'E_1$ in the diagram is shown on far too small a scale. The distance of the nearest fixed star, namely, *α Centauri*, is, according to the best observations of its

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annual parallax, about 274,000 times that of the sun. Thus, in Fig. 2, the distance $S'E_1$ ought to be 274,000 times SE_1, which for SE_1, an inch, would give $S'E_1$ about 4 miles! The distance of the nearest fixed star is thus about 25 million million miles, and the angle subtended at the star by the distance E_1E_2, which is 1,600,000 miles, cannot be more than ·0132 of a second of angle. The earth would turn through this angle in about $\frac{1}{11000}$ of a second of time; so it is no wonder that the discrepance between the length of the sidereal day and the period of the earth's rotation is not observed.

11. **Solar Day. Mean Solar Day.**—The interval between two successive passages of the sun's centre in the same direction across the meridian is called a *solar day.* As the sun's distance is only 93,000,000 of miles, the angle E_1SE_2 is quite appreciable, and this angle must be turned through (and a very little more, since the earth is continually going forward in its orbit while turning) to bring the sun's centre to the meridian. The time required is a little less than four minutes, and thus the solar day is rather less than four minutes longer than the sidereal day. The interval, however, is from two causes not quite invariable: (1) the earth moves in an ellipse round the sun, and thus moves fastest in its orbit when at mid-winter it is nearest to the sun, and slowest at mid-summer when the sun's distance is greatest; (2) the earth's axis of rotation, though approximately always parallel to itself, is not at right angles to the plane of the orbit. How these causes produce variation in the length of the solar day will be explained in the chapter on astronomical dynamics.

What is called a *mean solar day* is, however, used for civil purposes, while sidereal time is employed in observatories from its manifest convenience for stellar observations. The mean solar day is the proper average length of the actual solar days over a sufficiently great interval of time, and since the positions of stars or of the sun at any given instant of time thus reckoned have been computed, the actual mean solar time of any instant can be obtained by observations of the heavenly bodies. Thus, an accurate clock set to keep as nearly as possible mean solar time at an observatory has its error determined by daily observations, and can be arranged to give a signal that a chosen instant has arrived.

12. **Time Signals.**—Time signals are given at many places by arranging an observatory clock to pass a current of electricity which drops a ball or fires a gun, or in some other way makes a visible or audible signal. Time signals are also automatically distributed daily to the chief provincial post offices of England from the Royal Observatory at Greenwich by means of the telegraph lines radiating from the General Post Office in London, and these signals are retransmitted to the subordinate offices by clerks standing ready to signal when the time indication from Greenwich arrives at the provincial offices.

At various places in London and elsewhere clocks are to be found in private establishments which are said to be kept regulated to exact mean solar time. These are often many seconds in error, and even

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a clock continuously controlled may indicate wrongly, inasmuch as it is only kept in beat, and an error of a whole number of seconds left at starting will remain unchanged. Dependence can only be placed on the public time signals, or the hourly private signals which are received at many business establishments. The first stroke of each hour also on the clock at Westminster Palace may generally be taken as the exact time to within less than a second, if heard at a point in or very near Palace Yard.

13. **Numerical Relations of Solar and Sidereal Time.**—The mean solar day is 1·00273791 sidereal day. A sidereal second of time is defined as the interval in which the earth turns through $\frac{1}{86400}$ of a complete turn, that is, through the angle 360°/86400 or $\frac{1}{240}$ of 1°, a sidereal minute as the interval in which the earth turns through $\frac{1}{4}$ of 1°, and a sidereal hour as the interval in which the earth turns through 15°.

The mean solar day is thus 24h. 3m. 56·556s. of sidereal time.

A mean solar day is divided likewise into 86,400 mean solar seconds, in the sense that a mean solar second is the time taken by the earth to turn about its axis through $\frac{1}{86400}$ part of the angle it turns through in a mean solar day, and so for the other intervals, the minute and the hour. A sidereal day is 23h. 56m. 4·090s. of mean solar time.

14. **Question of Slowing of the Earth's Rotation.**—It will be observed from what has been stated above that the equality of intervals of time is nothing ultimate or absolute, but is a matter of definition. Equal intervals of time are here defined as those in which the earth turns through equal angles about its axis. As has been already stated a dynamical method of defining equal intervals of time will be described later, and some further information will be given as to the possible discrepance between intervals of time so defined and the measures given for them by the rotation of the earth. Owing to tidal friction the earth's rotation must be very gradually falling off, and this influence is only very slightly counteracted by shrinkage of the earth's volume, an effect which we shall see increases the speed of rotation.

It is clear that any slowing of the earth's rotation would become apparent by an advance of the heavenly bodies beyond their calculated places. This has been observed in the case of the moon, which appears to be in advance of its calculated place by an amount which increases as the square of the time. Part of this, there is reason to believe, is due to gradual diminution of the eccentricity of the earth's orbit, but there is an outstanding remainder which, if put down to tidal friction, shows that the earth falls behind a watch or clock keeping exact time by about twenty-two seconds in a century. (See the Chapter below on *Astronomical Dynamics*.)

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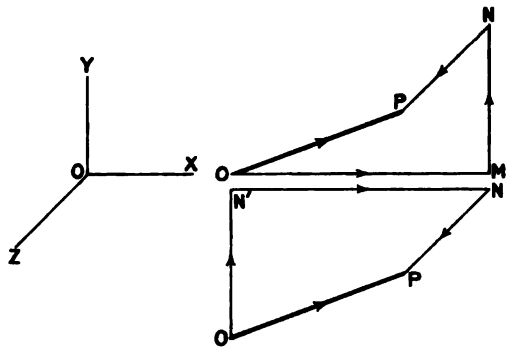
CHAPTER II.

KINEMATICS, OR THE GEOMETRY OF MOTION.

15. THE physical phenomena which first attract attention are those presented by the motions of bodies. But these motions are only relative, that is, they are the changes of position of some bodies relatively to others which we regard as fixed. Absolute or non-relative motion seems to be an impossible idea.

16. **Position of a Point. Steps.**—It is necessary therefore to

FIG. 3.



suppose first that the position of a point is expressible with reference to some geometrical system supposed known and fixed in space. Thus, if we have a plane, a line in that plane, and a point in the line all given, we may define the position of a point P not in the plane in the following manner:—Let the line beginning at O and ending at P be called the step OP. This step is equivalent to three other steps which may be made with reference to the given plane and line as follows. First a step OM, Fig. 3, is made along the line OX, then a step MN in the given plane and perpendicular to OX, and lastly a step NP at right angles to the plane. These steps are perfectly determinate in magnitude and direction, but they may be taken in any order. For example we might take first the step ON' equal in

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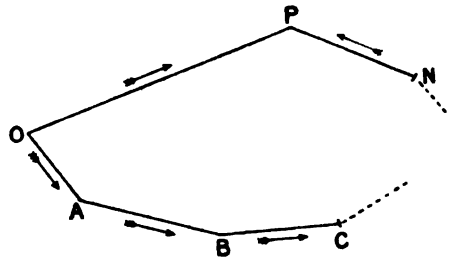
length and parallel to MN, then $N'N$ equal and parallel to OM, and finally NP as before.

17. **Addition and Equivalence of Steps.**—It is clear that the step OP might be specified by any number of successive steps OA,

FIG. 4.



AB, BC, ..., NP, Fig. 4, taken in any manner whatever, subject to the single condition that the series starts from O and ends at P. Thus we may write, using the sign of equality to denote equivalence,

$$\left.\begin{aligned} & OP = OA + AB + BC + \ldots + NP \\ \text{or,}\quad & OA + AB + BC + \ldots + NP + PO = 0 \\ \text{or,}\quad & OP + PN + \ldots + CB + BA + AO = 0 \end{aligned}\right\} \qquad (1)$$

since either of the last two arrangements of steps forms a closed chain starting from O and returning to the same point.

From these we see that any step OP must be taken as equivalent to the step $-PO$, inasmuch as we have

$$OP + PO = 0. \qquad (2)$$

It is to be observed that here all steps are taken as equivalent which are the same in the two respects, magnitude and direction, without regard to the position of the points which mark the beginning and end of the step. Steps, however, will occur later which are *localised*, that is act along specified lines.

Consider now the single step which is equivalent to, or is, as we shall term it, the resultant of two steps, $p\alpha$, $q\beta$, where p, q are any real numbers, and α, β steps of given length in assigned directions. If AB (Fig. 5) represent $p\alpha$ (α being a step in the direction AB), and BC in the same way represent $q\beta$, we have

$$AC = p\alpha + q\beta.$$

If, then, we have also

$$AC = p'\alpha + q'\beta$$

we get

$$(p - p')\alpha + (q - q')\beta = 0. \qquad (3)$$

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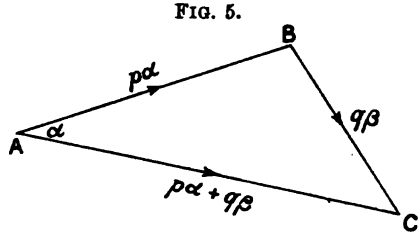
Now it is clear that no step in the direction BC can annul a step in the direction AB. Hence (2) gives the *two* equations

$$p = p', \quad q = q'. \tag{4}$$

Again, if $p\alpha$, $q\beta$, $r\gamma$ (Fig. 6) be three successive steps AB,

FIG. 5.



BC, CD, and the four points A, B, C, D do not lie in one plane, the equivalence

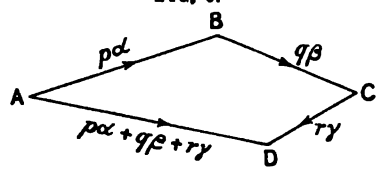
$$p\alpha + q\beta + r\gamma = p'\alpha + q'\beta + r'\gamma,$$

where p, q, r, p', q', r' are numbers, yields three equations connecting these multipliers, namely,

$$p = p', \quad q = q', \quad r = r'. \tag{5}$$

For the two steps $p\alpha$, $q\beta$ give a resultant AC, and this combined with $r\gamma$ makes up the step AD, and no other single step in the

FIG. 6.



plane ABC than AC combined with a step along CD (which is not in that plane) would give the resultant AD.

The numerical multipliers p, q, r, which may be positive or negative, integral or fractional, are frequently called *scalar* quantities, or, simply, scalars.

The theorem just proved is of great importance in the algebra of displacements. The first simple case of it stated in (3) is represented in Fig. 7, and is expressed by the theorem that the opposite sides of a parallelogram are equal to one another.

18. **Examples of the Geometry of Steps.**—The results already obtained afford easy proofs of geometrical theorems: but we shall confine ourselves here mainly to those which are of direct use in dynamics.

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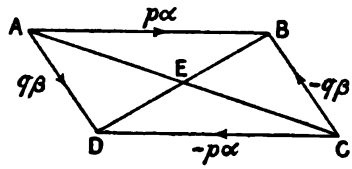
Draw the diagonal BD of the parallelogram in Fig. 7, and let the two diagonals intersect in E. Then we have

$$AE = m(p\alpha + q\beta)$$

FIG. 7.



where m is some multiplier to be determined. [Of course also $EC = (1-m)(p\alpha + q\beta)$.] But also

$$AE = AD + DE = q\beta + m'(p\alpha - q\beta)$$

where m' is another multiplier. Hence we obtain

$$m(p\alpha + q\beta) = q\beta + m'(p\alpha - q\beta),$$

or,

$$m - m' = 0, \quad m + m' = 1, \text{ that is}$$
$$m = m' = \tfrac{1}{2}.$$

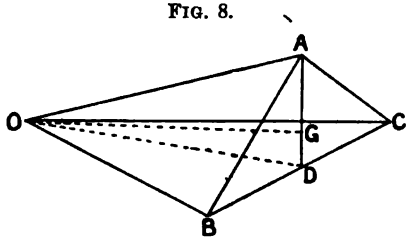
This proves that

$$AE = \tfrac{1}{2}(AB + BC), \quad DE = \tfrac{1}{2}(DC + CB)$$

or the diagonals of the parallelogram bisect one another.

Again, let steps α, β, γ be drawn from an origin O to three points

FIG. 8.



A, B, C (Fig. 8), not necessarily in the same plane with O. Let D be the point midway between B and C. Then

$$OD = OC + CD = \gamma + \tfrac{1}{2}(\beta - \gamma) = \tfrac{1}{2}(\beta + \gamma). \tag{6}$$

D is called the *mean point* of B, C.

If now G be a point on AD we shall have

$$OG = OD + DG = \tfrac{1}{2}(\beta + \gamma) + k[\alpha - \tfrac{1}{2}(\beta + \gamma)]$$

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where k is a scalar multiplier. If G be taken so that $DG/GA = \frac{1}{2}$, we have $k = \frac{1}{3}$, and

$$OG = \tfrac{1}{3}(\alpha + \beta + \gamma). \tag{7}$$

G is called the mean point of A, B, C.

19. **Mean Point of a System of Points.**—Consider now any number n of points $A_1, A_2, \ldots, A_n$ not necessarily in one plane, the steps to which from O are $\alpha_1, \alpha_2, \ldots, \alpha_n$. A point G can be found the step to which from O is given by the equivalence

$$OG = \frac{1}{n}(\alpha_1 + \alpha_2 + \ldots\ldots + \alpha_n). \tag{8}$$

G is called the mean point of the system $A_1, A_2, \ldots, A_n$.

The point G thus found is independent of the choice made of the point O, the origin from which the steps to $A_1, A_2, \ldots, A_n$ are measured. For let any other point O' be chosen as origin, and the step from it to O be ρ. We have then $O'A_1 = \alpha'_1 (= \alpha_1 + \rho)$, $O'A_2 = \alpha'_2 (= \alpha_2 + \rho)$, ... Let the mean point found in the same way as before be G'. The step from O' to G' is

$$O'G' = \frac{1}{n}(\alpha'_1 + \alpha'_2 + \ldots + \alpha'_n) = \frac{1}{n}(\alpha_1 + \alpha_2 + \ldots + \alpha_n) + \rho,$$

which is obviously the step from O' to G. G and G' are thus coincident.

If any number k of the points $A_1, A_2, \ldots$, form a group coincident at A, say, k others form a group coincident at another point B, and so on, and $\alpha, \beta, \gamma, \ldots$ be the steps from O to the respective groups, we have clearly by (8)

$$(h + k + \ldots.)OG = n.OG = h\alpha + k\beta + \ldots. \tag{9}$$

The step $n.OG$ is the resultant of the steps $\alpha_1, \alpha_2, \ldots, \alpha_n$, that is, it is the single step which is equivalent to the whole set of steps taken in succession, each in its own proper direction, and each having its initial point coinciding with the final point of the preceding step. The step $n.OG$ is also the resultant of the steps $h\alpha + k\beta + \ldots.$

We may call the numbers $h, k, \ldots$, for the present, the multiplicities of the steps to the points $A, B, \ldots$.

20. **Replacement of Groups of Points by Single Points.**—It is worth noticing that if we have say, m, points of any given multiplicities, we may divide these in any chosen way into groups, find the mean point of each group, then suppose each group replaced by its mean point, supposed to have a multiplicity equal to the sum of the multiplicities of the group, and find the mean point of the final system, which will be that of the original system. For putting $h_1, h_2, \ldots$ for the multiplicities of the points of one group the steps to which from the origin are $\alpha_1, \alpha_2, \ldots, k_1, k_2 \ldots$ for the multiplicities

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of another group the steps to which are $\beta_1, \beta_2, \ldots$, and so on, we see by (9) that the mean point of the system is given by

$$(h_1 + h_2 + \ldots + k_1 + k_2 + \ldots + \ldots)OG$$
$$= h_1\alpha_1 + h_2\alpha_2 + \ldots + k_1\beta_1 + k_2\beta_2 + \ldots + \ldots\ldots$$

or as it may be written

$$(h_1 + h_2 + \ldots + k_1 + k_2 + \ldots + \ldots)OG$$
$$= (h_1 + h_2 + \ldots.)\frac{h_1\alpha_1 + h_2\alpha_2 + \ldots.}{h_1 + h_2 + \ldots.}$$
$$+ (k_1 + k_2 + \ldots.)\frac{k_1\beta_1 + k_2\beta_2 + \ldots}{k_1 + k_2 + \ldots}. \tag{10}$$

The quantity on the right is the sum of the products obtained by multiplying the step to the mean point of each group by the sum of the multiplicities of the points of the group. Thus the theorem stated above is proved.

This theorem is continually employed, as will be seen later, in the ordinary process of finding the centre of inertia, or centre of mass, of a system of particles, whether discrete or forming a continuous body.

As an example, we may find from (9) how to construct the resultant of the steps OA_1, OA_2, where A_1, A_2 have multiplicities h, k respectively. Let OG (where G lies on A_1A_2) be the direction of the resultant. G is the mean point of A_1, A_2, as we have seen. Then

$$A_1G = -\alpha + \frac{h\alpha + k\beta}{h+k} = \frac{k}{h+k}(\beta - \alpha)$$

and

$$GA_2 = \beta - \frac{h\alpha + k\beta}{h+k} = \frac{h}{h+k}(\beta - \alpha);$$

or

$$\frac{A_1G}{GA_2} = \frac{k}{h}. \tag{11}$$

Thus G divides A_1, A_2 into two parts A_1G GA_2, which are inversely as the multiplicities of the initial and final points. The resultant of the steps OA_1, OA_2 is represented by $(h+k)OG$.

21. **Properties of the Mean Point.**—Returning to the case of a system of particles at points A, B, ... and of multiplicities h, k, ... let the mean point of the system be taken as origin, and any straight line be drawn through this point. The step from G to the point A may be compounded of two steps, a step ρ_1 from G along this straight line, and a second step α' perpendicular to this straight line from the final point of ρ_1 to A. Thus $\alpha = \rho_1 + \alpha'$. Similarly the step GB may be compounded of ρ_2 and β', the former along, the latter perpendicular to the straight line through G. Thus we get by (9)

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$$h(\rho_1+\alpha')+k(\rho_2+\beta')+\ldots\ldots=0,$$

or

$$h\rho_1+k\rho_2+\ldots+h\alpha'+k\beta'+\ldots=0.$$

It is obvious that $h\rho_1+k\rho_2+\ldots$ is equivalent to a step along the given straight line, while $h\alpha'$, $k\beta'$, ... are steps in the direction of α', β', ... respectively. The interpretation of the equation just found is of course that if these steps be taken in succession, in their proper directions, they will give a closed polygon. But the steps $h\alpha'$, $k\beta'$, ... thus taken will lie in a plane perpendicular to the straight line, and hence we must have

$$h\rho_1+k\rho_2+\ldots.=0.$$

Thus if each of the points, A, B, ... were transferred to the foot of the perpendicular let fall from the actual position of the point to any straight line through the mean point of the system, the mean point would not be altered.

Now consider any number n of steps A_1B_1, A_2B_2, ... drawn from a set of initial points A_1, A_2, ..., A_n, to a set of final points B_1, B_2, ..., B_n. The steps OB_1, OB_2, ..., OB_n to the points B_1, B_2, ..., B_n, from any origin O, have a resultant $n.OG_B$ where G_B is the mean point of B_1, B_2, ..., B_n. But

$$\begin{aligned} n.OG_B &= OB_1+OB_2+\ldots.+OB_n \\ &= OA_1+A_1B_1+OA_2+A_2B_2+\ldots\ldots+OA_n+A_nB_n. \end{aligned}$$

If O be the mean point of A_1, A_2, ..., A_n, we have identically

$$OA_1+OA_2+\ldots\ldots+OA_n=0,$$

and therefore

$$n.OG_B=A_1B_1+A_2B_2+\ldots.+A_nB_n.$$

The resultant of the n given steps is thus n times the step joining the mean points of the groups of initial and final points.

Any step OA may be resolved into any number of steps of which it is the resultant. For these, if taken in succession, will form an unclosed polygon, not necessarily plane, the initial point of which is O, and the final A.

22. **Calculation of Length and Direction of Resultant Step.**—It is frequently necessary to calculate the numerical length of the resultant of two steps, when their lengths and the angle between them is given. Thus, taking the arrangement of Fig. 5, let s_1, s_2 be the lengths of AB, BC. The angle between these steps we take as the angle between AB produced and BC. Let it be denoted by θ. Then if s be the length of AC the resultant, we have by an elementary theorem of the trigonometry of the triangle,

$$s^2=s_1^2+s_2^2+2s_1s_2\cos\theta, \tag{12}$$

which gives s.

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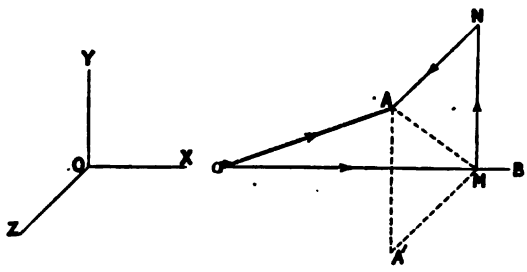
The angle BAC which s makes with AB must also clearly fulfil the relation

$$\left.\begin{aligned} \cos BCA &= \frac{s_1 + s_2 \cos\theta}{s}, \\ \text{similarly we have} \\ \cos BCA &= \frac{s_2 + s_1 \cos\theta}{s}. \end{aligned}\right\} \quad (13)$$

If we have given any number of steps, and it is required to find the length and direction of their resultant, we may use this theorem to find, first, the length and direction of the resultant of two of the given steps, which may then be supposed replaced by their resultant. The resultant of this and a third step may then be calculated in the same way, and so on until the whole system given is exhausted.

23. **Rectangular Resolution of Steps. Mean Point found by Rectangular Steps.**—As a particular case of such resolution, we may resolve OA into two steps at right angles to one another, or into

FIG. 9.



three steps which are mutually at right angles. This last mode of resolution will be very often useful in the treatment of a system of steps, and we shall choose for each step the same directions of resolution. Thus in Fig. 9 the three steps, OM, MN, NA, are supposed to be in the directions OX, OY, OZ respectively, and their sum is equivalent to OA. We shall denote unit steps along OX, OY, OZ respectively by i, j, k, and the numbers of such units in OM, MN, NA by x, y, z, so that $OM = xi$, $MN = yj$, $NA = zk$. Thus

$$OA = xi + yj + zk.$$

OX, OY, OZ are called the axes of co-ordinates, and are referred to as the axes of x, y, z respectively.

The projection of a step OA, on another OB, is the step OM, intercepted between O and the foot of a perpendicular AM let fall from A on OB; x, y, z are the lengths of the projections of OA on these axes of co-ordinates.

We may exemplify this mode of resolution by applying it to the discussion of the mean point of a system of particles, § 19.

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Let three mutually rectangular directions, OX, OY, OZ, be chosen, along which i, j, k are unit steps, and let x_1, y_1, z_1, x_2, y_2, z_2, ... be quantities such that

$$OA = x_1 i + y_1 j + z_1 k$$
$$OB = x_2 i + y_2 j + z_2 k$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

The equation

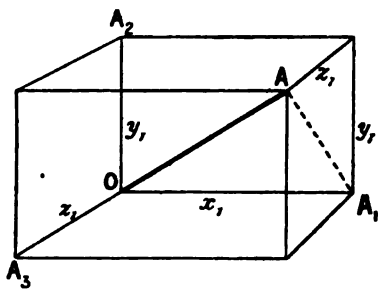
$$(h_1 + h_2 + \ldots)OG = n.OG = h_1.OA + h_2.OB + \ldots$$

gives, if $\bar{x}$, $\bar{y}$, $\bar{z}$ be the lengths of the steps along OX, OY, OZ which taken together are equivalent to OG,

$$\left.\begin{aligned} n\bar{x} &= h_1 x_1 + h_2 x_2 + h_3 x_3 + \ldots \\ n\bar{y} &= h_1 y_1 + h_2 y_2 + h_3 y_3 + \ldots \\ n\bar{z} &= h_1 z_1 + h_2 z_2 + h_3 z_3 + \ldots \end{aligned}\right\} \quad (14)$$

If $\bar{x} = \bar{y} = \bar{z} = 0$, that is if O be the mean point of the system the quantities on the right of (14) are each zero. Equations (14)

FIG. 10.



will be of great use in the determination of the centres of mass of systems of particles.

24. **Connection between Length and Direction of Step and its Rectangular Components.**—If the number of units of length in OA be l, it is easy to prove that

$$l_1^2 = x_1^2 + y_1^2 + z_1^2. \quad (15)$$

For let Fig. 10 represent a rectangular prism, the edges of which, meeting at O, lie along the axes OX, OY, OZ, and which is such that $OA_1 = x_1$, $OA_2 = y_1$, $OA_3 = z_1$, so that $OA = l_1$. Let r_1 be the length of the step A_1A, then since OA_1A is obviously a right angle, we have $l_1^2 = x_1^2 + r_1^2$; and since $r_1^2 = y_1^2 + z_1^2$, we obtain the relation stated above.

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Again, if α_1, β_1, γ_1 be the angles between the step OA and the axes, it is easy to see by Fig. 10 that

$$\cos\alpha_1 = \frac{x_1}{l_1},\ \cos\beta_1 = \frac{y_1}{l_1},\ \cos\gamma_1 = \frac{z_1}{l_1}. \tag{16}$$

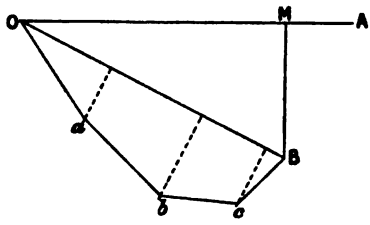
These cosines (called usually the *direction-cosines* of the step) are connected by the relation

$$\cos^2\alpha_1 + \cos^2\beta_1 + \cos^2\gamma_1 = 1, \tag{17}$$

as is obvious from (15). Similarly the direction-cosines α_2, β_2, γ_2. Of OB are x_2/l_2, y_2/l_2, z_2/l_2, where l_2 is the length of OB.

The sum of the projections on a step, OA, of any number of successive steps beginning at O, and ending at B, is also OM. This is

FIG. 11.



obvious from Fig. 11, in which the steps Oa, ab, ... may be regarded as not coplanar.

But we have

$$OB = x_2 i + y_2 j + z_2 k,$$

and the lengths of the projections of $x_2 i$, $y_2 j$, $z_2 k$ on OA are respectively x_2x_1/l_1, y_2y_1/l_1, z_2z_1/l_1. Thus if θ denote the angle AOB between the two steps OA, OB, we obtain the equation

$$\cos\theta = \frac{x_1x_2}{l_1l_2} + \frac{y_1y_2}{l_1l_2} + \frac{z_1z_2}{l_1l_2},$$

or

$$\cos\theta = \cos\alpha_1\cos\alpha_2 + \cos\beta_1\cos\beta_2 + \cos\gamma_1\cos\gamma_2. \tag{18}$$

25. **Relative Displacement.**—We shall now consider displacements of a system of particles relatively to a particle which itself has suffered displacement. Let the system consist first of a single particle B, and let A be the particle relatively to which the displacement is to be determined. Let α, β be the displacements to which A, B have been respectively subjected. We define the displacement of B relatively to A as the total step made by B when, after the displacements α, β have been effected, both particles are given a displacement equal and opposite to that of A, that is a displacement $-\alpha$. Thus it is $\beta - \alpha$.

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The displacement of B with respect to A is the displacement to which β is changed by imposing on both A and B displacements equal and opposite to α. By this process A is brought back to its original position with respect to the geometrical system of reference, while the configuration of the particles in space is not changed.

26. **Relative Displacement of Mean Points of Two Groups.**—Now consider two groups of points $A_1, A_2, \ldots, A_m, B_1, B_2, \ldots, B_n$, all of the same multiplicity, unity say, and let these receive displacements $\alpha_1, \alpha_2, \ldots, \alpha_m, \beta_1, \beta_2, \ldots, \beta_n$. The displacement of the mean point G_A of the first group is $(\alpha_1+\alpha_2+\ldots+\beta_m)/m$, and that of G_B the mean point of the second group is $(\beta_1+\beta_2+\ldots+\beta_n)/n$. The displacement of G_B relatively to G_A is

$$(\beta_1+\beta_2+\ldots+\beta_n)/n - (\alpha_1+\alpha_2+\ldots+\alpha_m)/m,$$

and that of G_A relatively to G_B is the same quantity with the opposite sign. If this quantity vanishes the mean points of the groups suffer no relative displacement.

If the multiplicities of the points of the groups be $g_1, g_2, \ldots, g_m, h_1, h_2, \ldots, h_n$ we must write in the formulæ just obtained $g_1\alpha_1, g_2\alpha_2, \ldots, g_m\alpha_m$ for $\alpha_1, \alpha_2, \ldots, \alpha_m$, $h_1\beta_1, h_2\beta_2, \ldots, h_n\beta_n$ for $\beta_1, \beta_2, \ldots, \beta_n$ and $g_1+g_2+\ldots g_m$, $h_1+h_2+\ldots+h_n$ for m, n respectively. Thus the displacement of G_B relatively to G_A becomes

$$\frac{1}{h_1+h_2+\ldots+h_n}(h_1\beta_1+h_2\beta_2+\ldots+h_n\beta_n)$$
$$-\frac{1}{g_1+g_2+\ldots+g_m}(g_1\alpha_1+g_2\alpha_2+\ldots.+g_m\alpha_m).$$

27. **Rotation of a System of Particles round an Axis.**—We now consider the displacement of a system of n particles by simple revolution round an axis. The displacement is expressed by the statement that the perpendiculars let fall from the particles of the system to the axis are all turned (while remaining perpendicular to the axis) round their extremities at the axis through the same angle θ. It is clear that since the particles are not relatively displaced and the perpendiculars remain of the same length as before, the perpendicular let fall from the mean point of the system to the axis also remains of the same length, and is turned through the same angle. Also the mean point of the system is not changed by turning the whole, without relative displacement of the particles, round an axis which passes through the mean point in the initial position of the system.

28. **Multiplication of Steps by Complex Numbers.**—If a step ρ be turned through 180° it becomes $-\rho$. Thus we may call multiplication by -1 the operation which reverses the direction of a step. This reversion of direction may, however, be regarded as the result of two successive turnings of the step through a right angle in any definite plane. Thus if $i^2=-1$, i is an operator multipli-

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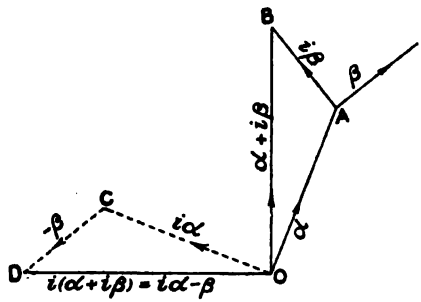
cation by which turns a step through 90° in that plane. The operator may be positive or negative, that is the step may be turned through 90° in the positive or in the negative direction. The direction of turning is positive when to an observer regarding the line it appears to turn oppositely to the hands of a watch, the face of which is seen in the plane of turning by the observer. If i is the operator which turns a step through 90° in this direction, $-i$ is the operator which turns it through 90° in the reverse direction. Thus

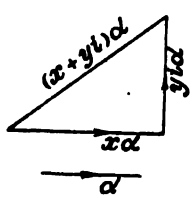
FIG. 12.



multiplication by $+i$ followed by multiplication by $-i$, or *vice versa*, is equivalent to multiplication by $+1$, that is the process leaves the step unaltered.

Thus if α, β be any two steps $\alpha+i\beta$ is the resultant of the step α and a step β turned through a right angle in the positive direction from that of β. The step $i(\alpha+i\beta)$ or $i\alpha-\beta$ is a step of the same length as $\alpha+i\beta$ but turned through $+90°$ from the direction of that step. This is easily verified by a diagram (Fig. 12): $i\alpha$ is α turned through $+90°$, and the combination of this with β reversed must give a resultant perpendicular to $\alpha+i\beta$.

FIG. 12'.



If a, b be the number of units of length in α, β respectively, the number of units of length in each of the steps $\alpha+i\beta$, $\alpha-i\beta$, $i\alpha-\beta$, $i\alpha+\beta$ is the same, namely $\sqrt{a^2+b^2}$. The positive numbers a, b may be called the *moduli* of α, β, and $\sqrt{a^2+b^2}$ (always taken positive) is the *modulus* of each of the resultant steps exhibited.

Multiplication of a step by a positive number x simply multiplies the modulus of the step by x, that is it alters the length of the step to x times its former length. Multiplication by $-x$ alters the length of the step in the same ratio, and reverses its direction. Hence multiplying by xi alters the length in the ratio of x to 1, and turns the step through $+90°$. Multiplication by $-xi$ effects the same *stretch*, as we may call it, and turns through $-90°$.

A quantity $x+yi$ where x, y are any positive or negative real

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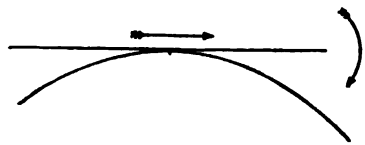
numbers, and i is the operator just defined, is called a *complex number*. The positive number $\sqrt{x^2+y^2}(=r)$ is its modulus, and the number x^2+y^2 is called its *norm*. If θ be such that $\cos\theta = x/r$, $\sin\theta = y/r$, θ is called the *argument* of the complex number.

When a step a is multiplied by the complex number $x+yi$, where x and y are positive real numbers, the result is the step $xa+yia$. The first part is the original step a stretched in the ratio of x to 1; the second part is a stretched in the ratio of y to 1, and turned through $+90°$ (see Fig. 12′). If the complex number be $x-yi$, the result is the same as before except that the angle of turning for the second part is $-90°$. If the number is $-x+yi$ the first part is reversed in direction, the second part remains yia. If the number is $-x-yi$, the result is $-(xa+yia)$ that is the first result reversed. The length of the new step in each case is $\sqrt{x^2+y^2}\times$ *the length of the step* a.

Again, multiplication by two complex numbers, $x+yi$, $x'+y'i$, in succession, where x, x', y, y' are any real numbers, is equivalent

FIG. 13.



to multiplication by the complex quantity $xx'-yy'+(x'y+xy')i$, which is the product of the complex numbers. This will be seen at once by performing the multiplications. The modulus of this complex number is $\sqrt{(xx'-yy')^2+(xy'+x'y)^2}$ or $\sqrt{x^2+y^2}\sqrt{x'^2+y'^2}$, the product of the moduli of the separate numbers. In the same way the result of multiplying by n complex numbers in succession can be found. It is the same as that of multiplying by a single complex number which is the product of the n complex factors: the modulus of this is the product of the moduli of the factors; its argument is the sum of the arguments of the factors.

29. **Continuous Displacement. Generation of a Curve by Point moving along a Turning Line. Speed.**—We have now to consider displacement progressing in time. With the metaphysical idea of time we are not here concerned: it must be taken for granted. The succession of ideas and of events gives a notion of the flux of time, and we shall here consider it as a quantity capable of being divided into successive measurable intervals of any degree of smallness. The practical definition of equal intervals of time is considered above in chapter i., and it will be further dealt with in the fuller discussion to follow of *Measures and Instruments*.

Let a geometrical curve be conceived as generated by the motion of a point along a straight line which is turning round the point, as in Fig. 13. If the curve lies in a plane, or is, as it is called, a plane

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curve, the line lies always in one plane. The straight line in any one of its positions is called the *tangent* to the curve at the point.

We have first to consider the motion of the point, then that of the line. Let it be supposed that in an interval of time t the point describes a distance along the curve equal to s. Then we call the ratio s/t the *average speed* of the point during the interval t. If the motion is such that when t is divided into any number n however great of equal parts the distance described by the point in each of these has the same value, s/n, the speed of the point is said to be uniform during the interval t.

The speed may, however, be variable. If it varies, we suppose that the amount of change of speed which takes place in any interval of time t/n, however small, bears always a determinate ratio to the interval, that is we suppose the variation of speed to be *continuous*.

30. **Angular Velocity.**—The motion of the line in any interval of time is measured by the angle through which the line has turned in that interval. This angle is measured by drawing from a point A, say, two lines parallel to the positions of the turning line at the beginning and end of the interval. Then if a circle be described from A as centre so as to cut the two lines, the angle is the ratio of the length of the arc intercepted between the lines to the length of the radius. Or if the radius have unit length the angle is simply the length of the arc. The unit angle is thus the angle subtended at the centre of the circle by an arc of length equal to the radius.

Let the angle turned through in an interval t be θ, then θ/t is called the *mean angular* velocity of the line during the interval t. If when t is divided into any number n however great of equal parts, the angle turned through is always in the same plane, and of amount θ/n, the angular velocity is said to be uniform during the interval t.

As is the case with the speed of the point, the angular velocity of the line may be variable. If it varies we suppose that the amount of change which takes place in any interval of time t/n, however small, bears always a determinate ratio to the interval, that is we suppose the variation of angular velocity to be continuous.

31. **Continuity of Motion of Line and Point.**—The angles turned through in the successive small intervals of time t/n are in the general case neither equal nor in one plane. We exclude here not only sudden changes in the amount of the angular velocity, but also sudden changes of the plane of turning, that is, if $\phi' - \phi$ be the angle between the planes in which the line is turning at the beginning and end of an interval t/n, the value of $(\phi' - \phi)/(t/n)$ has always a finite and determinate value.

The subject of varying velocity, whether linear or angular, will be considered more fully later, after some further general considerations have been stated.

We impose the condition upon the motion of the line and point that the line does not in any interval of time during which the motion of the point is considered make an infinitely great number of

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finite changes of direction. Thus if in any interval of time τ chosen however short, the point describes a distance CD (Fig. 14) on the curve AB, we exclude all such cases as that in which if the element

FIG. 14.

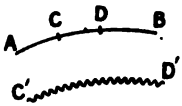


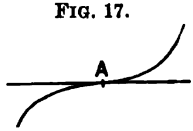
FIG. 15.

CD of the curve were sufficiently magnified it would be found to be of the form $C'D'$.

32. **Special Cases.**—Some special cases of the mode of generating a curve mentioned above may be here shortly referred to. If the point comes to rest on the line at a point A (Fig. 15) and then begins to gain speed in the opposite direction along it, while the line continues to turn in the same direction as before, we have the case represented in the figure. If the line reverses its angular motion at the same time as the point reverses its linear motion at the point A, the curve has a form like that shown in Fig. 16. The points A, A are called cusps.

FIG. 16.

FIG. 17.



If, however, the point continues in motion while the angular motion of the line becomes zero at A, and is thereafter reversed, we have the case shown in Fig. 17, where A is what is called a point of inflection.

33. **Velocity.**—The speed of a point moving uniformly has been defined above. We now consider the *velocity* of a moving point. The idea of velocity is wider than that of speed, as it includes the direction of the motion as well as its amount. In what has been stated above we have excluded the case of an abrupt change of direction of motion, that is the line along which the point is moving is not (except in some special cases to be dealt with when they arise) to suffer a finite change of direction in an interval of time which for practical dynamical purposes is infinitely short.

A velocity thus has a definite directed value at each instant of time: how that value and its variation are to be reckoned we now proceed to discuss.

In the case of varying velocity, as in that of uniform velocity, the numerical value of the rate of displacement is reckoned by the number of units of distance travelled per unit of time. That unit of

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time may be anything—a year, an hour, a minute, a second, or any fraction of a second. Thus the speed of a point may be equally expressed by saying that it is a mile per minute, 88 feet per second, or 8·8 feet per $\frac{1}{10}$ of a second. Also that particular speed may be possessed by the point only at one particular instant, and it may be retained unaltered for no interval of time however short. Thus the speed of a body falling freely from rest under gravity at a certain place is 32·2 feet per second at the instant at which the first second of its fall ends and the next second begins, is 64·4 feet per second at the end of the second second of fall and the beginning of the third, and so on, while the body does not retain any of the successive speeds it attains for any *interval* of time however short.

34. **Definition of Measure of Varying Velocity.**—In order, when the velocity is varying, to define it for a particular instant, we proceed as follows. An interval of time including the instant in question is taken, and the distance travelled by the body in that interval is supposed measured. Let it be expressed by s units of distance. Then if t be the measure of the interval of time in terms of a chosen unit, the average speed during the interval is expressed numerically by s/t. Let now the interval of time be made smaller and smaller, but be always taken so as to include the instant in question, then, if the velocity vary continuously, the ratio s/t will, as t is made smaller and smaller, approach a limiting value, which we define as that of the speed at the instant. The direction of motion is, in the limit, that of the distance represented by s. Thus the velocity is completely defined.

The justification of this process rests in the fact that we have a previous notion that if the motion is continuous there must exist *at each instant* a true rate of displacement, and all that is here done is to render this idea definite. That the idea of continuity of motion involves the notion of a definite rate of displacement at each instant may be seen from the following illustration. Let two trains run on parallel lines of rails in the same direction, one running uniformly, the other gradually getting up speed from rest. Let now the trains be imagined to be so long that a passenger in the latter train has during his observation the uniformly moving train opposite to him. If he fixes his attention on the carriage opposite, and withdraws it from his own motion, he will be conscious at first only of a motion of that carriage past him in the direction in which both trains are moving. As, however, his own train gets up speed this relative motion will fall off to zero, and thereafter the carriages of the uniformly moving train will seem to him to be moving backward. There was no *interval* of time, however short, of transition, but at an *instant*, the dividing mark between two intervals, the two trains were relatively at rest. The speed of the train the motion of which varies was at that instant precisely the same as that of the uniformly moving train. As the latter is perfectly definite, and may be anything, the speed in the other case is perfectly definite at every instant, provided the motion is continuous.

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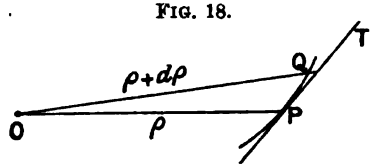
35. **Rate of Growth of a Step.**—Let P (Fig. 18) be the position of a particle at any instant, and let the direction of its motion at that instant be PT. In an interval of time dt, let the particle, at the rate of motion it has at P, have moved to Q, then the chord PQ is the amount by which the step OP must be increased to give OQ. We call this $d\rho$. As dt is taken more and more nearly equal to zero the chord, the arc, and the tangent approach more and more nearly to coincidence at P. Thus $d\rho/dt$, when dt is taken indefinitely small, represents in magnitude and direction the velocity at P. It is the rate at which a step of displacement in the direction PT is growing up. We shall usually denote this by $\dot{\rho}$.

The distinction between $\dot{\rho}$ and $\dot{s}$, or between $d\rho/dt$ and ds/dt, is that the former represents the rate of growth of step, that is of

FIG. 18.



displacement including direction, the latter usually denotes only the speed.

36. **Specification of Velocity.**—It is to be remarked that, while we must have recourse to an infinitely diminishing interval of time in *defining* the velocity of a particle at a particular instant, we do not generally specify a velocity as measured numerically by the ratio of the measure of a very small distance traversed to that of the very small interval of time in which it is described. The interval of time used in the specification may be any interval whatever. Thus in the expression ds/dt employed to denote some definite speed, v say, it is possible to consider dt as a time-interval not necessarily very small, but of quite arbitrary value. The value of ds, if this is done, is the corresponding distance given by the equation $ds = vdt$. The meaning of the expression is then that the rate of displacement at the instant is such that if it endured without alteration for *any* interval dt, a space ds would be described in that time.

37. **Varying Angular Velocity. Curvature.**—The same mode of defining the varying of a particle is applicable *mutatis mutandis* to the definition of varying angular velocity. The turning motion of the line by which the change of direction of the motion of a point in a curve is defined is supposed to vary continuously, so that at each instant the rate of description of angle is perfectly definite.

We shall sometimes denote angular velocity by the symbol $\dot{\theta}$, θ denoting the angle, at the instant under consideration, between the line and some fixed direction in the plane in which the line is then turning.

The ratio $\dot{\theta}/\dot{s}$ of the value of $\dot{\theta}$ at any instant to that of $\dot{s}$ at the

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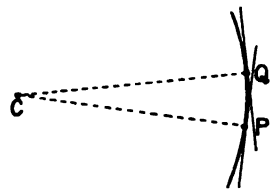
same instant is called the *curvature* of the curve at the position of the moving point. This is clearly the limiting value of the ratio of the angle between the tangents at the extremities of an element ds, of the curve (Fig. 19) and the length of the element, when the latter is diminished infinitely towards zero.

The reciprocal of this ratio $\dot{s}/\dot{\theta}$ is called the radius of curvature of the curve at the same point. This radius is supposed drawn from the point of contact in the direction perpendicular to and in the plane of motion of the tangent, and towards the concave side of the curve. Thus in Fig. 19, C, the point of intersection of lines drawn from P and Q in the plane of, and at right angles to, the tangents at these points, is approximately the centre of curvature for P or Q, if the element of arc PQ, which we denote by ds, is small.

FIG. 19.



For a circle $\dot{\theta}/\dot{s} = 1/r$, so that the centre of the circle is the centre of curvature for every element. Then for an element ds of any curve we may take C as the centre of a circle an element of which coincides with ds, and for which the ratio of the rate of turning of the tangent to the rate of displacement along the circle of the point of contact is the same as in the curve at the element ds.

38. **Illustrations of Varying Velocity.**—To illustrate what has been stated let us suppose that by observation it is found that the space s described by a moving particle in any time-interval t is expressed by the equation

$$s = v_0 t + \tfrac{1}{2}at^2, \tag{19}$$

where v_0 and a are constants. In an immediately succeeding interval, τ, the space described, σ say, is given by

$$\sigma = v_0(t+\tau) + \tfrac{1}{2}a(t+\tau)^2 - v_0 t - \tfrac{1}{2}at^2 = v_0\tau + \tfrac{1}{2}a(2t\tau + \tau^2),$$

which lies between $(v_0 + at)\tau$ and $[v_0 + a(t+\tau)]\tau$. If a be positive σ will be greater than the former of these values, and less than the second, and *vice-versâ* if a be negative. Now when τ is made smaller and smaller the difference, $a\tau^2$, between these values becomes smaller and smaller *in comparison with either*, that is the ratios

$$\frac{\tfrac{1}{2}a\tau^2}{(v_0 + at)\tau} \quad \text{and} \quad \frac{\tfrac{1}{2}a\tau^2}{[v_0 + a(t+\tau)]\tau}$$

approach more and more nearly the same limit, so that

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$$\text{Limit } \frac{\sigma}{\tau} = v_0 + at,$$

is the speed at time t and uniformly increases at rate a.

For any actual interval τ the distance $\frac{1}{2}a\tau^2$ represents the distance travelled in virtue of the change of speed during the interval. If the particle were to move throughout the interval τ, (1) with the speed it has at the beginning, *viz.* $v_0 + at$, (2) with the speed it has at the end, *viz.* $v_0 + a(t+\tau)$, the spaces travelled would be respectively $(v_0 + at)\tau$, $[v_0 + a(t+\tau)]\tau$. The true value lies between these limits, and in comparison with it the difference $a\tau^2$ (which represents twice the extra space just considered) becomes smaller and smaller as τ is diminished.

39. **Calculation of Velocities.**—In general let the space described in time t from some chosen era of reckoning be denoted by $f(t)$, that is be given by a certain algebraic expression involving t and constant quantities. Then the space σ described in time τ is accurately given by the equation

$$\sigma = f(t+\tau) - f(t),$$

since after the further lapse of τ, t has become $t+\tau$ and the space described from the era of reckoning has become $f(t+\tau)$.

Now let this be capable of being expressed by the equation

$$\sigma = [f'(t) + \epsilon]\tau, \tag{20}$$

where $f'(t)$ is another function of t. Then if, as τ is made smaller and smaller, ϵ becomes smaller and smaller, and ultimately vanishes when τ does,

$$\mathrm{L}\,\frac{\sigma}{\tau} = f'(t), \tag{21}$$

where, by $\mathrm{L}\,\frac{\sigma}{\tau}$ we denote the limit to which σ/τ approaches as τ is continually diminished.

If we take ds, dt any values of σ and τ which fulfil the equation

$$\frac{ds}{dt} = f'(t), \tag{22}$$

then the numerical value of the ratio ds/dt is that of the speed at time t. As stated above, it is possible to regard the interval dt as having any magnitude, whether one considered finite, or one considered infinitesimal, provided ds have the corresponding value given by the equation $ds = f'(t)dt$, where $f'(t)$ is the limiting value of the ratio $\delta s/\delta t$ when δt is made to vanish. In general, however, when ds and dt are considered as actual quantities, for example in processes of integration, they are taken as indefinitely small quantities fulfilling the relation just stated.

We may remark here, that when the relation between the distance

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traversed in a given time and the time is known, the speed at any instant can at once be calculated by the help of rules which have been constructed for algebraic functions. These rules are indeed those for what is called the *differentiation* of functions, and the theory and applications of the processes of differentiation form what is called the *differential calculus*. The fundamental notion of this calculus is, however, simply that of a rate of variation, which we have endeavoured to explain in connection with the idea of speed. The reader may usefully exercise himself in finding the value of the speed by first principles when s is given by simple functions of t.

The simplest case is that in which the relation is $s = vt + c$, where v and c are constants. In this case the speed $f'(t)$ is the constant v, and the distance s increases in simple proportion to the time. But in the general case, when $s = f(t)$, the value which $f'(t)$ has for any assigned value of t is the speed for that instant in precisely the same sense as that in which v is here the speed for every instant.

A very important case is that of a particle moving in a straight line in such a manner that its distance s from a fixed point in the line is given by the equation

$$s = a\cos nt, \tag{23}$$

where a and n are constants. Increasing t by any interval δt say, and supposing the corresponding increase of s to be δs, we have $s + \delta s = a\cos n(t + \delta t)$; and hence

$$\delta s = a\cos n(t + \delta t) - a\cos nt = -2a\sin n(t + \tfrac{1}{2}\delta t)\sin(\tfrac{1}{2}n\delta t).$$

Therefore
$$\frac{\delta s}{\delta t} = -2a\sin n(t + \tfrac{1}{2}\delta t)\sin(\tfrac{1}{2}n\delta t)\frac{1}{\delta t}.$$

But if δt be made smaller and smaller towards zero $\sin(\frac{1}{2}n\delta t)/\delta t$, approaches more and more closely the limiting value $\frac{1}{2}n$, so that the limiting value of the ratio $\delta s/\delta t$, which as before we denote by ds/dt, is given by the equation

$$\frac{ds}{dt} = -an\sin nt. \tag{24}$$

Instead of ds/dt the symbol $\dot{s}$ is very frequently used with exactly the same signification. This is known as the *fluxional* notation, from the fact that it was adopted by Newton in his theory of Fluxions, or the rates of flow, that is of variation, of varying quantities. From its brevity and convenience as serving to keep in mind the relation between velocity and displacement we shall often employ it.

40. **Graphical Representation of Velocity.**—It is clear that since a velocity is a step of displacement effected per unit of time, a velocity is capable of representation by a straight line drawn in the direction of motion, and made as many units of length as there are units of velocity. The unit of length chosen may, of course, be any whatever; what is essential is that when different velocities are repre-

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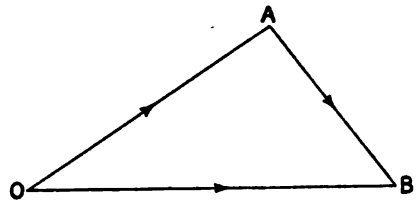
sented in a diagram, the same unit should be employed for all. The discussion of the composition and resolution of steps, given above, is directly applicable in all its processes and results to velocities.

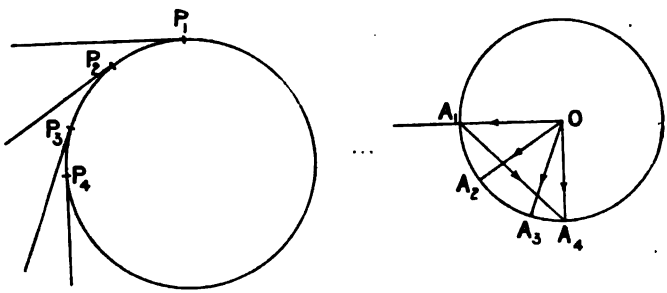
Let then OA (Fig. 20) represent the velocity of a particle at some specified instant of time. It is to be understood that the motion is in a direction parallel to that given from O to A, and that its magnitude is the number of times the length OA contains the chosen unit of distance. At a subsequent instant let the velocity be represented by OB, where, to take the general case, OB is not merely different in direction from OA, but is of different length. The velocity OB is evidently equivalent to the two velocities OA and AB. Of these OA is the previously existing velocity, AB is, whatever the manner of change has been, the resultant of the changes in the interval. Since when compounded with OA it gives OB, AB is in a very proper sense the *change* of velocity that has taken place in the interval.

FIG. 20.



The student may consider a case in which OA and OB are of the

FIG. 21.



same length, and note the direction and amount of AB in such a case. Clearly if the change of direction be θ, the length of AB is $2v \sin\theta/2$, if v denote the length of OA. Thus, the change of the velocity v in the interval in which the angle θ is turned through by the direction of motion is a velocity $2v\sin\theta/2$, inclined at an angle $\pi/2 + \theta/2$ (the supplement of OAB) to the positive direction of the initial velocity. The resultant velocity OB is unchanged in amount, but it is nevertheless a very different velocity from OA.

41. **Example. Particle moving in Circular Path.**—Take now the case of a particle moving with uniform speed round a circle, and

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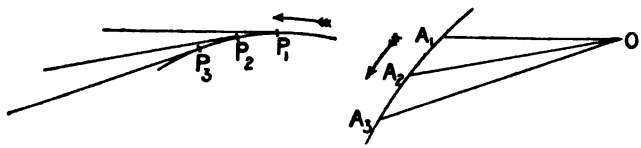
draw lines OA_1, OA_2, OA_3, ... from any point O in the direction of the velocities at successive points P_1, P_2, P_3, ... in the circle, and all of the same length, that expressing the speed of the particle. The extremities A_1, A_2, A_3 lie on a circle. The chords A_1A_2, A_2A_3, ... represent the changes of velocity in the successive intervals.

For an interval of a quarter of a period of revolution of the point round the circular path, as from P_1 to P_4 the velocity changes in direction through a quarter of a revolution, from OA_1 to OA_4. To the velocity OA_1 at the beginning of the interval there has been added a velocity A_1A_4 of amount $2v\sin 45°$, inclined at an angle of 135° with the positive directions of the initial and final velocities as shown in the right-hand diagram of Fig. 21.

After an interval of half the period of revolution, the velocity has been reversed in direction; that is to say, in that interval a velocity $2v$ in the direction opposed to the original velocity has been generated.

FIG. 22.



These examples may help to correct the ordinary impression of the beginner in dynamics, that when a particle moves with unvarying speed in a circle, the velocity of the particle does not undergo change.

42. **Hodograph.**—If a particle move in any path, whether a plane curve, or a curve in space, and P_1, P_2, ... (Fig. 22) be successive positions of the particle, and lines OA_1, OA_2, ... be drawn from a chosen point O, in the proper directions and of the proper lengths to represent the velocities at P_1, P_2, ... respectively, the points A_1, A_2, ... will, if the motion vary continuously, lie on a curve of continuous curvature, which will be fully specified when the motion in the path is known. This curve is called the *hodograph* of the motion of the particle, and is of great importance in discussions of the kinematics of a particle.* Examples of its use will be given later when the subject of acceleration has been dealt with.

43. **Simple Harmonic Motion.**—Returning now to the example of § 39, for which (23) gives the displacement, let AA', BB' be perpendicular diameters of a circle, of radius a and centre O, and let P be a point on the circumference which moves with uniform speed round the circle in the direction of the arrow. Let p be the foot of the perpendicular let fall from the point P on the diameter AA'. Then as P moves round the circle from A to B, p moves from A to O;

* The name is due to Sir William Rowan Hamilton who invented and made great use of the hodograph, especially in the discussion of elliptic motion. It had, however, previously been discovered and used by Möbius, *Mechanik des Himmels*, § 22 (1843).

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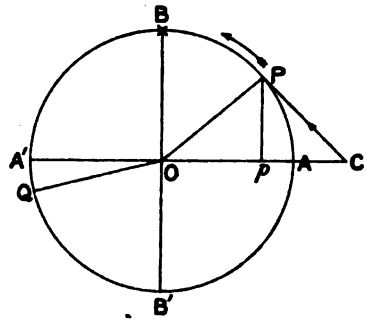
as P moves from B through A' to B', p moves from O along the diameter to A' and back to O; and as P returns from B' to A, p moves from O to A. It is also to be observed that P has a velocity which may be regarded as compounded of two simultaneous velocities, one parallel to the radius OA, and one perpendicular to OA. The former of these is the velocity with which the line pP is advancing perpendicularly to itself, the other that at which pP is shortening or lengthening. For example for the position of P shown in Fig. 23, the velocities are one from A to O, and another from p to P. The former is the velocity of the line pP from A to O, the other is the rate at which pP is lengthening, that is at which P is moving away from the line AA'.

FIG. 23.



Let now in Fig. 23 CP, which is in the direction of motion of the point in the circle when it is at P, represent, on a proper scale, the velocity of P, namely v, then Cp will, on the same scale, represent the velocity of p. But (dealing in what follows only with numerical magnitudes) we have $Cp = CP \sin pPC = CP \sin pOP$. If a be the length of the radius of the circle, and ω the angular velocity of the radius OP (which also, it is to be noticed, is the angular velocity of the turning line CP along which the point P is moving), the length of CP, that is, the magnitude of v, is $a\omega$. Also the angle pOP is the angle turned through from the instant at which P was at A. Let that instant be the zero of reckoning for t, then $\angle pOP = \omega t$. Thus

$$\dot{s} = -a\omega \sin \omega t. \tag{25}$$

This is exactly the same as the expression given in (24) (§ 39) if we put $\omega = n$. Thus we have the interpretation of n as the angular velocity of the circular motion by which that of p is defined. The value of s given in (23) is Op of Fig. 23, which clearly is $a \cos pOP$.

The motion of p here defined is of great importance, and is generally called *simple harmonic motion*. The time taken by the point P to move once round the auxiliary circle is called the period of the motion; clearly its value is $2\pi/n$. This is usually denoted by T. The reciprocal $1/T$ of the period, that is, the number of periods per second, is called the *frequency* of the motion.

If instead of reckoning the time from the instant at which P is at A (Fig. 23) as zero, we reckon it from the instant at which P is at Q say, then the angle POQ is now ωt. Denoting AOQ by ε, we have $pOP = \omega t - \varepsilon$, and the equations for s and $\dot{s}$ now become

$$s = a \cos(\omega t - \varepsilon) \tag{26}$$

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and

$$\dot{s} = -\omega a \sin(\omega t - \epsilon). \tag{27}$$

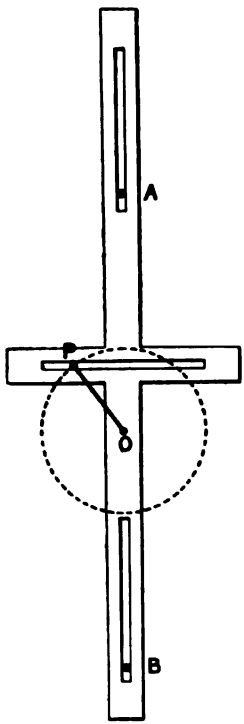
The angle ϵ is called the *epoch* of the motion. If measured in time it is ϵ/ω.

The distance OA, that is a, is called the amplitude of the motion.

FIG. 24.

44. **Modes of producing S.H.M.**—There are various modes of realising simple harmonic motion. The to-and-fro motion of the cross-head at the junction of the piston-rod of a steam engine with the connecting-rod is approximately simple harmonic, if the crank runs with uniform angular velocity. This it nearly does if the speed of the engine is equalised throughout the stroke by a fly-wheel. The motion would be exactly simple harmonic if the connecting-rod were infinitely long. The arrangement is shown in Fig. 24, in which the radius OB is supposed to revolve with uniform speed, while the extremity A of the rod AB moves in a straight line. The point A would clearly describe simple harmonic motion if AB were infinitely long; as it is, the varying inclination of AB to the line in which A moves is considerable.

FIG. 25.



An arrangement which gives an accurate simple harmonic motion is shown in Fig. 25. A crank represented by the radius OP turns with uniform angular velocity, while a pin at P moves along a slot, the successive positions of which are kept parallel to one another by guides, A, B, in a line at right angles to the slot.

The up-and-down motion of a weight attached to a spiral spring which has been elongated beyond the equilibrium position and then left to itself is very nearly, if not perfectly simple harmonic. So also is the backward-and-forward motion of the prongs of a vibrating tuning-fork.

45. **Resolution of any Periodic Motion to S.H.M.**—It will be shown later that any periodic motion whatever may be resolved into

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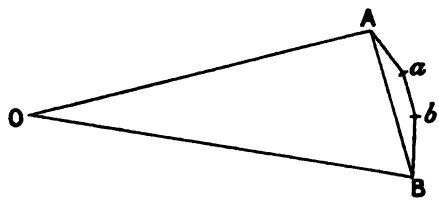
simple harmonic motions of different periods, epochs, and amplitudes compounded together. On this fact depends, among other things, the theory of musical sounds, and their analysis by the ear and resonators, and the analysis of the varying height of the tide, into the parts due to the various tide-producing constituents of the action of the sun and moon.

46. **Acceleration.** — We now consider more fully change of velocity and acceleration. It is necessary in the first place to distinguish between the change which the velocity sustains in a given time, and the rate at which it is changing at a given instant. This distinction is important, and must be clearly apprehended at the outset.

Let the velocity of a particle at a given instant be represented by OA, and at the end of a certain interval of time be represented by OB, then, as we have explained above, the change of velocity which

FIG. 26.



has taken place in the interval is represented by AB. The change may conceivably have taken place in any steps, Aa, ab, bB (Fig. 26), the resultant or total change is still AB. Let τ be the interval of time in which AB has grown up, then we define the ratio AB/τ as the *average acceleration* during the interval.

It is to be noticed that this ratio as stated, like that by which velocity has been defined above, transcends the ordinary arithmetical signification of a ratio, inasmuch as the quantity denoted by it has direction, namely, that of AB.

We shall suppose now that the velocity changes continuously, that is suffers sudden change neither of direction nor of amount. The change that has taken place in any interval of time is then perfectly definite. This amounts to assuming that at each instant of time the rate of turning of the line specified above (§ 29), along which the particle is supposed to be moving at each instant, and the rate of motion of the particle along it, are both quite definite and continuously vary.

47. **Measure of Acceleration defined.** — To define then the acceleration at a given instant, let an interval of time τ be taken including the instant, and let it be found that AB/τ for that interval can be brought as near a determinate limit as we please by taking τ sufficiently nearly equal to zero; that limiting value is the acceleration at the instant.

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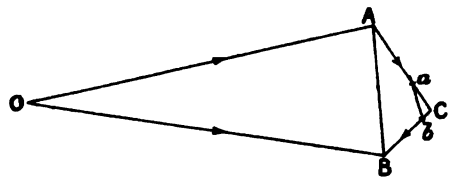
This definition assumes that such a limiting value exists, and it is essential that this assumption should be mentioned. For let, during a certain interval of time, velocity be continually generated in the direction AC (Fig. 27), and then let the direction of change alter suddenly to CB, and endure for another interval. If then we took a short interval of time which the instant of change of direction from AC to CB divides into two parts, we should get ab for the direction of the acceleration at the instant of change, and that direction would entirely depend on the ratio of the two parts of the interval, in the limit when it is taken infinitely short, and would not be definite at all. We could, of course, define the acceleration for an instant infinitely near the instant of change, before it or after it; for the former it would be along AC, for the latter along CB. The acceleration *at* the instant of change itself would have no meaning if the change of direction of velocity at C were perfectly sudden.

FIG. 27.



It is necessary to exclude the existence of such cases as this, or to divide the whole interval of change into parts, to any element of each of which the definition can be applied, and take account of the sudden changes, if any, where they occur.

48. **Acceleration in Path is Velocity in Hodograph.**—It is now possible to exhibit the elegant graphical method of discussing acceleration which the hodograph affords. Let a line OA turn round the extremity O so as to remain always parallel to the direction of motion of the particle in its path, and change in length so as always to represent the amount of the velocity. The point A will then trace out the hodograph of the particle's motion, which will, as explained above, be either a curve of continuous curvature, or one made up of parts each of continuous curvature.

Now, clearly, from the definition of acceleration given above, the rate of motion of A in the hodograph is in magnitude and direction precisely the acceleration in the path, or, as it is sometimes put, "the velocity in the hodograph is the acceleration in the path." For the rate of motion of A is to be finally interpreted as an acceleration. It represents the rate of change of OA, and OA, though a distance in the diagram, represents distance travelled per unit of time, that is a velocity. The rate of change of OA, the so-called velocity in the hodograph, is thus the rate of change of velocity. This remark seems necessary to obviate the difficulty which beginners seem often

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to have in understanding how what is spoken of as a velocity in one curve can be an acceleration in another.

49. **Examples of Hodograph.**—We now consider one or two simple cases of the hodograph.

(1) For a particle moving in a straight line with uniform velocity. The hodograph is plainly a point, the outer extremity of the straight line which represents the velocity. The acceleration is zero.

(2) For a particle moving in a straight line with varying speed. The hodograph is a straight line in the same direction.

(3) For a particle moving in any manner under the action of gravity only. In this case, as we shall see later, the direction of the

FIG. 28.

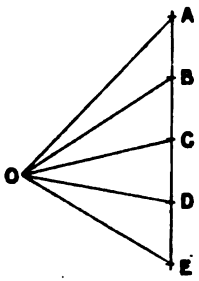
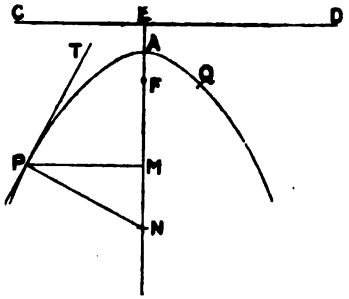


FIG. 28′.



acceleration is always vertical. Thus, unless the particle like a meteorite traverses an extensive path, the hodograph is a straight line drawn in the vertical direction.

The formation of the hodograph for the case of a football or projectile describing a parabolic path, is shown in Fig. 28. There the velocities after equal successive intervals of time are shown by OA, OB, OC, ..., for which the vertical steps AB, BC, ... are all equal, and represent the equal changes of vertical velocity which gravity produces in those intervals.

50. **Path of a Body under Constant Acceleration. Theory of Unresisted Projectile.**—That the path of a body moving with uniform acceleration is a parabola can easily be seen from the facts that the velocity perpendicular to the acceleration remains constant in magnitude and direction, while equal increments of velocity in the direction of the acceleration are produced in equal successive intervals of time. To fix the ideas we take the equivalent case of a free unresisted projectile, under the action of gravity, which we suppose to produce a constant downward acceleration. The lines OA, OB, ... in Fig. 28, represent the directions of motion in this case, that is the tangents to the path, after successive equal intervals of time; in other words, the tangents at the extremities of vertical ordinates which are at successive equal horizontal distances apart. Thus, if p be the tangent of the angle, MPT, θ say, of Fig. 28′, which

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the direction of motion makes at any instant with the horizontal, then $p = p_0 - at$ where p_0 is the value of the tangent at a point P_0 on the curve which is the position at any chosen instant of reckoning. But if u be the horizontal velocity and v the vertical velocity of the particle at the point P where the tangent of the inclination of the direction of motion to the horizontal is p we have

$$p = \frac{v}{u} = \frac{v_0}{u} - \frac{v_0 - v}{u}.$$

But $v_0 - v$ is the change of vertical velocity effected in time t, and this is gt, if g be the downward acceleration due to gravity. Thus $a = g/u$. If x be the horizontal distance of P from P_0 we have $x = ut$, and $t = x/u$. Thus we get

$$p = p_0 - \frac{g}{u^2}x.$$

That is the change of $\tan\theta$ is directly proportional to the change of abscissa of the curve. This is well known to be a property of the parabola.

The equation of the curve can be found as follows. In any time t from the instant of projection (with velocities u, v, say) the particle has travelled a distance $x = ut$ horizontally and a distance $vt - \frac{1}{2}gt^2$ vertically upwards (§§ 38, 90). We shall denote this by $-y$, so that y is taken positive when measured downwards. Replacing t in this expression by x/u we obtain

$$y = \frac{g}{2}\frac{x^2}{u^2} - \frac{v}{u}x, \tag{28}$$

which is the equation of the curve.

If the point of projection be the highest point the velocity of projection must be u, and $v = 0$. Thus the equation of the curve becomes

$$y = \frac{g}{2u^2}x^2. \tag{28'}$$

The equation of the curve can also be found by remembering that $p = -dy/dx$, and integrating the equation

$$-p = \frac{dy}{dx} = \frac{g}{u^2}x.$$

The integral is

$$y = \frac{g}{2u^2}x^2 + C,$$

where C is a constant which must be zero if y is measured from the highest point of the curve.

Since (28′) gives for any possible value of y two values of x, which are numerically equal but of opposite signs, it is evident that

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the curve is symmetrical about the line AB (Fig. 28′) drawn perpendicular to the (horizontal) tangent at A. This line is called the *axis* of the curve.

The resultant velocity at any distance y below the highest point is $\sqrt{(u^2+g^2t^2)}$, since gt is the vertical velocity acquired in the fall from that point. Hence denoting the resultant velocity by V, and taking account of $y=\frac{1}{2}gt^2$, we have

$$V^2=2g\left(y+\frac{u^2}{2g}\right).$$

Thus V is the velocity that would be acquired by a particle in falling under uniform downward acceleration g through a distance $y+u^2/2g$, that is in falling to the point x, y in the curve from a horizontal line CD in its plane at a distance $u^2/2g$ above the highest point.

It may easily be verified by the reader that if a point F on the axis of the curve at a distance $u^2/2g$ below the highest point be taken, the vertical distance of any point P on the curve from the straight line CD is equal to the distance PF, and that if V be the resultant velocity at P

$$PF=V^2/2g.$$

CD is called the *directrix* of the curve and F its *focus*.

We have evidently $u=V\cos\theta$, $v=V\sin\theta$; and as the particle moves freely under the action of gravity, V and θ both vary so that $V\cos\theta$ remains constant. This may be contrasted with the fact, which will be noticed again later, that if a particle be guided by a curve so as to travel under gravity from one given point to another at a lower level in the shortest time, the curve (which lies wholly in the vertical plane through the two points) fulfils the condition $(\cos\theta)/V=$ constant.

Let t be the time of flight of a particle, that is the interval between the particle's leaving any level and its return to the same level, and V the velocity of projection, then $t=(2V\sin\theta)/g$; the range R on a horizontal plane through the point of projection is given by

$$R=\frac{2V^2}{g}\sin\theta\cos\theta=\frac{V^2}{g}\sin 2\theta$$

This is clearly a maximum when $2\theta=\pi/2$, or $\theta=\pi/4$.

For a given range R other than the maximum there are evidently two values of θ given by this equation. For if $\pi/4-\alpha$ be a value of θ which satisfies the equation, $\pi/4+\alpha$ must clearly be another, since these two values make up $\pi/2$.

In order to find the direction in which a particle must be projected from a given point P, with a given speed V, in order that it may pass through another given point Q (not necessarily on the same level), it is to be noticed, first, that the path must be in the vertical plane through the two points, and that PF and QF must be equal to

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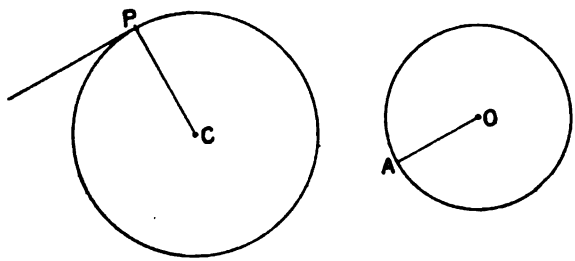
the distances of P and Q respectively from the directrix. But the distance PF is fixed by the value of V, and is $V^2/2g$. If the ordinates of P and Q be y and y' respectively, the distance of Q from the directrix is $V^2/2g-(y-y')$. The distance QF is also fixed and has this value.

If now the inclination θ of the direction of V to the horizontal be varied F moves on a circle of radius $V^2/2g$ round P as centre. Let this circle be described and also a circle with QF as radius. These two circles, if there is a solution, will intersect in two points, which may of course coincide. These points are the foci of two parabolic paths by which the particle, projected as stated from P, will reach Q. If the two points coincide there is only one possible path. If the circles do not intersect there is no solution.

FIG. 29.



51. **Acceleration of a Particle moving in a Circular Path.**—For a particle which describes a circle with uniform speed, clearly the hodograph is another circle, each radius of which represents the speed of the particle at a particular instant. Let (Fig. 29) the circle of centre C and radius r represent the path, that of centre O the hodograph, the radius of which is the speed in the path. Let the radius OA of the hodograph represent the velocity of the particle when in the position P, and a denote the rate of motion of the point A. The two points P and A must describe the circles in the same time, since OA must always remain parallel to the tangent at P, that is perpendicular to CP. But $2\pi r$ being the circumference of the circle, the time of describing it by P is $2\pi r/v$, and similarly the time of description of the hodograph is $2\pi v/a$, and these give the equation $2\pi r/v=2\pi v/a$; or,

$$a=\frac{v^2}{r}. \tag{29}$$

This gives the amount of a. As to its direction, it is that of the tangent to the hodograph at A; that is, the acceleration of P is from P toward C, the centre of the path.

Equation (29) may be put into other convenient forms by using the angular velocity of the radius CP, and the period of revolution.

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Thus if ω be the angular velocity, and T the period, we have $v = \omega r$, and $T = 2\pi/\omega$. Thus (29) becomes

$$a = r\omega^2 = \frac{4\pi^2}{T^2} r. \tag{29'}$$

That the acceleration in this case has a finite value v^2/r, and is directed towards the centre of the circular path, are results which give much trouble to students who have conceived hitherto acceleration only as affecting a body moving continually in a straight line, for example, a body falling under the action of gravity; and their ideas having been prematurely fixed by this imperfect treatment of the subject, they ask how a body can be continually accelerated towards the centre of the path while its velocity remains unchanged. The fallacy lies in supposing the velocity to be unaltered. The velocity is continually undergoing change. In describing any portion of the circular path, the particle has the direction of its velocity altered through the angle between the radii drawn to the extremities of that portion of the circle. If the portion, for example, be half the circle, the velocity, so far from remaining the same, has actually been reversed. The change in any interval is the integral effect of the centreward acceleration during the interval.

We shall see presently (§ 55) that when a particle moves in any curved path, it has in general at each point an acceleration in the line of motion, and an acceleration of amount v^2/R towards the centre of curvature for that point of the path, where v is the speed of the particle at the point, and R is the length of the radius of curvature.

52. **Elliptic Motion.**—We shall consider later the hodograph of a particle moving in an elliptic path with an acceleration always directed towards one of the foci of the ellipse, and varying inversely as the square of the length of the radius drawn from the focus to the particle. This is the case of the undisturbed motion of a small satellite round a primary body attracting the satellite according to the known law of gravitational attraction. It will be seen that in this case also the hodograph is a circle, and that the origin O, from which the lines representing the velocities are drawn, is an eccentric point.

53. **Acceleration in S.H.M. Analytical Formulæ for S.H.M.**—We may now resume the consideration of the particle describing simple harmonic motion, and investigate its acceleration. Just as the point p (Fig. 23, § 43) has at each instant the velocity which P has along the diameter AA', so it has also the acceleration of P in the same direction. But the total acceleration of P is from P towards O, and is of amount $\omega^2 r$. The component of this along AO is of amount $\omega^2 r \cos POp$, and this is the acceleration of p, which thus is directed towards O. But $r \cos POp$ is the displacement Op, and the acceleration of p is thus the product of ω^2 into the displacement, and is directed towards the fixed point O. The acceleration

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bears thus a fixed ratio to the displacement, which may be expressed numerically (without regard to sign) by the equation

$$\frac{acceleration}{displacement} = \omega^2 = \frac{4\pi^2}{T^2}. \tag{30}$$

The student should bear this relation in his memory.

We may, for convenience of reference, group together here the formulæ we have stated above for a particle describing simple harmonic motion. In addition to s and $\dot{s}$ for the displacement and the velocity respectively, we shall use the symbol $\ddot{s}$ for the acceleration or time-rate of change of $\dot{s}$. We have then

$$\left.\begin{aligned} s &= a\cos(nt-\varepsilon). \\ \dot{s} &= -na\sin(nt-\varepsilon). \\ \ddot{s} &= -n^2a\cos(nt-\varepsilon). \end{aligned}\right\} \tag{31}$$

The minus sign on the right in the second and third equations, it will be seen, is at once given by Fig. 23. The component velocity along AA' of the point in the circular path is from A towards O—that is in the direction to diminish the displacement—whereas in the first equation the displacement is taken as positive. Also the acceleration of P, being from P towards O, has a component from p towards O, which is increasing numerically the velocity in the same direction, that is the acceleration as well as the velocity must have the negative sign.

The reader will do well to account for the signs for any possible value of t, that is, for a position of P in each of the four quadrants of the circle.

54. **Harmonic and Exponential Functions.**—It will be noticed that the third equation of (31) is derivable from the second by a precisely similar analytical process to that by which in § 39 the second of (31) was derived from the first. It will be useful to note that the successive differentiation with respect to t of the function $a\cos(nt-\varepsilon)$ is effected by alternately changing the function from *cos* to *sin*, and from *sin* to *cos*, multiplying each time by n, and changing sign when the function is altered from *cos* to *sin*. Thus if we denote the time-rate of variation of ds/dt by d^2s/dt^2, that of d^2s/dt^2 by d^3s/dt^3, and so on, and if $s = a\cos(nt-\varepsilon)$; then

$$\frac{ds}{dt} = -na\sin(nt-\varepsilon), \quad \frac{d^2s}{dt^2} = -n^2a\cos(nt-\varepsilon), \quad \frac{d^3s}{dt^3} = n^3a\cos(nt-\varepsilon),$$

and so on.

We have not space here to discuss this subject at length, but it will be useful later to note that these results are connected with another which it is easy to obtain. Let the function e^{mt} be given by the equation

$$e^{mt} = 1 + mt + \frac{m^2t^2}{1.2} + \frac{m^3t^3}{1.2.3} + \dots \tag{32}$$

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where the quantity on the right is an infinite series the law of formation of the terms of which is indicated by the examples given.

If we suppose now t to be changed to $t+\tau$, on both sides, and take the limit towards which the quantity on the right approaches when τ is made to approach infinitely near to zero, we see at once that

$$\frac{d(e^{mt})}{dt}=m\left(1+\frac{mt}{1}+\frac{m^2t^2}{1.2}+\frac{m^3t^3}{1.2.3}+\dots\right),$$

that is

$$\frac{d(e^{mt})}{dt}=me^{mt}. \tag{33}$$

In the same way we have

$$e^x=1+x+\frac{x^2}{1.2}+\dots \tag{34}$$

and

$$\frac{d(e^x)}{dx}=e^x. \tag{35}$$

Thus the rate at which e^x increases with x is e^x itself: x is called the *logarithm* of e^x to the base e. The value of e, as the reader may verify, is 2·71828.... It is an incommensurable number, and the base of the system of logarithms invented by Napier.

This result, though we cannot stop to prove it, holds strictly, whether m is positive or negative, integral or fractional, real or complex. Now it is shown in books on trigonometry that, e^{mt} having the value indicated, and i denoting $\sqrt{-1}$.

$$\cos nt=\tfrac{1}{2}(e^{int}+e^{-int}),\quad \sin nt=\frac{1}{2i}(e^{int}-e^{-int}). \tag{36}$$

The differentiation of trigonometrical functions is thus reduced to that of exponential functions, and the reader may verify from these formulæ that $d(\cos nt)/dt=-\sin nt$, and $d(\sin nt)/dt=n\cos nt$. The relations between the two kinds of function will be again referred to in connection with the subject of vibrations damped out by friction or other retarding forces.

55. **Acceleration in any Path.**—We now consider the acceleration of a particle moving in any path. Let s, as heretofore, denote the length of the curve from a fixed point, chosen as zero, up to the position of the moving point P, Fig. 30, and let ρ denote the step from the origin O to P. Let Q be an adjacent point on the curve, distant ds from P, which P will have reached after a short interval of time dt, and let $d\rho$ be the step from P to Q (the chord PQ), so that the step OQ is $\rho+d\rho$. Let $d\varpi$ be a step, also of length ds, along the tangent at P. Then, as dt is made smaller and smaller without limit, $d\rho$ and $d\varpi$ both approach without limit of closeness to coincidence with the element of the path the length of which is ds. Hence any very short step whether along PQ or along the tangent, of any length δs, is given in magnitude and direction by the product

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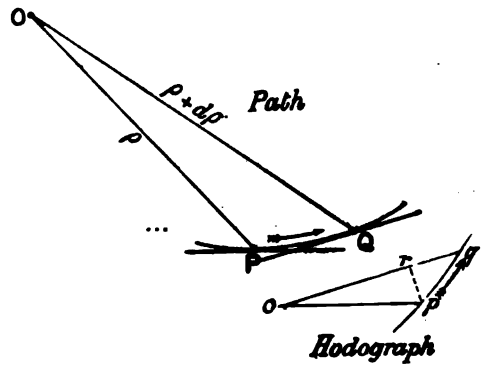
$d\rho/ds.\delta s$; that is $d\rho/ds$ represents a unit step along the tangent at P.

The rate then at which the particle is proceeding along the curve,

FIG. 30.



that is at which the step ρ is increasing, is clearly the ratio $d\rho/ds$ multiplied by $\dot{s}$. Thus we have (see § 35 above)

$$\dot{\rho} = \frac{d\rho}{ds}\dot{s}. \tag{37}$$

Now let the hodograph of the particle's motion be drawn, Fig. 30, and let the step Op represent $\dot{\rho}$ for the point P of the path. Then Oq will represent the velocity for the adjacent point Q of the path, that is $\dot{\rho} + d\dot{\rho}$. Hence pq must represent $d\dot{\rho}$, and we have just seen that

$$d\dot{\rho} = d\left(\frac{d\rho}{ds}\dot{s}\right).$$

But pq consists of two parts, one rq along oq, and another part pr at right angles to oq. Now, if the element of time dt from P to Q be diminished without limit, oq will approach without limit of closeness to op, and rp to perpendicularity to op. Hence in approaching the limit we may regard rq as being in the direction of the tangent to the path at P, and pr as at right angles to that direction, in the direction, as will be seen from the diagrams, of the centre of curvature of the path.

The latter part of $d\dot{\rho}$ brings about no change in the length of op, that is in $\dot{s}$, and involves only the change in the unit step $d\rho/ds$ along the tangent at P caused by the change of direction poq. Therefore approximately for an element of time dt

$$pr = \dot{s}.d\frac{d\rho}{ds}, \tag{38}$$

where $d\,(d\rho/ds)$ is the change in $d\rho/ds$.

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The other part rq, which in the limit is along Op, involves no change in $d\rho/ds$, but arises from the change in $\dot{s}$. Hence again approximately for the element of time dt

$$rq = \frac{d\rho}{ds}\, d\dot{s}. \tag{39}$$

Hence for the total change $d\dot{\rho}$ of the velocity in the time dt we have, with approximation which is without limit of closeness as the interval dt is diminished

$$d\dot{\rho} = \dot{s}d\frac{d\rho}{ds} + \frac{d\rho}{ds}d\dot{s}.$$

In the same way we have

$$dt = \frac{ds}{\dot{s}}.$$

Hence

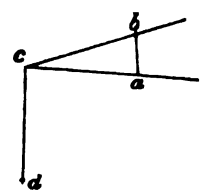
$$\frac{d\dot{\rho}}{dt} = \dot{s}^2 \frac{d}{ds}\frac{d\rho}{ds} + \frac{d\rho}{ds}\frac{d\dot{s}}{dt}$$

or with the notation explained in § 53.

$$\ddot{\rho} = \frac{d^2\rho}{ds^2}\dot{s}^2 + \frac{d\rho}{ds}\ddot{s}. \tag{40}$$

To interpret this result we note that $d^2\rho/ds^2$ is the rate of variation with s of $d\rho/ds$, and the latter, as we have seen, is a unit step along the tangent at the extremity P of ds. This for the point Q has become $d\rho/ds + d(d\rho/ds)$, but is still a *unit* step along the tangent. Hence if ca, cb, Fig. 31 represent these unit steps $d(d\rho/ds)$ is ab. But since ca and cb are each unity, the length of ab is the angle acb between the two tangents at the extremities PQ of ds in Fig. 30, and the ratio of this length to ds is the curvature of the path, § 37. Thus $d^2\rho/ds^2$ which is from a to b, that is at right angles to the tangent, and towards the centre of the curvature, is the ratio of a unit step at right angles to $d\rho$ to the length of the radius of curvature. If σ denote a unit step directed from P to the centre of curvature, and R be the length of the radius of curvature, we have

FIG. 31.



$$\frac{d^2\rho}{ds^2} = \frac{\sigma}{R}, \tag{41}$$

and therefore

$$\ddot{\rho} = \sigma\frac{\dot{s}^2}{R} + \varpi\ddot{s}, \tag{42}$$

where ϖ, as before, denotes a unit step along the tangent at P.

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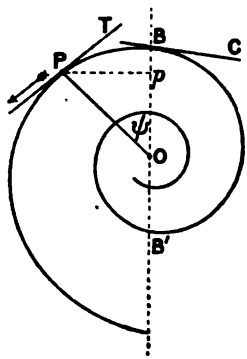
The total acceleration thus consists of two parts, one numerically $\dot{s}^2/R$, towards the centre of curvature, the other numerically $\ddot{s}$, in the direction of motion.

56. **Curves in Space. Osculating Plane. Tortuosity.**—The case of motion along a curve in space (or a *tortuous curve* as it is called) requires a little further explanation. We have defined the motion along any curve as that given by the motion of a point along a straight line which itself is turning round the point. Now though at any instant of course the plane in which the line is turning is perfectly definite, the plane may be changing its direction by turning round the line. This plane for any position of the point is called the *osculating plane* for that position of the plane.

FIG. 32.



Returning to Fig. 31 we have there the osculating plane coincident with the plane of the paper. Let a line cd of unit length be drawn in the osculating plane, from the fixed point c, at right angles to ca. The extremity d as P moves, lies always on a sphere of unit radius described from c as centre. The rate at which cd is turning round ca per unit of distance travelled along the curve, is the measure of the tortuosity of the curve.

57. **Motion in Equiangular Spiral with Uniform Angular Velocity.**—The case of a particle moving in an equiangular spiral with uniform angular velocity round the pole is of great interest for its application to resisted vibrational motion, and will form an excellent illustration of the theorems established above. We shall treat the problem by means of the theorem of tangential and normal resolution of acceleration, then show how the same results may be very easily obtained by means of the hodograph.

An equiangular or, as it is sometimes called, a logarithmic spiral, is a plane curve such that the tangent at any point of it, P, is inclined at a constant angle to the line drawn to P from a fixed point called the pole. Thus in Fig. 32, O is the pole, and the angles OBC, OPT are equal, each being about 82°. The angle of the curve, taken as in the diagram, for the tangent drawn in the direction along the curve towards the pole, cannot exceed $\pi/2$.

The relation between the length r, of the radius-vector drawn from the pole to any point, and the angle θ, which the radius makes with a fixed line in the plane of the curve is

$$r = a^{\theta} \tag{43}$$

where a is a constant for the curve. The fixed line is thus the radius-vector of length equal to 1.

We can easily prove that the curve given by this equation fulfils

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the condition stated. For let P, Q be adjacent points on the curve, O the pole, then $OP = r = a^{\theta}$, and as θ is supposed to increase as P is taken further from the pole, $OQ = r + dr = a^{\theta + d\theta}$. Hence

$$dr = r(a^{d\theta} - 1). \tag{44}$$

For equal increments $d\theta$ of θ, therefore, the increments of r vary directly as r. But if q (Fig. 33) be the foot of the perpendicular let fall from P on OQ, Qq is dr, and if $d\theta$ be the very small angle POQ, Pq is approximately $rd\theta$. But if OQP be denoted by ϕ

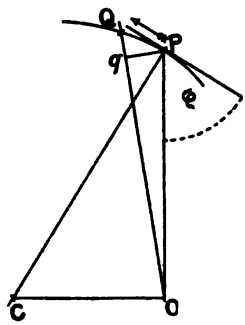
$$\tan\phi = \frac{rd\theta}{dr}, \text{ or } dr = rd\theta . \cot\phi ; \tag{45}$$

and

$$\dot{r} = r\dot{\theta}\cot\phi. \tag{46}$$

Equation (45) agrees with the former result that for equal small

FIG. 33.



successive increments of θ, the corresponding increments of r are proportional to r. A comparison of the two values of dr found in (44) and (45) gives the equation

$$\cot\phi\, d\theta = a^{d\theta} - 1.$$

which since $d\theta$ is small is equivalent to $\cot\phi = \log a$.

Now consider a particle moving in an equiangular spiral, so as to recede from the pole, with uniform angular velocity $\dot{\theta}$ round the pole. The velocity of the point is $\dot{s}$. But from Fig. 33, resolving $\dot{s}$ along r, that is along OP, we have

$$\dot{s} = \dot{r}\frac{1}{\cos\phi} = r\dot{\theta}\frac{1}{\sin\phi}. \tag{47}$$

If then, as supposed above, $\dot{\theta}$ be constant, $\dot{s}$ is proportional to r.

The centreward and tangential components of acceleration are therefore

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$$\dot{s}^2 \times \text{curvature} = r^2\dot{\theta}^2 \frac{1}{\sin^2\phi} \frac{\sin\phi}{r} = \dot{\theta}^2 \frac{r}{\sin\phi}, \quad \ddot{s} = r\dot{\theta}^2 \frac{\cos\phi}{\sin^2\phi}. \tag{48}$$

The student may verify at once by resolution that these are equivalent to a component $r\dot{\theta}^2/\sin^2\phi$ in the direction PO, Fig. 33, and a component $2r\dot{\theta}^2\cos\phi/\sin^3\phi$ along the tangent and with the motion.

It may also be noticed that if the point move towards the pole the component accelerations are not changed. The tangential acceleration however now acts against the motion.

58. **Derivation of Resisted S.H.M. from Motion in an Equiangular Spiral.**—Consider now a point moving in a fixed straight line BB' through the pole so as to be always at the foot, p, of the perpendicular let fall from P upon the straight line, while P moves along the curve *towards* the pole. Let ψ be the angle between OP and OB (Fig. 32). The component along BB' of the velocity of P is the velocity of p, and the component in the same direction of the acceleration of P is the acceleration of p. Let the distance of p from O be s', and denote its velocity by $\dot{s}'$. Then from Fig. 32 we get

$$s' = r\cos\psi, \quad \dot{s}' = \dot{s}\cos(\psi+\phi) = \frac{r\dot{\theta}}{\sin\phi}\cos(\psi+\phi)$$

by (47). Also $\dot{\psi} = \dot{\theta}$, so that, by (46), $\dot{s}' = r\dot{\theta}(\cos\psi\cot\phi - \sin\psi)$.

The velocity $\dot{s}'$ is regarded here as in the direction of increasing s', that is from O to B, but the actual direction of motion of p will, of course, depend on the sign of the numerical value of the quantity on the right. When $\psi+\phi=\pi$, the point in the spiral is abreast of the pole and is moving parallel to BB'. When $\psi+\phi=\pi/2$, p is at the extremity of its range and is at the instant at rest.

Resolving along BB' the tangential and normal accelerations of P we obtain for the total acceleration of p in the direction from B towards B':

$$r\dot{\theta}^2\frac{\sin(\psi+\phi)}{\sin\phi} - r\dot{\theta}^2\frac{\cos\phi}{\sin^2\phi}\cos(\psi+\phi).$$

This may be written in the form

$$s'\frac{\dot{\theta}^2}{\sin^2\phi} - 2\dot{s}'\dot{\theta}\cot\phi.$$

The acceleration of p consists therefore of two parts, one proportional to and in the opposite direction to the displacement s' of p from O, the other proportional to and opposed to the speed $\dot{s}'$ of p.

If we put

$$n^2 = \frac{\dot{\theta}^2}{\sin^2\phi}, \quad k = \dot{\theta}\cot\phi, \tag{49}$$

and $-\ddot{s}'$ for the whole acceleration of p we have the equation

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$$\ddot{s}' + 2k\dot{s}' + n^2 s' = 0, \tag{50}$$

which is the differential equation of vibrational motion of a particle subject to a retardation proportional to the speed of the particle.

If α be the angle between BB' and the line from which θ is measured we have $\theta = \alpha + \psi$, and therefore $\dot{\theta} = \dot{\psi}$.

Thus we get

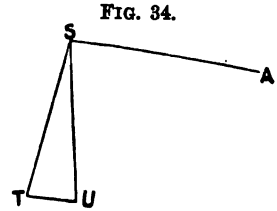
$$s' = r\cos\psi = a^{\theta}\cos(\alpha - \theta). \tag{51}$$

The displacement s' has a numerical maximum when $\dot{s}'$ is zero, that is whenever $\cos(\phi + \psi)$ is zero, and, as the diagram shows, this maximum is diminished each time the radius-vector makes a half-turn. To find the rate of diminution we notice that as ψ changes from π to 0, θ changes from $\alpha + \pi$ to α, and s' changes from $a^{\alpha+\pi}\cos\pi$ to $a^{\alpha}\cos 0$, that is from $a^{\pi}a^{\alpha}$ to a^{α}. The amplitude is therefore diminished in the ratio of a^{π} to 1 in each half-turn. The successive amplitudes are as indicated in Fig. 32.

The rate of diminution of r for each half-turn is exactly the same. The amount of diminution per unit of time is by (46) $r\dot{\theta}\cot\phi$ or kr, so that the rate of diminution of r per unit of time per unit of length is k.

59. **Calculation of Acceleration by Hodograph.**—The method of the hodograph leads much more briefly and elegantly to the value found above for the acceleration of the particle moving in the spiral. Since the velocity is proportional to r the hodograph is an equiangular spiral of the same angle as the path. Let AS, Fig. 34, represent the velocity at P, then the acceleration is along the tangent ST to the hodograph. Also, if ST represent the magnitude as well as the direction of the acceleration it must be derivable from AS by precisely the same operation as gave AS from OP. But the magnitude of AS is $r\dot{\theta}/\sin\phi$, and therefore the magnitude of ST is $r\dot{\theta}^2/\sin^2\phi$. Thus the acceleration is completely determined.

FIG. 34.



Resolving ST into two components represented by UT in the direction with the motion, and SU in the direction PO (Fig. 33), we easily obtain (since clearly $\angle UTS = \angle SUT = \phi$) for the value of the former $2r\dot{\theta}^2\cos\phi/\sin^2\phi$, and for that of the latter $r\dot{\theta}^2/\sin^2\phi$, values which are identical with those obtained by the process first employed.

60. **Resultant of Two S.H.M.s in one Line.**—We now consider the motion which is the resultant of two or more simultaneously existing simple harmonic motions. First let the motions be of the same period and in one line, and let the concentric circles AA', BB'

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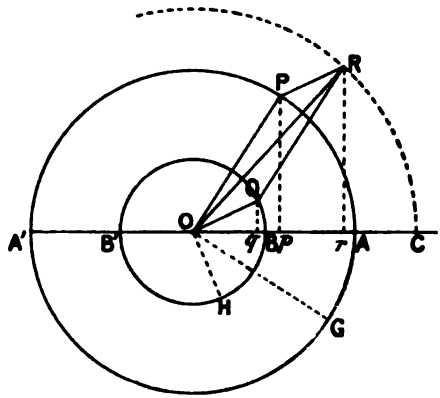
(Fig. 35) be those of the auxiliary circular motions which define the simple harmonic motions (of the points p, q, along the diameter AA'), which are to be compounded. Let P,Q be the positions of the points in the circular motion at time t, reckoned from the instant at which P was at G. The epoch of the motion of P is COG in angle, and clearly, if $GOH = POQ$, Q must have been at H at the zero of reckoning of time. Thus the epoch of Q is COH, and the difference of epochs GOH or POQ.

Complete the parallelogram $POQR$, and draw the diagonal OR; then as OP and OQ revolve at the same rate, OR remains throughout of the same length, and inclined at the same angles to OP and OQ. The displacement of the particle describing the resultant motion is

FIG. 35.



the sum of the displacements Op, Oq, which, being the sum of the projections of OP, OQ, that is, of OQ, QR, in OC, is equal to the projection of OR on OC, that is, Or. Clearly r describes a simple harmonic motion, and the resultant motion is one of amplitude equal to OR, of the same period as the constituent motions, and of phase $\angle ROP$ behind the motion of P, and $\angle QOR$ before the motion of Q.

If a, b, c be the lengths of OP, OQ, OR, and θ be the difference of phase of the motions compounded, α, β the angles POR, ROQ, we have

$$c = \sqrt{a^2 + b^2 + 2ab\cos\theta}. \tag{52}$$

$$\alpha = \cos^{-1}\frac{a + b\cos\theta}{c}, \quad \beta = \cos^{-1}\frac{b + a\cos\theta}{c}. \tag{53}$$

This result is easily obtained analytically. Let ε denote the angle AOG, u, v, the displacements of p, q. Then

$$u = a\cos(nt - \varepsilon), \quad v = b\cos(nt - \varepsilon - \theta)$$

$$\begin{aligned} u + v &= a\cos(nt - \varepsilon) + b\cos(nt - \varepsilon - \theta) \\ &= (a + b\cos\theta)\cos(nt - \varepsilon) + b\sin\theta\sin(nt - \varepsilon). \end{aligned}$$

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If we put $a + b\cos\theta = c\cos\phi$, and $b\sin\theta = c\sin\phi$, this may be written $u + v = c\cos(nt - \epsilon - \phi)$, where

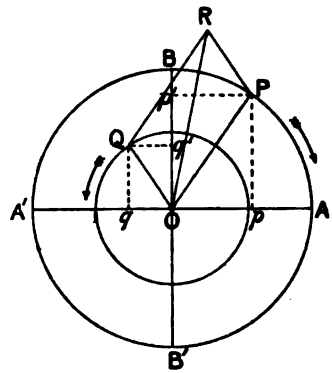
$$c = \sqrt{a^2 + b^2 + 2\,ab\cos\theta},\quad \phi = \tan^{-1}\frac{b\sin\theta}{a + b\cos\theta},$$

which, since ϕ is obviously the angle denoted above by α, agrees with the result already obtained.

In the same way a third simple harmonic motion, of the same period, and in the same line, may be compounded with that just found, and so on to the resultant of any number of simple harmonic motions of the same period and in the same line.

61. **Resultant of Two Opposite Circular Motions of Equal Periods.**—Another important case is that in which two concentric circular motions of the same period, but of different amplitudes and in opposite directions, are compounded. Let OP, OQ (Fig. 36) be the radii drawn to the points P, Q describing the circular orbits. Then if the radii were together along OB, the angles BOP, BOQ are equal. Describe the parallelogram $POQR$, then OR is the resultant displacement, and the motion of R is the resultant motion. But the displacement of R is compounded of the two displacements Op, Oq along the line OA, and the two Op', Oq' along the line OB. The motions of p, q are both simple harmonic in the diameter AA', and if Q' be the point in which Qq produced meets the smaller circle, these motions may be regarded as the projections on the diameters of motions of the points P, Q', which traverse the circles in the same direction. Hence the motions compound by the preceding theorem into a single harmonic motion. Similarly the motions of p', q' give a single harmonic motion in the diameter BB'. Thus the motion of B is compounded of two simple harmonic motions in lines at right angles to one another.

FIG. 36.



Let x, y be the displacements of R along OA, OB. Then if $OP = a$, $OQ = b$, $\angle BOP = \angle BOQ = nt$,

$$\left.\begin{aligned} x &= a\sin nt - b\sin nt = (a - b)\sin nt \\ y &= a\cos nt + b\cos nt = (a + b)\cos nt. \end{aligned}\right\} \quad (54)$$

The amplitudes along OA, OB are $a - b$, $a + b$ respectively.

The relation between the x and y of any point in the path of R may be found by eliminating nt between the two equations just written. Thus by squaring and adding we get

$$\frac{x^2}{(a-b)^2} + \frac{y^2}{(a+b)^2} = 1, \quad (55)$$

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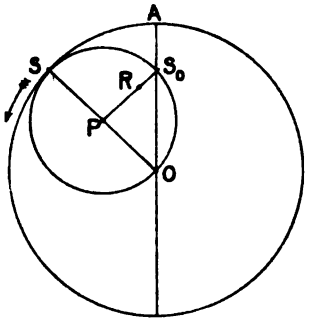
which represents an ellipse, the semi-axes of which are $a-b$, $a+b$ (see §§ 69, 75).

If b were equal to zero we should have a single uniform circular motion, which therefore is equivalent to two equal simple harmonic motions in lines at right angles to one another, and of 90° difference of phase.

62. **Elliptic Motion produced by rolling One Circle inside another.**—It is interesting to look at this case of motion in the manner indicated by Fig. 37. Let a circle of diameter OS in internal contact with a circle of double its radius *roll* along the latter in the direction indicated by the arrow, that is remain in contact while its centre P moves with angular velocity n round O, and each radius of the smaller circle turns with equal angular velocity in the opposite direction round P. Let PS_0 be a radius which was along OA when OP was in the same direction, and let R be any point on that radius, or on that radius produced. The angle SPS_0 is clearly $2 \angle SOA$. Hence the arc of circumference SS_0 of the smaller circle is equal to the arc SA of the larger, and therefore the successive points of SS_0 have come into contact with those of SA, without relative motion along the tangent, or *slipping* as it is called, that is the motion is one of pure rolling. Clearly the motion of R is precisely that of the point R in Fig. 36, which has just been discussed. Hence when a circle rolls internally on another of twice its radius, any point fixed in the rolling circle describes an ellipse; and if the rolling take place with uniform angular velocity, the motion is compounded of two simple harmonic motions at right angles to one another.

FIG. 37.



The ellipse becomes a straight line or diameter of the larger circle in the case in which the point R lies on the circumference of the rolling circle. The opposite extremity of the diameter through R describes the diameter perpendicular to that described by R. These results will be easily made out as particular cases of those just found.

63. **Resultant of Two Equal and Opposite Circular Motions.**—It follows from the above that the composition of two equal and opposite circular motions gives a simple harmonic motion of the same period and of amplitude equal to twice the radius of the circle, in the line bisecting the angle between the radii drawn to the positions of the particles at any instant. Thus in Fig. 38 let P, Q be the positions of the particles whose motions in the circle APQ are to be compounded, and let CA be drawn perpendicular to PQ. The components of the two motions taken perpendicular to CA cancel one another, so that the resultant is a motion in the radius CA equal to

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the sum of the equal components of the circular motions along that line. Obviously the direction of CA is fixed.

64. **Resultant of Opposite Circular Motions of Unequal Periods.**—If, however, the angular velocities of OP and OQ be unequal, the line CA will change its direction with angular velocity equal to half the difference of angular velocities of the two radii, and will turn round C in the same direction as the more quickly moving radius.

Thus, if T denote the period of P, T' that of Q, the angular velocities n, n' are $2\pi/T$, $2\pi/T'$, and half the difference, the angular velocity of CA, is $\pi(1/T-1/T')$. If $T<T'$ the line CA turns round C in the direction shown by the arrow in the period $2/(1/T-1/T')$,

FIG. 38.

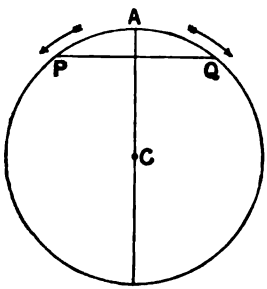
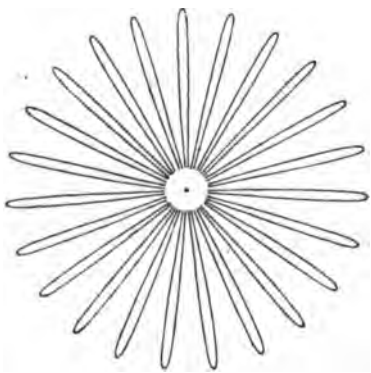


FIG. 39.



that is $2TT'/(T'-T)$. The path of the particle which has the resultant motion is a star-shaped curve like Fig. 39, in which, however, to avoid confusion, the rays are not carried into the centre. In the particular case here represented the period of revolution of the resultant motion is about twelve and a half times that of either of the components.

These cases of motion are of great importance in the theory of plane polarised light, and of the rotation of the plane of polarisation in media like quartz, solutions of sugar, tartaric acid, &c., and in magneto-optic theory generally.

65. **Resultant of Two Similar Circular Motions of Unequal Period and Radius.**—The case of a particle, the motion of which is the resultant of two concentric circular motions in the same direction, unequal both in period and in radius, is important and interesting. The displacement of the particle evidently varies from a minimum equal to the difference of the radii to a maximum equal to their sum and back again alternately.

The amount of the displacement is given by the equations

$$\left.\begin{aligned} x &= a\sin nt + b\sin n't \\ y &= a\cos nt + b\cos n't, \end{aligned}\right\} \quad (56)$$

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which are what (54) become when the period of P is $2\pi/n$, and that of Q is $2\pi/n'$, and the circular motions have the same sense.

To interpret these equations for this case, let the zero of reckoning of t be one of the instants at which the displacements in the circular motions are in the same direction. Then clearly at any subsequent instant $\theta=(n-n')t$ supposing $n>n'$. Thus, for the amplitude of the resultant motion and the phase-angle ϕ ($=\angle POR$ of Fig. 35), we have

$$c=\sqrt{a^2+b^2+2ab\cos(n-n')t}\,; \tag{57}$$

and

$$\tan\phi=\frac{b\sin(n-n')t}{a+b\cos(n-n')t}. \tag{57'}$$

When $(n-n')t=m\pi$, where m is a whole number, we have $c=a+b$ or $c=a-b$, according as m is even or odd, and when $(n-n')t=(2m+1)\,\pi/2$ we have $c=\sqrt{a^2+b^2}$.

The value of ϕ is always such that the resultant displacement lies between the two moving radii in the angle which is less than π, except in the critical positions, when, in passing through zero, it changes over from one side of the more quickly moving radius to the other.

The reader may work out from first principles (by assigning a small change dt to t, calculating the corresponding change $d\theta$ of θ, and finding the limiting value of $d\theta/dt$ when dt is made infinitely small) the angular velocity of the resultant displacement at each instant.

[The results for the case of two unequal motions in different periods in opposite directions are to be obtained, of course, from the results in the case just discussed by changing the sign of n'.]

66. **Case of Two Similar Circular Motions of Equal Radius and Unequal Period.**—In the particular case of the preceding motion in which the component displacements are equal, it is clear that the resultant displacement is then $2a\cos\frac{1}{2}(n-n')t$, and that its direction always bisects the angle (less than π) between the two moving radii. The angular velocity of this bisecting line is evidently $\frac{1}{2}(n+n')$; or, $\pi(1/T+1/T')$. In the infinitely short interval in which the revolving radii pass through a position of opposition, the line bisecting the angle between them passes through an angle π from one side to the other of the two lines, and then moves with angular velocity $\frac{1}{2}(n+n')$ until opposition is again attained, and so on.

The resultant motion may be regarded as simple harmonic with varying phase, and its amplitude changes from zero to double the amplitude of either circular motion, and back again alternately, as the direction changes (Fig. 40). When $2\pi t/T=2\pi t/T'+n\pi$ we have $\pi t(1/T+1/T')=2\pi t/T'+n\pi/2$. Thus, having started from coincidence, the directions of the components are then coincident or opposed, and by (57) the amplitude is the sum of the radii or zero, according as n is even or odd.

The variation of the amplitude can be traced from Fig. 40 by drawing radii to the curve there shown from the point O, at which

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the amplitude is zero. If the arrows are followed, it will be seen that the curve is a kind of spiral consisting of two branches, an outward and an inward, passing into one another at m. The outward is described by the particle as it moves away from O until it reaches m, when it returns on the inward branch to O, to again describe the outward path, and so on.

Figs. 39 and 40 may be regarded also as examples of cycloidal and trochoidal curves, and will be encountered again in connection with vibrational motion.

67. **Curve of Displacements.**—The varying displacement of a

FIG. 40.

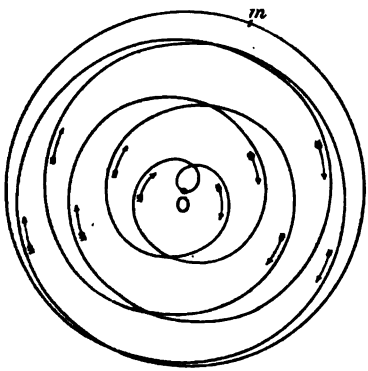
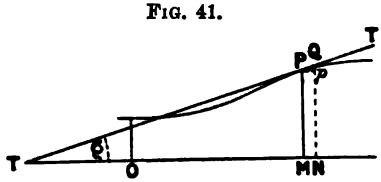


FIG. 41.



point is frequently represented as in Fig. 41, where distance OM along the line OX is proportional to the time from the chosen zero of reckoning, and the distance MP of the point P from OX the displacement or a quantity proportional to it. The displacement, of course, may be of any kind whatever, *e.g.*, distance of the moving point from a fixed point, a fixed line, or a fixed plane, or distance measured along any curve from a fixed point to the moving one.

If, as we suppose to be the case, the curve of displacements thus obtained be continuous, a definite tangent PT to the curve can be drawn at any point P, and this will in the limit coincide with an infinitesimal element PQ of the curve. The rate at which MP is changing per unit of increase of OM, that is Qp/MN is thus precisely $\tan\phi$, where ϕ is the inclination of PT to OX. Hence $\tan\phi$ is proportional to the particular speed of the point P corresponding to the displacement represented by MP.

68. **S.H.M. represented by Curve of Sines.** — An important application of this graphical process is to simple harmonic motion. Here OM (Fig. 42) represents the time from the zero of reckoning, and MP the displacement $s = a\cos(nt - \varepsilon)$. Thus OA represents in time the epoch of the motion, AB the maximum positive displacement, and M_0M_1, M_1M_2 half the period.

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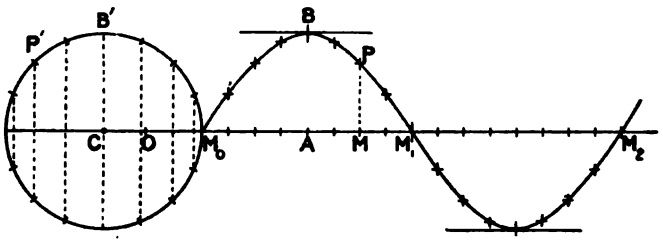
The distances M_0A, M_0M, ... are proportional to the corresponding angles $B'CM_0$, $P'CM_0$, shown in the circle of centre C, and radius equal to AB; and the ordinates are therefore proportional to the sines of these angles. Hence the name usually given to the curve traced out by P, the *curve of sines.*

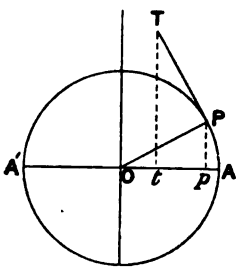
It will be seen at once from the figure that the velocity of the point is a numerical maximum at the middle position, that is, when P is on the axis OM_1. The velocities at the different points of the curve are capable of graphical representation also by a curve of sines. For,

FIG. 42.



draw a new circle (Fig. 43) with centre O, and radius equal to that of the circle in Fig. 42. Let the point P represent the particle in the defining circular motion, and the line PT, carried round by the revolving radius OP, the constant speed of the particle in the circle. (If a be the radius of the circle, and n the angular velocity $2\pi/T$, PT represents na.)

FIG. 43.



The projection pt of PT on the line OA represents the velocity of P resolved parallel to OA, that is the velocity in the simple harmonic motion, and is in amount $PT \sin POp$, or $na \sin(nt - \epsilon)$.

Thus the displacement being represented by Op, that is, $OP \cos POp$ or $a \cos(nt - \epsilon)$, the corresponding velocity is $-na \sin(nt - \epsilon)$, and its successive values also give a curve of sines. This curve is sometimes called the first derivative of the former curve. It is, in fact, the former curve with its ordinates all multiplied by n, and shifted towards the left along OX through a distance representing three quarters of a period.

In the same way the first derivative of this curve, the second derivative of the former, may be plotted, and since the ordinates of the curve so obtained are proportional to the rates of variation of those of the second curve, they represent on the scale adopted in the diagram the accelerations in the first curve. We return, in fact, to the curve originally plotted, with the change that each of its ordinates is multiplied by n^2, and is turned in the opposite direction.

69. **Generation of Curve of Sines from Circular Cylinder.**—The

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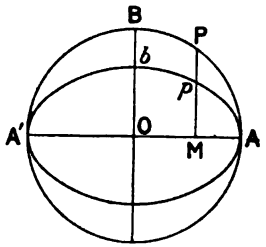
curve of sines is the curve generated by marking on a sheet of paper, wound round a right circular cylinder, the outline of a plane section of the cylinder taken oblique to the axis, and then bringing the paper again into a plane sheet. A plane perpendicular to the axis and passing through the centre of the oblique section intersects the paper in a line which, in the plane sheet, is the axis of the curve. This construction is obvious since the ordinates of the curve are projections of the ordinates of the circle of Fig. 42 drawn in a circular section of the cylinder, and projected on the oblique section by the generators of the cylinder.

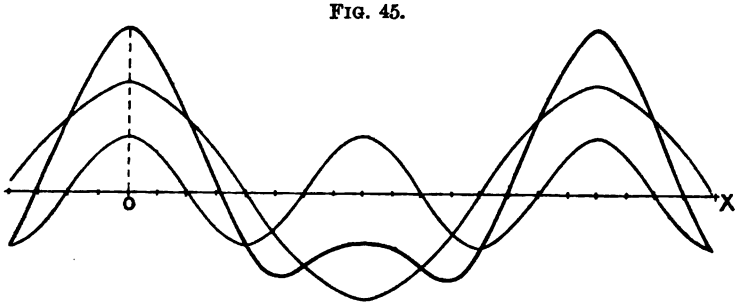
FIG. 44.



The curve of section of the cylinder is that called an ellipse, and is of great importance in physical astronomy, the theory of elasticity and other branches of physics. It is worth noticing here, though the curve will be discussed in §§ 74–80, and in connection with its applications, that an ellipse may be derived from a circle, by shortening (or lengthening) all ordinates at right angles to a perpendicular diameter in the same ratio, as in Fig. 44. There the ellipse AbA' is derived from the circle ABA' by diminishing all the ordinates (as MP to Mp) in the ratio of OB to Ob. This is obvious from the mode of derivation just explained.

70. **Composition of S.H.M.s by Means of Their Sine-Curves.**—Two or more simple harmonic motions of different periods and

FIG. 45.



epochs, but in the same line, are compounded by simply adding together the displacements in the different motions for each instant. The resultant displacement is the displacement at that instant in the resultant motion. This may be done graphically by drawing with a common axis, OX (Fig. 45), the curves of sines representing the different motions, and then drawing the curve the ordinates of which are each the sum of the corresponding ordinates of the curves representing the constituent motions. Thus Figs. 45 and 46 represent the resultant of the two motions.

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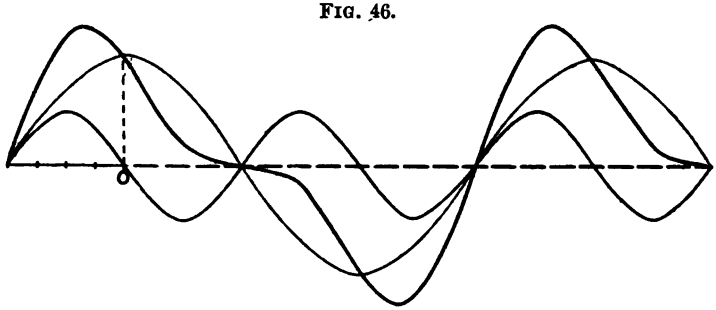
$$s_1 = a_1 \cos(nt - \epsilon_1), \quad s_2 = a_2 \cos(2nt - \epsilon_2),$$

for the cases in which $a_2 = \frac{1}{2}a_1$, and (1) $\epsilon_1 - \epsilon_2 = 0$, (2) $\epsilon_1 - \epsilon_2 = \frac{1}{2}\pi$. The period of the second motion is half that of the first.

71. **Mechanical Composition of S.H.M.s. Tide Gauges and Tide Predicters.**—The resultant of a series of simple harmonic motions in one line can be traced graphically by giving, by means of proper

FIG. 46.



mechanism, the resultant displacement at each instant to a pen or pencil which inscribes the resultant curve on a sheet of paper, drawn uniformly past the pen by clock-work in a direction perpendicular to the line in which the marking point is free to move. Thus in a tide-gauge, a pen, moved in a vertical line by a float on the surface of the water in a harbour or dock, marks the varying height of the tide on a convenient scale on a sheet of paper constantly drawn past the pen by a properly regulated clock, which likewise registers time at equal intervals by a slight motion of the pen. Data are thus obtained for the setting of another machine which also compounds harmonic motions, and draws curves, the ordinates of which represent the heights of the tide at successive instants of a considerable interval of future time.

In this tide-predicting machine, which is the invention of Lord Kelvin, a fine chain passes half round each of a number of pulleys, all of the same size, arranged in two rows along the top and bottom of a vertical plate, and so placed that the chain passes from row to row as it passes from one pulley to another, and that the parts of the chain not on the pulleys are all parallel. The centres of the pulleys are carried on pieces which are each constrained to describe simple harmonic motion, in lines parallel to the free parts of the chains, by a crank-pin traversing a circle and working in a transverse slot as described above, § 44. These crank-pins, having been set to the proper relative amplitudes and epochs are driven by a train of wheel-work in the proper relative periods from a single handle turned by the operator. The motion of each crank-pin corresponds to a tidal constituent, and as the pulleys move the length of the free end of the chain, to which the marking point is attached, rises or falls

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in any time through a distance which is the resultant of the displacements of the pulleys in that time, and the varying height of the water is registered upon a ribbon of paper drawn past the marking point by the driving mechanism. Of course, variations of the speed of driving cannot affect the result, as the speeds in the different harmonic motions and that of the paper are all affected in the same ratio.

72. **Uniform Circular Motion derived from Two S.H.M.s.**—As was pointed out in § 62 a uniform circular motion is equivalent to two simple harmonic motions in perpendicular diameters of the circle. These motions are such that the displacement in either is zero when that in the other is a maximum; in fact, if the simultaneous displacements in the motions be represented by x, y, we have

$$\left.\begin{aligned} x &= a\cos(nt-\varepsilon) \\ y &= a\sin(nt-\varepsilon). \end{aligned}\right\} \quad (58)$$

These equations may be put in the same form thus:

$$\left.\begin{aligned} x &= a\cos(nt-\varepsilon) \\ y &= a\cos(nt-\eta) \end{aligned}\right\} \quad (59)$$

where $\eta = \varepsilon + \pi/2$. The phases ε, η of the two motions thus differ by $\pi/2$.

Conversely it is clear that two simple harmonic motions of the same period in perpendicular lines, and as here differing in phase by $\pi/2$, give uniform circular motion when compounded. Squaring the equations of (58) and adding we obtain

$$x^2 + y^2 = a^2.$$

which shows that the point P lies on a circle of radius a.

73. **Projection of Circle into Ellipse.**—Now consider the projection by parallel lines of a circle on a plane inclined at any angle to the plane of the circle. The projection is a closed curve, called an *ellipse*, some of the properties of which we now investigate.

It is clear that straight lines in the plane of the circle project into straight lines, and that parallel lines project into parallel lines. Also it is clear that the lengths of all parallel lines are altered by the projection in the same ratio, and therefore the ratio of any two parallel lines is equal to the ratio of their projections.

The angle between the projections of two lines is, however, in general different from the angle between the lines. Thus two chords of the circle at right angles to one another are in general projected into two chords of the ellipse inclined at a different angle. Thus the diameters AA', BB' of the circle are in Fig. 47 projected into the lines (also diameters) of the ellipse, aa', bb'.

74. **Properties of Ellipse.**—If PQ be any chord of the circle divided in any ratio by a point C, the projection pq of the chord is divided in the same ratio by the point c, which is the projection of C. A diameter of a circle bisects all chords at right angles to

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itself, and is projected into a line bisecting the projections of the chords of the circle. Any two diameters of a circle are projected into two chords of the ellipse which bisect one another, and the point of intersection is the projection of the centre of the circle. All chords passing through this point are there bisected; it is therefore called the centre of the ellipse, and any line drawn through it to meet the ellipse on both sides is called a diameter.

A line cutting the circle is projected into a line cutting the ellipse, and a line touching the circle into a line touching the ellipse.

FIG. 47.

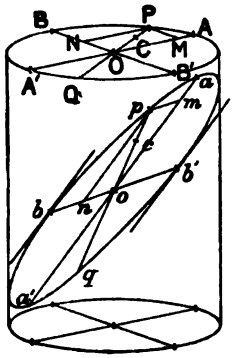
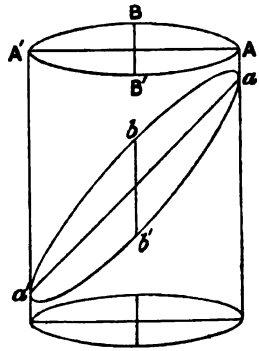


FIG. 48.



For if a line cutting the circle in two points be turned round one of these points until the other has moved up to coincidence with the former, the projection of the moving point will move along the ellipse to coincidence with the projection of the former point, so that the line and its projection become tangents simultaneously.

Since a diameter of a circle bisects all chords parallel to the tangents at its extremities, and parallel lines project into parallel lines, a tangent to an ellipse is parallel to the chords bisected by the diameter drawn to the point of contact, and the tangents at the opposite extremities of a diameter are parallel. Thus if bb', Fig. 47, be a diameter of an ellipse it is bisected at o by the centre, and the tangents at b, b' are parallel to one another and to the chords bisected by bb'. The diameter aa', which is parallel to the tangents at the points b, b' and the diameter bb' are said to be conjugate, inasmuch as the tangents at the points a, a' are parallel to bb', and each diameter bisects all chords parallel to the other.

Let AA', Fig. 48, be a diameter of the circle and aa' the corresponding diameter of an ellipse obtained by projection perpendicular to the plane of the circle. As the plane $AA'a'a$ is turned round the line of centres, the length aa' of the diameter changes, and is clearly a maximum when the lines Aa, $A'a'$ have their greatest difference in length, and a minimum when they are of the same

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length. But a line of given length as AA', to have its maximum projection on the oblique plane, must be drawn at right angles to the line of intersection of the two planes, that is at right angles to those lines of the two planes which are parallel. Its projection is therefore at right angles to such lines. Hence the maximum diameter of the ellipse is perpendicular to the minimum.

They are also conjugate diameters. For let BB', Fig. 48, be the diameter of the circle which projects into the minimum diameter bb' of the ellipse. The tangent to the circle at B projects into the tangent to the ellipse at b, and since bb' is parallel to BB' it is perpendicular to the plane defined by Bb and the tangent at B. Hence it is perpendicular to the tangent at b which lies in that plane. Similarly bb' is perpendicular to the tangent at b'. The conjugate diameter aa' therefore cuts bb' at right angles, and the tangents at its extremities are parallel to bb'. It coincides with the maximum diameter.

75. **Equation of Ellipse.**—Let the points marked by small letters in Fig. 47 represent the projections of the points marked by capital letters in the circle indicated at the top of the diagram; then if AA', BB' be perpendicular diameters of the circle aa', bb', their projections, are conjugate diameters of the ellipse. Let P be a point on the circle, OM, $ON(=MP)$ the co-ordinates of P taken with respect to OA, OB as axes, om, $on(=np)$ their projections. Now, since the ratios of the parallel lines are not altered by projection, we have

$$\frac{OM}{OA} = \frac{om}{oa}, \quad \frac{ON}{OB} = \frac{on}{ob}.$$

But

$$\frac{OM^2}{OA^2} + \frac{ON^2}{OB^2} = 1,$$

and therefore the preceding equalities give

$$\frac{om^2}{oa^2} + \frac{on^2}{ob^2} = 1.$$

If we write a, b, for the lengths of the conjugate semi-axes oa, ob, and x, y for the lengths of om, mp, respectively, the last equation becomes

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{60}$$

which is the relation fulfilled by the co-ordinates of a point on an ellipse referred to a pair of conjugate axes, as axes of co-ordinates.

The relation just found may be written.

$$\frac{on^2}{ob^2} = 1 - \frac{om^2}{oa^2} = \frac{(oa + om)\,(oa - om)}{oa^2} = \frac{a'm.ma}{oa^2};$$

or

$$a'm.ma = \frac{oa^2}{ob^2} mp^2, \tag{61}$$

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which may be contrasted with the relation fulfilled by points on the circle

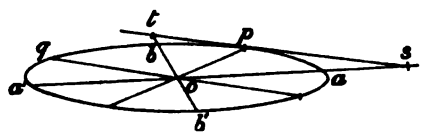
$$A'M.MA = \frac{OA^2}{OB^2}MP^2 (= MP^2).$$

It follows also from the permanence of the ratios of parallel lines that the rectangles under the segments of intersecting chords are proportional to the squares on the parallel diameters. Thus let pcq, $p'cq'$ be any two chords intersecting in c, and oa', ob', the lengths of the parallel semi-axes then

$$pc.cq = \frac{oa'^2}{ob'^2}p'c.cq'_1. \tag{62}$$

76. **Principal Diameters of Ellipse. Polar Equation of Ellipse.**—Let a tangent be drawn to a point p of the ellipse and let it meet the conjugate diameters aa', bb', Fig. 49, in s and t, and draw the semi-diameter oq parallel to st. In the circle from which these lines

FIG. 49.



are projected the diameters AA', BB' are at right angles to one another, and OQ is the radius parallel to the tangent, and we have

$$SP.PT = OQ^2.$$

Since the ratios of parallel lines are not altered by projection we have also in the ellipse,

$$sp.pt = oq^2.$$

This relation enables the principal axes of the ellipse to be determined when a pair of conjugate axes are given in position and magnitude.

For, Fig. 50, take a point r on oa produced such that $oa.ar = ob^2$, and from a centre c in the tangent at a draw a circle through o, r, cutting the tangent ac in the points d and e. Then, since $ea.ad = oa.ar = ob^2$, oe, od must be along conjugate semi-axes, and they are perpendicular, since the angle eod is in a semicircle. The longer principal axis is called the major axis, the other is called the minor axis. We shall denote the lengths of the principal semi-axes by a, b.

Putting now ρ for the length of the radius-vector drawn from the centre to any point on an ellipse, θ for the angle the radius-vector makes with a fixed line in the plane of the ellipse, α for the angle the positive semi-axis major makes with the fixed line, we

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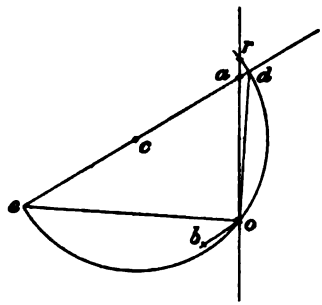
have $\angle oap = \theta - \alpha$. We have also $x = \rho\cos(\theta - \alpha)$, $y = \rho\sin(\theta - \alpha)$. Hence equation (60) becomes

$$\rho^2 = \frac{b^2}{1 - e^2\cos^2(\theta - \alpha)}, \tag{63}$$

where $e^2 = (a^2 - b^2)/a^2$. This gives ρ for any value of θ. It will be observed that the equation gives two equal and opposite values of ρ for a given value of θ.

Let ϕ be the angle between the perpendicular from the centre

FIG. 50.



on a tangent and the radius-vector to the point of contact, then by calculating $\rho\sin\phi\left[=\rho^2 d\theta/(d\rho^2 + \rho^2 d\theta^2)^{\frac{1}{2}}\right]$ the reader may verify that the length, ϖ, of the perpendicular from the centre on a tangent touching at the extremity of a radius-vector ρ is (if $\alpha = 0$) given by

$$\varpi = \frac{b\sqrt{1 - e^2\cos^2\theta}}{\sqrt{e^2\cos^2\theta(e^2 - 2) + 1}}. \tag{64}$$

77. **Area of an Ellipse.**—If A be the area of any plane figure the area of its projection, by lines at right angles to its plane, on a plane inclined at an angle θ to the former (that is for which the angle between two lines drawn in the planes, and perpendicular to the line of their intersection, is θ) is $A/\cos\theta$. For imagine the area divided into very narrow strips having their sides all perpendicular to the line of intersection of the planes. Each strip is a rectangle plus a small portion at each end given by the curved boundary of the area. The length of any rectangular portion if l is changed to $l/\cos\theta$ while its breadth remains unchanged. If A' be the sum of the area of the rectangles, the sum of the areas of their projections is $A'/\cos\theta$. In comparison with A' the difference $A' - A$ may be made as small as we please by taking the strips sufficiently narrow, and hence we obtain accurately for the projected area the value $A/\cos\theta$.

Similarly, if the area be projected on the second plane by lines perpendicular to the latter, the area of the projection is $A\cos\theta$.

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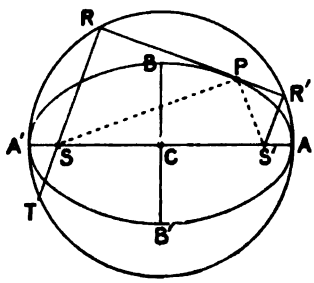
These results it is to be observed are independent of the position of the given figure in its plane.

We see, therefore, that if a square be described round a circle, and the figure be projected, the circle will become an ellipse, and the sides of the square four tangents forming a circumscribing parallelogram.

If a', b' be the lengths of the conjugate semi-axes parallel to the tangents, and θ the angle between them, we have for the area of the parallelogram the value $4a'b'\sin\theta$, and this holds for all pairs of conjugate axes. Therefore, if a, b be the lengths of the principal semi-axes, $a'b'\sin\theta = ab$.

Also the area of the circle being πb^2 (supposing the semi-axis b to be equal to the radius of the circle) the area of the ellipse is πab. For the semi-axes a', b' this becomes $\pi a'b'\sin\theta$.

FIG. 51.



78. **Foci and Focal Properties of Ellipse.**—Space does not permit of further discussion of the geometry of the ellipse, but from the results given above various properties of the curve can be deduced with great ease. For example, taking the principal axes as axes of co-ordinates, and fixing two points SS' (Fig. 51) on the major axis, such that $S'C = CS = \sqrt{a^2 - b^2}$, the reader may prove at once that if as before $e = \sqrt{(a^2 - b^2)}/a$, then $S'P = a - ex$, $SP = a + ex$, so that

$$SP + S'P = 2a.$$

The ratio e is called the *eccentricity* of the ellipse.

The points S, S' are called the foci of the ellipse, and play a great part in the discussion of the properties of the curve. An ellipse of semi-axes a, b, that is of eccentricity $\sqrt{(a^2 - b^2)}/a$, may be generated in the following manner. Fix two pins on a sheet of paper at a distance apart of $2ea$, and place over them a thin string, the ends of which have been knotted together so as to give a closed length of $2(1 + e)a$. Then keeping the string taut against the pins by pressing against it with one side of a sharp pencil point, while the point is carried round pressing against the paper, a closed curve will be described, which will be found to be an ellipse of the dimensions required.

Again, if a tangent to an ellipse be drawn at any point P as in Fig. 51, and produced so as to meet the circle described on the major axis of the ellipse as diameter in the points RR', the lines SR, $S'R'$ drawn from the foci, are perpendicular to the tangent, and $SR.S'R' = b^2$; or, which is the same thing, if RS be produced to meet the circle again in T, $RS.ST = b^2$. This property will be of great service in the discussion of elliptic motion, that is the motion of a particle in an elliptic orbit, under an acceleration constantly directed towards one of the foci.

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The student may also verify that the lines $S'P$, SP drawn from the foci to any point make equal angles with the tangent at P, and that, therefore, the normal at any point bisects the angle between the focal radii of the point. Also he may verify the property, generally made the definition of the curve, that the distance of any point P on the curve from one of the foci, S' say, bears a constant ratio e (less than unity) to its distance from a straight line called the directrix, which is drawn perpendicular to the major axis, and cuts it in a point X at a distance AX from the vertex equal to FA/e.

79. **Focal Polar Equation of Ellipse.**—The polar equation of the ellipse with one focus as pole, that is the relation between the length of the radius vector, drawn from the focus to any point of the curve, and the angle θ its direction makes with a fixed direction in the plane, is easily obtained. Take as pole the focus S on the left of the centre in Fig. 51, and let the angle which the radius-vector makes with the positive direction of the major axis be $\theta - \alpha$. Since $\rho = a + ex$, and $x = \rho\cos(\theta - \alpha) + ae$, we have $\rho = a(1 - e^2) + e\rho\cos(\theta - \alpha)$, which gives

$$\rho = \frac{a(1 - e^2)}{1 - e\cos(\theta - \alpha)}. \tag{65}$$

If $\alpha = 0$, that is if θ be measured from oa, the equation is

$$\rho = \frac{a(1 - e^2)}{1 - e\cos\theta}. \tag{66}$$

The perpendicular from the focus on the tangent at any point is $\rho^2 d\theta / \sqrt{\rho^2 d\theta^2 + d\rho^2}$. The last equation gives

$$\varpi = \frac{b^2}{a\sqrt{1 - 2e\cos\theta + e^2}}. \tag{67}$$

80. **Sections of a Cone.**—In the ellipse e is always less than 1, but another curve, the *hyperbola*, exists, defined also by the focus and directrix property, with the difference that $e > 1$. The properties of this curve correspond exactly to those of the ellipse, though they are by no means identical with the latter. A third curve, the *parabola*, is obtained by making $e = 1$, that is the distance of any point of the curve from the focus equal to its distance from the directrix.

The curves may be obtained by cutting sections of a double right circular cone. A section by a plane making an angle with the axis greater than the semi-vertical angle of the cone gives an ellipse, a section by a plane making an angle with the axis less than the semi-vertical angle gives a hyperbola, which it will be seen has two infinite branches (since the sheets of the cone are supposed to be infinite), while a parabola is given by a section parallel to a generating line. Thus the parabola consists of a single infinite branch.

This mode of generating curves accounts for their ordinary name—the conic sections. The reader may observe the curves, by throwing, by means of a small source of light, such as the flame of a small

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candle, the shadow of a circular disk (a soup-plate for example) on the floor, or on the wall of a room. In the former case, if the flame be higher than every part of the disk, the shadow on the floor will be an ellipse, becoming a straight line of course when the disk is edge-on to the source of light. If the flame be at a height above the floor intermediate between those of the highest and lowest points of the disk, the edge of the shadow will be one branch of a hyperbola thrown by the part of the edge of the disk below the flame. The rays produced backward from the upper part of the edge of the disk would mark out on the floor the other branch of the curve. It is clear that the branches are infinite, since the horizontal rays only meet the floor at an infinite distance.

If the luminous source be on a level with the highest point of the disk the second branch cannot be regarded as existing at all. The curve is then a parabola.

81. **S.H.M.s in Conjugate Diameters of Ellipse giving Elliptic Motion.**—Returning now to simple harmonic motion producing circular motion, we see that the rectangular components

$$x = a\cos(nt - \epsilon), \quad y = a\cos(nt - \epsilon + \pi/2)$$

project into simple harmonic motions in the conjugate diameters of an ellipse of amplitudes a', b' the lengths of the conjugate semi-axes, and differing in phase by $\pi/2$ in angle, or a quarter of a period in time. Further, the resultant of the two simple harmonic motions thus specified is an elliptic motion which can be obtained by projection from a uniform circular motion.

It is to be noticed that the difference in phase is not the angle between the conjugate diameters. The time taken by the radius, drawn from the centre of the ellipse to the projection of the particle moving in the circle, to pass through any of the four angles in the ellipse between one conjugate semi-axis and the other, is, of course, a quarter of a period. The angular velocity of the radius to the particle in the elliptic motion is a maximum at the extremities of the minor axis, and a minimum at the extremities of the major axis.

82. **Velocities in the Elliptic Motion.**—The magnitudes of the linear and angular velocities at any point in the elliptic orbit may be found as follows. The radius drawn from the centre of the circle to the particle moving in the circumference has angular velocity n, and describes equal areas in equal times. Thus in a short time dt, the area swept over is $\frac{1}{2}r^2ndt$, since it is very approximately a triangle of base $rndt$ and height r. Thus $\frac{1}{2}r^2n$ is the rate at which the radius describes area. If successive intervals all of one length dt be taken, the successive areas $\frac{1}{2}r^2ndt$ will be equal, and their projections on the plane of the ellipse will be equal. If now ω be the angular velocity of the radius-vector for any position of the point in the ellipse, v the linear velocity of the point, ρ be the length of the radius-vector, and ϖ that of the perpendicular let fall from the centre on the tangent at the point, the area described in time dt is $\frac{1}{2}\rho^2\omega\, dt = \frac{1}{2}\varpi v dt$. Thus we have

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$\omega = r^2 n/\rho^2 \cos\theta$, and $v = r^2 n/\varpi \cos\theta$, where θ is the angle between the planes of the circle and ellipse in Fig. 47. The angular velocity is thus inversely proportional to the square of the length of the radius-vector, the linear velocity is inversely as the perpendicular from the centre on the tangent. The lengths of the radius-vector and the perpendicular from the centre on the tangent at its extremity are given in § 76 above.

83. **Composition of S.H.M.s in Rectangular Directions.**—When the motions to be compounded are in directions at right angles to one another, but do not differ in phase by a quarter of a period, the resultant motion is still elliptic, as may be seen in the following manner. Let the given components be

$$\left.\begin{aligned} x &= a\cos(nt - \varepsilon) \\ y &= b\cos(nt - \eta). \end{aligned}\right\} \quad (68)$$

These expanded give $x = x_1 + x_2$, $y = y_1 + y_2$ where

$$\begin{aligned} x_1 &= a\cos\varepsilon\cos nt, & x_2 &= a\sin\varepsilon\sin nt \\ y_1 &= b\cos\eta\cos nt, & y_2 &= b\sin\eta\sin nt. \end{aligned}$$

The components x_1, y_1 give simple harmonic motion in a straight line, the equation of which is

$$b\cos\eta . x_1 - a\cos\varepsilon . y_1 = 0.$$

The components x_2, y_2 give also a rectilinear simple harmonic motion in the line

$$b\sin\eta . x_2 - a\sin\varepsilon . y_2 = 0.$$

Thus we have in these lines the two simple harmonic motions

$$\left.\begin{aligned} s_1 &= \sqrt{a^2\cos^2\varepsilon + b^2\cos^2\eta}.\cos nt. \\ s_2 &= \sqrt{a^2\sin^2\varepsilon + b^2\sin^2\eta}.\sin nt. \end{aligned}\right\} \quad (69)$$

They are inclined at the angle

$$\tan^{-1}(b\cos\eta/a\cos\varepsilon) - \tan^{-1}(b\sin\eta/a\sin\varepsilon)$$

and differ in phase by a quarter of a period. They therefore give motion in an ellipse of which these two lines are conjugate diameters. The lengths of the conjugate semi-axes are the co-efficients of $\cos nt$, $\sin nt$ in the values of s_1, s_2 above.

When the periods of the two simple harmonic motions to be compounded are not quite exactly equal, the amount of inequality is equivalent to a difference of epoch added each period. The motion is very closely in an ellipse at any instant of time, but in consequence of the change of phase thus growing up the position of the axes of the ellipse is continually undergoing change. Thus with an apparatus for recording the continually varying path this phenomenon, which is analogous to that of the revolution of the apses of the orbit of a planet, can be studied.

84. **General Case of Composition of S.H.M.s.**—If we have any number of simple harmonic motions of any amplitudes and epochs

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and in any directions whatever, but of the same period, it can be shown that they are equivalent to two components having a difference of phase of $\pi/2$, and therefore give elliptic motion of the kind just discussed. For let $s = a\cos(nt - \varepsilon)$ be a specimen motion in a line the direction cosines of which are l, m, n, then we can resolve it into components along the axes of x, y, z, which are

$$\begin{aligned} x &= la\ \cos\varepsilon\cos nt + la\ \sin\varepsilon\sin nt \\ y &= ma\cos\varepsilon\cos nt + ma\sin\varepsilon\sin nt \\ z &= na\ \cos\varepsilon\cos nt + na\ \sin\varepsilon\sin nt. \end{aligned}$$

Doing this for all the motions and adding we get (denoting the sum of the values of any quantity x by $\Sigma(x)$)

$$\left.\begin{aligned} \Sigma(x) &= A\cos nt + A'\sin nt \\ \Sigma(y) &= B\cos nt + B'\sin nt \\ \Sigma(z) &= C\cos nt + C'\sin nt \end{aligned}\right\} \quad (70)$$

where $A = \Sigma(la\cos\varepsilon)$, $B = \Sigma(ma\cos\varepsilon)$, $C = \Sigma(na\cos\varepsilon)$, $A' = \Sigma(la\sin\varepsilon)$, $B' =$ &c.

We may now compound the motions given by the three first terms on the right into a single harmonic motion, the motions given by the three second terms into another. The former is along the line the direction cosines of which are

$$A/\sqrt{A^2 + B^2 + C^2},\quad B/\sqrt{A^2 + B^2 + C^2},\quad C/\sqrt{A^2 + B^2 + C^2},$$

and is represented by

$$s = \sqrt{A^2 + B^2 + C^2}\cos nt. \quad (71)$$

The latter is along the line the direction cosines of which are $A'/\sqrt{A'^2 + B'^2 + C'^2}$, &c., and is given by

$$s' = \sqrt{A'^2 + B'^2 + C'^2}\sin nt. \quad (72)$$

Thus the motion is reduced to two in lines inclined to one another at the angle

$$\cos^{-1}\frac{AA' + BB' + CC'}{\sqrt{A^2 + B^2 + C^2}\sqrt{A'^2 + B'^2 + C'^2}}$$

with a difference of phase of a quarter of a period. Thus again the resultant motion is motion in an ellipse of which the lines just determined are the conjugate axes.

85. **Composition of S.H.M.s in Rectangular Directions and of Periods in the Ratio of 1 : 2.**—Let us consider now very briefly two simple harmonic motions in rectangular directions, in periods which are in the ratio of two whole numbers, and with epochs differing by any specified amount. Thus let the motions be represented by

$$x = a\cos(mt - \varepsilon),\quad y = b\cos nt,$$

where m, n are whole numbers. The equation of the resultant path

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is to be obtained by eliminating t between the two equations, and so finding a relation connecting x, y and the three constants a, b, ε.

We may take one or two particular cases. Thus let $m/n = 2$ (corresponding to the interval of an octave in music), the equations of the motions are

$$x = a\cos(2nt - \varepsilon), \quad y = b\cos nt.$$

Expanding the value of x and substituting y/b for $\cos nt$, $\sqrt{1 - y^2/b^2}$ for $\sin nt$, we obtain

$$x = a\left\{\left(\frac{2y^2}{b^2} - 1\right)\cos\varepsilon + \frac{2y}{b}\sqrt{1 - \frac{y^2}{b^2}}\sin\varepsilon\right\} \tag{73}$$

for the equation of the path.

If $\varepsilon = 0$, the equation reduces to

$$y^2 = \frac{b^2}{2a}(x + a), \tag{74}$$

which represents a parabola. The path is thus an arc of a parabola symmetrical about the axis of y, and limited by the extreme values of x and y. It is shown in the first diagram of the series in Fig. 52.

If $\varepsilon = \pi/2$ the equation of the path is

$$b^4x^2 = 4a^2y^2(b^2 - y^2). \tag{75}$$

This represents a curve passing through the origin. Its nature near the origin may be made out by considering what the equation becomes for very small values of y. Then the term in y^4 is very small in comparison with that in y^2, and the equation is approximately

$$b^2x^2 = 4a^2y^2,$$

which is equivalent to the two $bx - 2ay = 0$, $bx + 2ay = 0$. These represent two lines inclined to the axis of x at the supplementary angles $\tan^{-1}b/2a$, $\tan^{-1}(-b/2a)$ to the axis of x. Thus the curve at the origin has the form of a St. Andrew cross. These lines are the tangents at the origin, which is also a point of inflection for each of the two portions of the curve that cross there.

Again, when $x = 0$ we have $y^2(y^2 - b^2) = 0$, which gives $y = \pm 0$, $y = \pm b$. Thus the axis of y cuts the curve in four points, two coincident at the origin (one on each part of the curve), and normally in two others equidistant from the origin. Lastly, since the curve is symmetrical about the axes of x and y, for its equation is satisfied by equal positive or negative values of x for a given value of y, and there are two pairs of such values of y for a given value of x, it has the form of a "figure of eight." When $a = b$ the amplitudes of the component motions are equal, and the curve is shown in the fifth diagram of the series in Fig. 52. The same curve would be obtained if the difference of epochs were any odd multiple of $\pi/2$, as the reader may verify.

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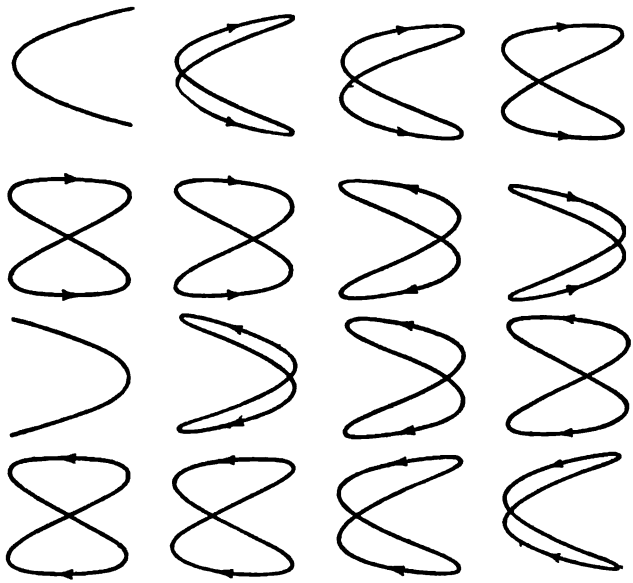
When the difference of epochs is neither a multiple of π nor an odd multiple of $\pi/2$, the curve lies in form between a parabola and the figure of eight. The series of sixteen curves in Fig. 52 are drawn for values of ε increasing by steps of $\frac{1}{8}\pi$ from 0 up to 2π. Further like increase of ε would only result in a repetition of the same series of paths.

When the periods of the simple harmonic motions are very nearly, but not quite exactly, in the ratio of 1 to 2, the resultant motion is

FIG. 52.



at each instant, and even for a moderate interval, very approximately along the curve which would be followed throughout if the relation were exact; while the non-fulfilment of the relation has its effect in a continual change of the difference of epochs, so that the path is carried in succession through every possible variety of form. This succession is illustrated in Fig. 52.

86. **Two Rectangular S.H.M.s of Periods in the Ratio of 2 : 3.**—When the periods are in the ratio of 2 : 3 (corresponding to the interval of a perfect fifth in music) the resultant path is more complicated. Fig. 53 gives the curve for a difference of epoch varying from 0 to $5\pi/8$ by steps of $\frac{1}{8}\pi$. Fig. 54 shows the still more complex path for two motions of periods in the ratio 3 : 4, and of epochs differing as before.

87. **Realisation of Resultant Curves.**—These harmonic curves may be realised in different ways. We shall describe three, and

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it will be a good exercise for the student to try them, and follow out for himself the formation of the curve in different cases.

Take a glass cylinder (a wide glass bottle will do) and lay down on good transparent tracing-paper a curve of sines on such a scale that say two periods of the curve are obtained in a distance AB along the

FIG. 53.

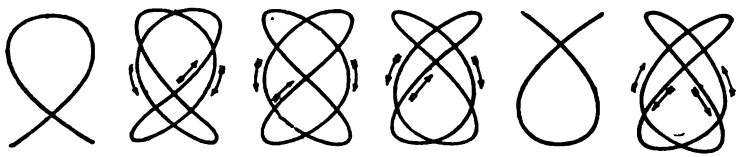
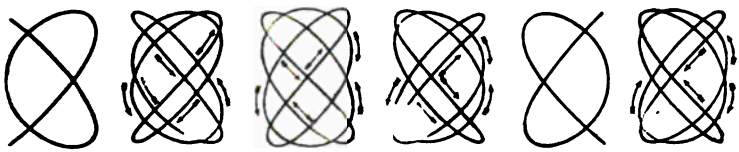


FIG. 54.



axis of x equal to the circumference of the tube. Then roll the paper round the tube so that the line ABC forms a circumference. If the tube is placed in a vertical position and looked at horizontally from a sufficient distance, the curve on the cylinder will appear to be projected by parallel lines on a vertical plane perpendicular to the

FIG. 55.

FIG. 56.

FIG. 57.

direction of vision. If the tube be then turned round its axis the projected curve will pass through its successive forms, of which specimens are given in Figs. 55, 56, 57. If two diametrically opposite points in which the curve cuts the line of abscissae are in line, the figure seen is the figure of eight; if two maximum ordinates or two minimum ordinates are placed in the vertical plane through the axis of the tube and the eye, the curve is the parabola; and so on. Figs. 55, 56, 57 were obtained in this way by winding two periods of a

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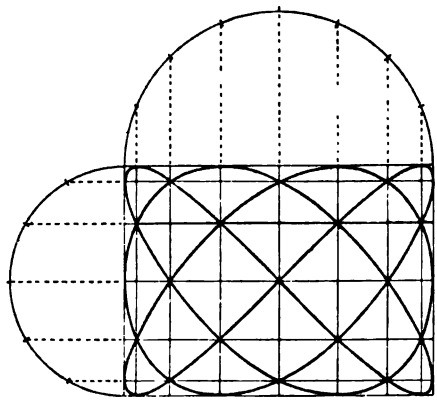
curve of sines round a cylindrical glass bottle, and photographing the whole from a distance.

If the curve drawn on the tracing-paper has three periods, and is just long enough to pass twice round the tube, the ratio of periods is 2 : 3 ; and a series of curves comprising those given in Fig. 53 will be observed in viewing the tube as before, and turning it round its axis.

The curves may also be drawn as follows. Lay down on paper a rectangle having sides proportional to the ranges of motion $2a$, $2b$. Then if the period of the motion of range $2a$ be to that of the motion of range $2b$ in the ratio m/n, divide, in the following manner, the side $2a$ of the rectangle into pm parts, and the side $2b$ into pn parts, where p is any convenient whole number. On the sides

FIG. 58.



$2a$ and $2b$ as diameters describe semi-circles, and divide the circumferences respectively into pm and pn equal parts. Then project the points of division of each semi-circle on the diameter by lines perpendicular to the latter. Lines then drawn through these points thus found parallel to the sides of the rectangle will divide the latter into p^2mn small rectangles, of sizes varying from the centre to the sides of the large rectangle.

Then starting from any corner of one of these rectangles draw a line diagonally across an adjacent rectangle, and continue it diagonally across the next, and so on until one of the rectangles at the outside is reached. The line should then be continued, but made to curve round, touching the side at the remote corner of this rectangle so as to turn back, and go diagonally across the next. The line is then continued as before until either one of the corners of the large rectangle is reached or a symmetrical figure is completed by the line returning to the first point, going in the same direction as at starting. Further motion of the tracing point would merely retrace the curve. If a corner is reached, the curve is to be retraced and continued

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beyond the starting-point until a corner is again reached. After this the curve is again merely retraced.

Fig. 58 illustrates this construction for the case of $m=4$, $n=3$, and $p=2$.

88. **The Blackburn Pendulum.**—But perhaps the easiest way of drawing the curves is by means of the double pendulum invented by Professor Blackburn. The arrangement is shown in Fig. 59.* Cords of equal length are attached at points AB in a horizontal beam, and meet at C, where there is attached a single cord carrying a heavy bob P. The bob is generally a ring of lead encircling a funnel containing fine dry sand,† which may be confined by a small plug in the mouth of the tube until an experiment is to be made. Below on the table is a board painted a dull black, which can be removed when required and replaced by another.

FIG. 59.

It will be seen that this contrivance really consists of two pendulums of different lengths, free to vibrate in planes at right angles to one another. For the whole suspended system may turn about the line AB, or the lower part alone about the point C in the plane ABC. Any motion given to the lower part perpendicular to this plane would cause motion of the upper part as well, so that there are only the two independent motions specified.

By means of a double cord and a small ring which can be pushed along it the position of C and the relative lengths of the two pendulums can be altered at pleasure. Any variation in the total vertical height caused by changing the length of the fork can be corrected by carrying the ends of the cord through rings or over pulleys at AB, and securing the ends round cleats on the posts as shown.

It is proved in § 138 that the bob of an ordinary simple pendulum (that is a pendulum consisting of a massive particle suspended

* Fig. 59 is taken, with some changes, from *Practical Acoustics*, by C. L. Barnes (Macmillan and Co.).

† It is sometimes difficult to get fine white sand. The fine powdered sugar called "castor sugar" is very cheap, and does quite as well.

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by a fine thread) describes approximately simple harmonic motion, and the more approximately the smaller the arc of oscillation, and that the period varies directly as the square root of the length of the pendulum. Thus, by adjusting the position of D, the periods may be brought very exactly into any required relation, except, of course, that of equality, which can only be roughly obtained owing to the impossibility of bringing AC, CB into a horizontal line.

The pendulum may be initially deflected to any desired extent, and secured by a fine thread attached to the bob and an upright, carried by a heavy foot so that it can be shifted if necessary after the thread has been fixed. The plug having been withdrawn and any motion of the bob set up having died out, the thread is burned by the flame of a match, and the bob released to begin the description of the path, which is shown by the white sand on the black ground of the board beneath.

As starting the pendulum from rest in this way involves $\epsilon = 0$, it is better, for the exhibition of the curve for any required difference of phase, to have the relation of periods slightly inaccurate, and, having started the pendulum, to sweep off the sand from the board without touching the bob, until by the accumulation of difference of epoch the motions are in the required phase. Then the pattern having been completed, the board can be slipped from under the bob, and the path photographed if a record of it is to be preserved.

The pendulum serves exceedingly well for the demonstration of the slowly varying elliptic motion produced by two motions not exactly in unison. The fork C is pulled up as near to the horizontal line AB as possible, so that the periods are rendered almost equal. It must be remembered, however, that with a heavy bob the tension of the nearly horizontal cords will be great, so that a strong cord must be chosen.

It is very interesting to watch the gradual alteration of phase as shown by the change of form of the path, and also the slow diminution of amplitude due to frictional resistance to the motion of the bob. As the stream of sand forms a perfectly flexible nearly frictionless style, the chief cause of diminution of amplitude is the action of the air on the bob. The motions compounded are thus cases of the resisted motion discussed above, § 59. A pen or pencil is sometimes substituted for the sand stream, and a simple harmonic motion of the record-sheet is sometimes superposed, as in Tisley's and other forms of apparatus for producing harmonic curves. It is perhaps worth while applying the method of obtaining a stream of ink from a fine nozzle, not touching the record-sheet, which is adopted in Lord Kelvin's "Siphon Recorder." There the ink is contained in a metal box, which is electrified. The paper rests on an uninsulated metal roller, and so the ink is forced out in a stream of fine drops by electric attraction.

The curves here described seem to have been first experimentally produced by Lissajous, and his method of exhibiting them will be explained later in the section of this work dealing with *Acoustics*.

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89. **Curve of Velocities. Distance travelled in Given Time.**—The distance travelled in any direction in any time by a particle moving in a given manner can be found by taking what is called the integral, that is simply the sum, of the distances travelled in successive exceedingly small intervals of time, the proper mean velocities for each of which are calculated from the known law of variation of the velocity. This process, which apart from the technical rules for carrying it out in certain cases is exceedingly simple in conception, will be easily understood from the examples which follow. Let a curve be constructed with as ordinates successive values of the velocity for which the space described in a given time is to be computed.

FIG. 60.

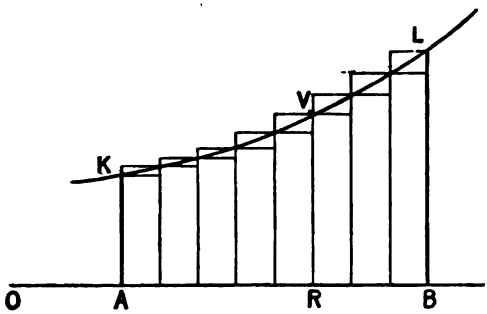
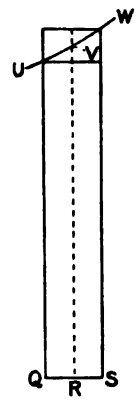


FIG. 61.



Let Fig. 60 represent such a curve, and let the ordinate AK represent the speed at time t_0, the beginning of the interval, BL the speed at time t_1, the end of the interval, and RV the speed at time t, an intermediate instant. The numerical value of the space described in a short interval of time dt including the instant which marks the time t lies between vdt the area of the rectangle SU, Fig. 61, and that of the rectangle QW. For these are obtained by multiplying the interval dt in the first case by the speed at the beginning of the interval, in the second by the speed at the end, and the interval is taken so short that the change of speed during the interval has been throughout an increase or a diminution.

Dividing then the whole interval $t_1 - t_0$ up into elements of length dt, not necessarily equal, but very small, we see that the whole distance traversed in the time $t_1 - t_0$ is less than the sum of the areas of the rectangles, the tops of which are the steps of, so to speak, the upper staircase, and greater than the sum of the areas of those which have as their tops the steps of the lower staircase. Clearly the narrow zig-zag string of small areas which forms the difference between these values approaches zero indefinitely

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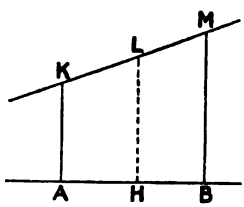
as dt is taken shorter and shorter. Thus, RV being denoted by v and denoting by

$$\Sigma_{t_0}^{t_1} vdt$$

simply the sum of the products, of which vdt is a specimen, for the elements into which the time $t_1 - t_0$ is divided, we see that the limiting value of this sum, when all the elements, dt, are taken infinitely short, is the area of the figure contained within the terminal ordinates AK, BL, the curve KVL, and the axis AB. Then to indicate that the limiting value is taken we use instead of the Greek *sigma* for sign of summation the sign $\int$, which is merely an old-fashioned long s, and is called the sign of integration. It has nothing more than the simple meaning which has just been explained. The process, however, by which the value of $\int_{t_0}^{t_1} vdt$ can be calculated when the relation of v to t is known, is a technical matter, and is part of what is called the *integral calculus*. The fundamental ideas on which this calculus is based are quite simple, and exist in a more or less latent state in the mind of nearly everybody.

FIG. 61′.



90. **Integral of Curve of Velocities. Uniformly Accelerated Motion.**—We may now take one or two examples. It will not matter to the results, but it will save printing if in these we take t_0 as zero, that is reckon the time from the beginning of the interval.

Let the speed vary directly as the time, that is let the rate of gain of speed be a constant, a, and let it be required to find the space traversed in time $t = t_1$. This is the case of constant acceleration in the direction of motion. If the speed at time $t = 0$ was v_0, at time t it will be $v_0 + at$. If A, Fig. 61′, represent t_0, which we take as zero, AB represents t_1. Let AK and BM represent v_0 and v_1 respectively, then the speed at any intermediate instant, H, will be obtained by joining LM by a straight line, and taking HL parallel to AK or BM. Thus the distance traversed is represented by the area $ABMK$, which is $\frac{1}{2}(AK + BM).AB$ or $\frac{1}{2}(v_0 + v_1)t$. But $v_1 = v_0 + at_1$. Hence

$$s_1 = v_0 t_1 + \tfrac{1}{2} a t_1^2,$$

and of course for any time t we have

$$s = v_0 t + \tfrac{1}{2} a t^2. \qquad (76)$$

Substituting for t its value $(v - v_0)/a$, we obtain

$$s = \frac{v^2 - v_0^2}{2a};$$

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or, as we may write it,

$$as = \tfrac{1}{2}v^2 - \tfrac{1}{2}v_0^2. \qquad (77)$$

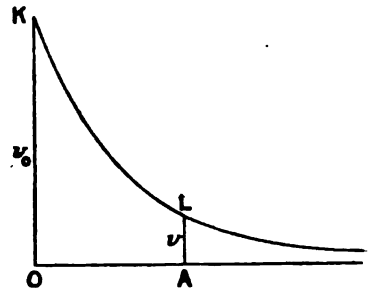
This last equation is of great importance, as will be seen in the sequel in connection with the theory of energy.

91. **Integral of Curve of Velocities. Resisted Motion.**—As another example, let us find the space described in an interval t by a particle the velocity of which is given by the formula $v_0e^{-\lambda t}$, where λ is constant. This is the case of a particle moving under a retardation proportional to the velocity. The curve of velocities is shown in Fig. 62, in which OA represents t, and OK and AL represent v_0 and

$$v(=v_0e^{-\lambda t})$$

respectively.

FIG. 62.



The curve represents a velocity diminishing in geometrical progression as the time increases in arithmetical progression. For if t be divided up into n successive intervals of equal length dt, so that $t = ndt$, the velocities at the end of the first, second, ..., n^{th} of these are by the expression given

$$v_0e^{-\lambda dt},\ v_0e^{-2\lambda dt},\ \ldots,\ v_0e^{-n\lambda dt}.$$

The space described in time t will thus be less than

$$v_0dt + v_0e^{-\lambda dt}dt + \ldots + v_0e^{-(n-1)\lambda dt}dt,$$

and greater than

$$v_0e^{-\lambda dt}dt + v_0e^{-2\lambda dt}dt + \ldots + v_0e^{-n\lambda dt}dt,$$

for the first of these is the space that would be described if the speed of the particle throughout each interval dt had the value given by the formula for the beginning of the interval, and the second is the space that would be described if the speed throughout each interval had the value given by the formula for the end of it. The first value exceeds the second by $v_0dt - v_0e^{-n\lambda dt}dt$, or

$$v_0dt(1 - e^{-\lambda t}),$$

which may be made as nearly zero as we please by making dt sufficiently small.

Call the sum of the first series s_1, and multiply it term by term by $e^{-\lambda dt}$, then we get $s_1e^{-\lambda dt}$, which is the second series. Thus by subtracting the second series from the first we obtain

$$s_1\left(1 - e^{-\lambda dt}\right) = v_0dt\left(1 - e^{-n\lambda dt}\right) = v_0dt\left(1 - e^{-\lambda t}\right);$$

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and therefore

$$s_1 = \frac{v_0 dt\left(1 - e^{-\lambda t}\right)}{1 - e^{-\lambda dt}} = \frac{v_0 dt\left(1 - e^{-\lambda t}\right)}{1 - e^{-\lambda t/n}}, \tag{78}$$

the limiting value of this when dt is made indefinitely small, that is when n is made indefinitely great, is the distance traversed. The denominator in the expression for s_1 can be written (see the exponential theorem in § 54 above) in the form of the infinite series

$$\frac{\lambda t}{n} - \frac{\lambda^2 t^2}{1.2.n^2} + \frac{\lambda^3 t^3}{1.2.3.n^3} - \ldots\ldots,$$

the terms of which are, after the first, each less than the corresponding terms of the geometric progression

$$\frac{\lambda t}{n} - \frac{\lambda^2 t^2}{n^2} + \frac{\lambda^3 t^3}{n^3} - \ldots.$$

This may be written

$$\frac{\lambda t}{n}\left\{1 - \frac{\lambda t}{n} + \frac{\lambda^2 t^2}{n^2} - \ldots\right\}.$$

The limiting value of the series in brackets is unity when n is made indefinitely great, and thus the limiting value of the sum of the geometric series is $\lambda t/n$. This is evidently, from what has been stated, also the limiting value of $1 - e^{\lambda t/n}$. Thus if s be the limiting value of s_1 we have instead of (78)

$$s = \frac{v_0 n dt}{\lambda t}\left(1 - e^{-\lambda t}\right) = \frac{v_0}{\lambda}\left(1 - e^{-\lambda t}\right). \tag{79}$$

The summation here effected is much more quickly carried out by the use of the rules of the integral calculus. The process is, however, given at length as an illustration of the theory of limiting values.

92. **Uniplanar Kinematics of a Rigid System.**—We now consider the kinematics of a rigid body, that is a body the particles of which have a constant relative configuration, or, in other words, a body in which the distance between every pair of particles remains unchanged. We also impose here the condition that every point of the body is compelled to move parallel to a given plane. We need consider only the displacements of points lying in a parallel plane of the body, since all lines in the body at right angles to the plane at any one instant preserve throughout the same direction. We shall consider then the motion of a plane of the body which slides on a plane fixed in space. We shall call the former the *moving plane*, or the *carried plane*, the latter the *fixed plane*.

93. **Displacement, with one Point fixed, effected by Rotation.**—First then any displacement of the body, from any initial to any

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final position, may be effected by rotating the body through an angle about a determinate axis perpendicular to the plane of motion. For let AB, Fig. 63, be any line in the plane of motion in the initial position of the body, and $A'B'$ the final position of the line. Join AA' and BB'. Through the points of bisection of AA' and BB' draw perpendiculars to these lines, and let these perpendiculars meet in I. Then AB may be transferred to $A'B'$ by a rotation round I, and the same will hold for the corresponding displacement of any other line in the body. For joining AI, BI and $A'I$, $B'I$, we see that the triangles AIB, $A'IB'$ are equal in all respects, and the angle AIA'

FIG. 63.

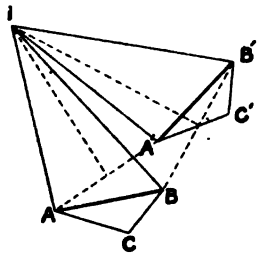
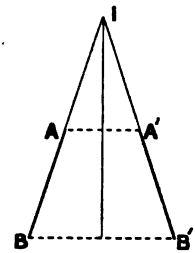


FIG. 64.



is therefore equal to the angle BIB'. Hence AB can be brought into the position $A'B'$ by a rotation of the body through the angle $\theta = \angle AIA' = \angle BIB'$. This proof breaks down in the particular case in which the lines bisecting AA', BB' coincide, as in Fig. 64. The axis in this case is at I where the lines AB, $A'B'$ meet. If AB, $A'B'$ be parallel, the axis I is at an infinite distance.

Also if C be any other point of the body the position of C must evidently be C', given by the same rotation. Since the body is rigid in the sense defined above, the configuration of any three points A, B, C must remain unaltered. The triangles ABC, $A'BC'$ must therefore be equal in all respects, and C is brought to C' also by a rotation about I through the angle θ.

94. **Continuous Uniplanar Displacement.**—In the continuous uniplanar displacement of a rigid body, the motion of the body at any instant is capable of being regarded as one of rotation round an axis perpendicular to the plane. In any such motion the extremities of any straight line in the plane of motion describe curves in that plane, and the motion may be regarded as one in which the straight line moves with its extremities on these two curves. This will specify the successive positions geometrically, since all parts of the body are rigidly connected, and all displacements are parallel to one plane. Let S, S', Fig. 65, be the curves on which the line AB moves, AB, $A'B'$ two positions of the line. A has moved to A', and B to B' along the elements AA', BB' of the curve. The change from AB to $A'B'$ may be effected by a rotation of the body about an axis found by

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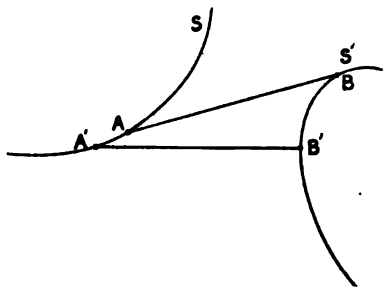
bisecting the chords AA', BB' of the curves and drawing perpendiculars to these chords through the points of bisection, as in Fig. 63. Different positions of the axis will be given by the different pairs of chords AA', BB', $A'A''$, $B'B''$, &c. Thus as the body moves the axis moves also, and changes its position both in the body and in space.

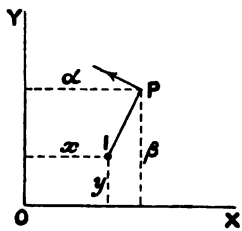
The motion of the axis is continuous if that of the body is con-

FIG. 65.



tinuous; and the position of the axis about which the body is moving at any instant, say when the line is in the position AB, is clearly the limiting position found by making the chords AA', BB' infinitely short, that is the intersection of the perpendiculars from the tangents to the curves at A, B. This is called the *instantaneous axis* of the body for its position as defined by AB. It is clear from what has been stated above, with respect to the displacement shown in Fig. 63, that there is at each instant only one such axis, and that it can be obtained by considering the *motion* of any line of the body in the plane of displacement.

FIG. 66.



95. **Centrodes.** — In general, as has already been stated, the intersection of the instantaneous axis with the fixed plane describes a curve in that plane, and likewise a curve in the plane of the body which is coincident with the fixed plane. These paths are called *centrodes*. The former is called the *space-centrode*, the latter the *body-centrode*. We shall denote the former curve by C_s the latter by C_b.

We may find the position of the instantaneous axis as follows. Let OX, OY, Fig. 66, be rectangular axes fixed in the plane of motion, and let x, y be the co-ordinates of the instantaneous axis I. Since the axis is at rest at the instant under consideration, and all other points are turning round it with angular velocity ω, we can easily find the component velocities of any other point P, the co-ordinates of which are α, β, say. If l be the distance IP, the velocity

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of P is in the direction shown in Fig. 66, and is of amount $l\omega$. Thus if θ be the angle IP makes with OX, and u, v be the velocity components of P, we have

$$-u = l\omega\sin\theta = \omega(\beta - y)$$
$$v = l\omega\cos\theta = \omega(\alpha - x).$$

Thus

$$\left.\begin{aligned} x &= \alpha - \frac{v}{\omega} \\ y &= \beta + \frac{u}{\omega} \end{aligned}\right\} \quad (80)$$

which determine the position of the axis in terms of α, β and their time-rates of change u, v and the angular velocity ω.

Equations (80) involve the time, inasmuch as α, β, u, v, ω are in general functions of t. The equation of the space-centrode is found by eliminating t between the two equations of (80).

96. **Equation of the Body-Centrode.**—The body-centrode, C_b, is as stated above the curve traced out, in that plane of the body which coincides with the plane of motion, by the successive positions of the instantaneous axis in the body. Take as axes two lines fixed in the body and let the co-ordinates of the point I just considered be, at time t, ξ, η with respect to these axes. Also let the angle which the axis of ξ then makes with OX be θ'. The co-ordinates of P with respect to these axes are

$$\xi + l\cos(\theta - \theta')$$
$$\eta + l\sin(\theta - \theta'),$$

that is, since $l\cos\theta = \alpha - x$, $l\sin\theta = \beta - y$,

$$\xi + (\alpha - x)\cos\theta' + (\beta - y)\sin\theta'$$
$$\eta + (\beta - y)\cos\theta' + (\alpha - x)\sin\theta'.$$

But since I is the instantaneous axis we have seen that

$$x = \alpha - v/\omega, \quad y = \beta + u/\omega,$$

so that these co-ordinates become

$$\xi + \frac{v}{\omega}\cos\theta' - \frac{u}{\omega}\sin\theta'$$
$$\eta - \frac{u}{\omega}\cos\theta' + \frac{v}{\omega}\sin\theta'.$$

If now the origin of ξ, η coincide with P these values are zero, and therefore

$$\left.\begin{aligned} \xi &= -\frac{v}{\omega}\cos\theta' + \frac{u}{\omega}\sin\theta' \\ \eta &= \frac{u}{\omega}\cos\theta' - \frac{v}{\omega}\cos\theta' \end{aligned}\right\} \quad (81)$$

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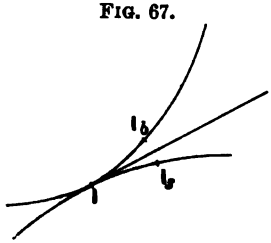
These equations give by elimination a relation between ξ, η and constants, which is the equation of C_b with reference to axes fixed in the carried plane.

The loci C_b, C_s are two sequences of points such that, as the body moves, each point of the former sequence in its order comes into coincidence with a corresponding point of the latter, and as it does so comes to rest, though it need not remain at rest for any interval of time however short. Thus a point I_b of C_b becomes the instantaneous axis when it comes to rest in coincidence with a point I_s of C_s. The position in space of I_s changes with velocity $\dot{x}$, $\dot{y}$, and acceleration $\ddot{x}$, $\ddot{y}$, to be calculated from the values of x, y given in equations (80).

FIG. 67.



97. **Velocity of Instantaneous Axis along Body-Centrode.**—The velocity in space of the point I_b, which is in coincidence with I_s at the instant, is of course zero. We shall show that the velocity with which the position of I_b is changing along C_b is the same as the velocity of I_s along C_s, and that the two curves have a common tangent at the instantaneous axis (Fig. 67).

Let I be the instantaneous axis, and I_s, I_b points of C_s, C_b, which after an infinitesimal interval of time dt, come into coincidence for the new position of I. Then since the body is turning round I, I_b is approaching I_s in a direction at right angles to the line II_b, with a speed $II_b.\omega$; and since ω is finite the direction of any part of the element, II_b of C_b can alter only infinitesimally in dt. Consequently, in the limit, as I moves up to I_b, the direction of motion of I_b does not change, and is throughout at right angles to II_b. But I_bI_s is small in comparison with II_b, being of the order $II_b.\omega dt$; hence II_s is in the limit coincident with II_b. The two curves have therefore a common tangent at I, and the speed with which the space-point at I is moving along C_s is equal to the speed $\dot{s}$, say, with which the coincident body-point is moving along C_b.

98. **Acceleration of Body-Point at Axis.**—Since II_b is $\dot{s}dt$, the speed of I_b in space is at the instant under consideration $\dot{s}\omega dt$, and, since I_b comes to rest at I_s, this must be annulled in time dt. Hence the acceleration of I_b is $\dot{s}\omega$ in the direction opposed to the motion due to rotation. The component of acceleration $\omega^2\dot{s}dt$, from I_b to I, is infinitely small in comparison. The acceleration of the body-point at I is thus $\omega\dot{s}$, and is towards the centre of curvature of C_b.

99. **Acceleration of any Body-Point.**—It is to be observed that this result gives for the acceleration of every body-point P, Fig. 68, at distance p from I besides the components $\dot{\omega}p$ perpendicular to IP in the direction of motion, and ω^2p from P to I, a component $\omega\dot{s}$ in the direction at right angles to the tangent at I as shown in Fig. 68.

The latter component is due to the motion of I along C_s. Its existence at P may be shown as follows. In the interval dt let the

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body-point move from P to P'. Let Pp, Fig. 68′, perpendicular to IP, represent the velocity of P round I, and Pp' the velocity of P' round I_s. [Since we otherwise take into account the acceleration due to change of ω, and change of position of P, we here suppose ω constant and P fixed, during the interval dt.] Pp is proportional to IP, Pp' to I_sP, and the angle pPp' is equal to IPI_s. The triangles

FIG. 68.

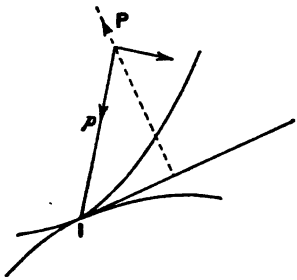
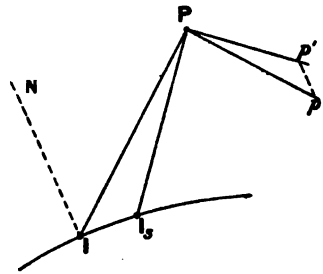


FIG. 68′.



IPI_s, pPp' are therefore similar, and since Pp, Pp' are perpendicular to PI, PI_s, pp' is perpendicular to II_s. Also $pp' = II_s.Pp/PI$. But $Pp = \omega.PI$, so that $pp' = \omega.II_s = \omega\dot{s}dt$. The change of velocity in time dt is thus $\omega\dot{s}dt$, that is the acceleration is $\omega\dot{s}$, and in the direction IN perpendicular to the curve at I.

The acceleration of P may be obtained by the algebra of steps as follows. Denoting IP by p we have for the velocity of P the value $i\omega p$, which represents the rate at which P is moving, and indicates that the motion is in the direction perpendicular to IP, as shown in Fig. 68. The rate of variation of this is $i\omega\dot{p} + i\dot{\omega}p$. But $\dot{p}$ is the rate of change of p, and therefore is equal to

$$\textit{velocity of } P - \textit{velocity of } I = i\omega p - \omega\dot{s}.$$

Hence $i\omega\dot{p} + i\dot{\omega}p = -\omega^2 p - i\omega\dot{s} + i\dot{\omega}p$, that is the acceleration of P consists of $\omega^2 p$ from P towards I, of $-i\omega\dot{s}$ at right angles to $\dot{s}$, and $i\dot{\omega}p$ in the direction of motion of P. Since ip is used to denote a step turned round from IP to the direction of motion, $-i\omega\dot{s}$ denotes an acceleration the direction of which is that of $\dot{s}$ turned through a right angle in the opposite direction, that is $-i\omega\dot{s}$ is in the direction of the arrow on the dotted line through P in Fig. 68.

100. **Curvature of Path of any Point deduced from Acceleration.**—The acceleration of any point P enables the curvature of the path described by P while the body rolls to be determined. The component at right angles to the direction of motion of P at any point is equal to the square of the velocity in the path multiplied by the curvature.

Let C, C', Fig. 70, be the centres of curvature of C_b, C_s for the point I, r, r' the radii of curvature. Then we have

$$\dot{s}dt = r.ICI_b = r'.IC'I_s.$$

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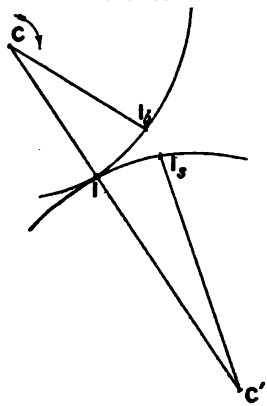
Hence
$$ICI_b + ICI_s = \dot{s}\,dt\left(\frac{1}{r} + \frac{1}{r'}\right),$$

and this is the angle between the tangents at I_b, I_s.

When I has moved to I_s, CI_b and I_sC' are in line, and the body has turned through the angle $ICI_b + IC'I_s$. This is accomplished in time dt. Hence

FIG. 69.



$$\omega = \dot{s}\left(\frac{1}{r} + \frac{1}{r'}\right).$$

For the acceleration of the body-point at I we have therefore

$$\omega\dot{s} = \omega^2 \frac{rr'}{r + r'}. \tag{82}$$

This is on the supposition that the curvatures are oppositely directed as in Fig. 69. If they be similarly directed, and r, r' denote as before the positive numerical values of the radii, the results just obtained become

$$\omega = \dot{s}\left(\frac{1}{r} - \frac{1}{r'}\right), \quad \omega\dot{s} = \omega^2 \frac{rr'}{r' - r}. \tag{83}$$

For the point P, Fig. 68′, the acceleration along the normal PI to the path of P is, by § 99, $\omega^2 p - \omega\dot{s}\cos\theta$, where θ is the angle between the normal IN to C_s at I and IP. Hence, since the speed of P is ωp at right angles to IP, the curvature of the path of P is $(\omega^2 p - \omega\dot{s}\cos\theta)/\omega^2 p^2$, that is by (82) and (83)

$$\textit{curvature of path of } P = \frac{1}{p} - \frac{1}{p^2}\frac{rr'}{r' \pm r}\cos\theta,$$

according as the curvatures of the centrodes are oppositely or similarly directed.

It will be observed that this acceleration vanishes for all points for which $p = rr'\cos\theta/(r' \pm r)$, which is a circle of diameter $rr'/(r' \pm r)$. This diameter is directed along the normal IN to the centrode. Since IP may be drawn in any direction, θ may have any value, and so every point of this circle is at any one instant passing through a point of zero curvature of its path, or, as it is called, a point of inflexion. The diameter of the circle is, however, constantly varying, and therefore the points of inflexion pass from circle to circle of points as the rolling proceeds.

It is clear that the acceleration tangential to the path of P is $\dot{\omega}p - \omega\dot{s}\sin\theta$, which vanishes for all points for which $p = \omega\dot{s}\sin\theta/\dot{\omega}$, which is a circle of diameter $\omega\dot{s}/\dot{\omega}$. This diameter is directed tangentially to the centrode at I, and hence the two circles intersect at right angles at I. At I, however, the length p is zero, as well as the acceleration normal to the path of P. The tangential accelera-

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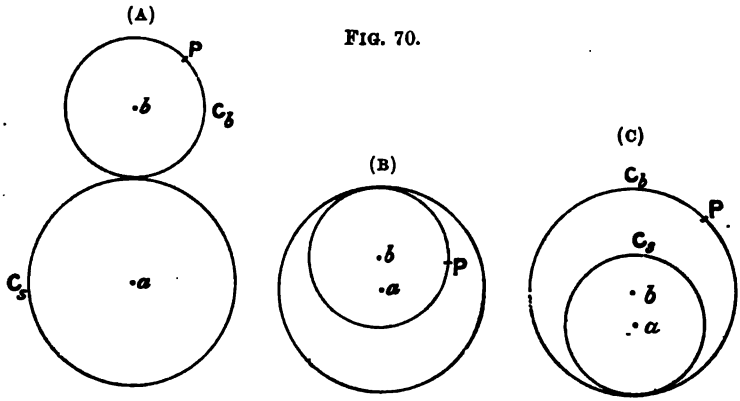
tion is, however, there $-\omega\dot{s}$. At every other point of the second circle the tangential acceleration is zero, and at the other point of intersection of the two circles the normal acceleration is zero as well.

101. **Rolling and Slipping.**—The motion of the body just discussed is one of what is called *pure rolling* of the curve C_b upon the curve C_s. At each instant the normals to the two curves at the common point I are in one line, the body is turning round the point of contact, and the speed at which the point of contact is changing its position in C_b is always equal to that at which it is moving along C_s, so that the length s_b of the part of C_b which has been brought

FIG. 70.



into contact with C_s, in any time is equal to the distance s_c which the contact-point has travelled along C_s. If s_b were greater or smaller than s_c, the body-centrode would be said to have *slipped* along the space-centrode, in the first case against the direction of motion along C_s, in the second case in the opposite direction.

A familiar example is a wheel rolling along a rail. When the rolling is pure there is no relative motion of the rail and the point of the wheel in contact with it. The body-centrode is the line round the rim of the wheel in which it touches the rail, the space-centrode is the line of contacts along the rail. If there be no slipping, the distance through which the centre of the wheel has advanced is just equal to the number of turns the wheel has made multiplied by the circumference of the wheel, that is by the length of C_b.

When the brakes are applied to some of the wheels of a railway train, these have their angular velocity suddenly reduced, and they slip along the rails in the direction of motion. On the other hand when a locomotive is starting a heavy train forward, the driving wheels slip on the rails in the backward direction, and sand is frequently thrown on the rails in front of the engine to make the wheels "grip," that is, to bring about more nearly pure rolling.

102. **Roulettes.**—The curve traced out in the fixed plane by any point in the moving plane as the curve C_b rolls on C_s is called a

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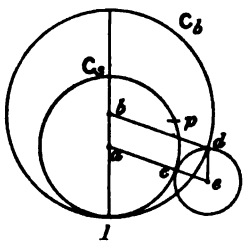
roulette. When C_b is a circle and C_s a straight line, the curve traced out by a point in C_b is called a *cycloid*, and that traced out by any other point in the moving plane is called a *trochoid.* If C_s be also a circle as in Fig. 70, a point in C_b is said to generate a cycloidal curve and any other point a trochoidal curve.

If the circle C_b touches the circle C_s so that each is outside the other, as in Fig. 70 (A), the curve traced out by a point P on C_b is called an epicycloid, and that by any other point Q an epitrochoid.

The curves traced by P and Q are called hypocycloids and hypotrochoids, if C_b be surrounded by C_s as in Fig. 70 (B), and epicycloids and peritrochoids if C_s be surrounded by C_b as in Fig. 70 (C). The name epicycloid is used in the last case instead of pericycloid since, as will be shown, a pericycloid is also an epicycloid as defined in the first case.

FIG. 71.



103. **Double Generation of Pericycloids.**—In Fig. 71 let a, b be the centres of the circles C_s C_b. Draw parallel radii ac, bd, and complete the parallelogram $abde$. The point e is the centre of a circle of radius equal to the difference of the radii of C_s and C_b, and intersecting C_b in d. Since each of the angles Ibd, dec is equal to Iac, and $bd = ac + ce$, the arcs Ic, cd of the circles centred at a and e are together equal to the arc Id of C_s. Let arc Ip = arc Id, then $dc = pc$. Therefore, as C_b rolls, d, regarded as a point fixed on C_b, comes to p, and as the circle, centre e, rolls in the same direction, d regarded as a point fixed on that circle also comes to p. Let C_b be rolled through any angle, so that the point upon it which was at d comes to d', while I comes to I', b to b', and the circle of centre e remains fixed. Now let the circle of centre e be rolled through the same distance along C_s in the same direction, so that c comes to a new position c', and the point formerly at d to a new position d''. It is clear that the two radii ac', $b'd'$ are parallel to one another, and likewise aI', ed''. The two points d', d'' must therefore be coincident. Thus as the two circles are rolled forward with equal angular velocities, their point of intersection remains the same point of both, and the path described by the point is given by the motion of either circle. Therefore the path traced out by any point of C_b, while that circle rolls on C_s, is also traced out by a certain point of the circle of centre e, while the latter rolls round C_s in the same direction. A pericycloid is thus also an epicycloid.

In the case in which the inner circle rolls on the outer a similar construction holds. There the circle of centre e rolls internally to C_s, and any point traces out a hypocycloid which can also be traced by the rolling of C_b.

104. **Involute and Evolute.**—Let the moving centrode be a straight line. It is then a tangent at I to the space-centrode C_s. I is the centre of curvature of the element of the path which is at the

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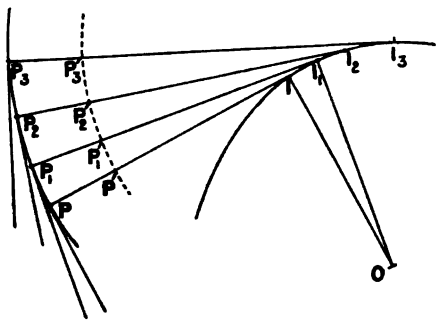
instant being described by any point P of the straight line. Every such path is called *an involute* of the curve touched by the straight line. A given pair of involutes are everywhere at the same distance apart along the common normal, so that the system of involutes is made up of an infinite number of parallel curves. The curve C_e touched by the moving straight line is called *the evolute* of this system of curves, and is evidently the locus of the centres of curvature of every one of the involutes.

The involutes may be supposed described by the points of a thread kept taut while being unwound from the evolute. It is clear

FIG. 72.



from this mode of description that the length IP of the normal to the involute equals the length of the arc IA of the evolute from which the string has been unwound.

Fig. 72 shows two involutes, $PP_1P_2\ldots$, $P'P_1'P_2'\ldots$, of a system produced by unwinding a thread from the curve C_e, which is their evolute. The points $I, I_1, I_2, \ldots$ are the centres of curvature corresponding to the points $P, P_1, P_2, \ldots$ of each of the involutes. The tangent at P to any involute is parallel to the normal at I (its centre of curvature) to the evolute.

The reader will observe that if the positions of the generating straight line be taken for points $P, P_1, P_2, \ldots$, which are very close together, each position contains the centres of curvature for two successive elements. This fact is taken advantage of in the method of graphically describing curves explained in the following section.

The fixed centrode is touched by the common normal IP to the system of involutes in every one of its positions. Hence the evolute is called the *envelope* of these normals.

105. **Drawing of Curves from their Curvature.**—When the velocity and acceleration are known for different points, the path can be laid down on paper in the following way. From any point O, Fig. 73, draw a line OP in length equal to a radius of curvature of

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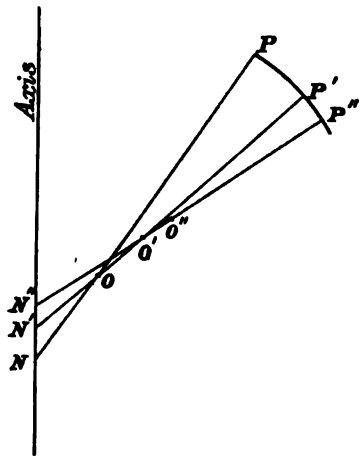
the path, from O as centre with OP as radius describe a short circular arc PP', and join O to P'. Then PP' is a short arc of length ds of the path. Next calculate the radius of curvature for an element beginning at the distance ds from P, that is at P'. Let this distance laid off from P' be $P'O'$;

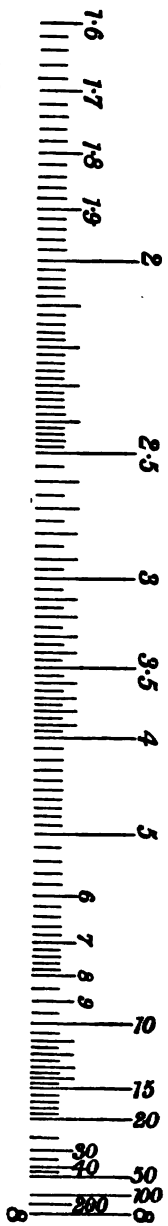
FIG. 73.



then O' is the centre of curvature for the next element $P'P''$, which is described as a short circular arc of centre O'. O'' in $P'O'$ is then found in like manner as centre for a succeeding element $P''P'''$, and so on. In this way Lord Kelvin and Professor Perry have drawn diagrams of capillary surfaces of revolution,* of which some account will be given below in the chapter on *Capillarity*. The lunar orbit shown in Fig. 131 below was also drawn by this method.

Mr. C. V. Boys has greatly facilitated this method of describing curves by the invention of his scale of reciprocals. This is represented in Fig. 74. The distance between successive scale divisions increases with distance from the starting-point, and so finer subdivisions are introduced as the openness of the graduation allows. The numbers marked at the divisions are the reciprocals of the distances of the divisions from the starting-point of the scale, which point is therefore marked ∞.

FIG. 74.



* *Nature*, July 22 and 29, and August 19, 1886.

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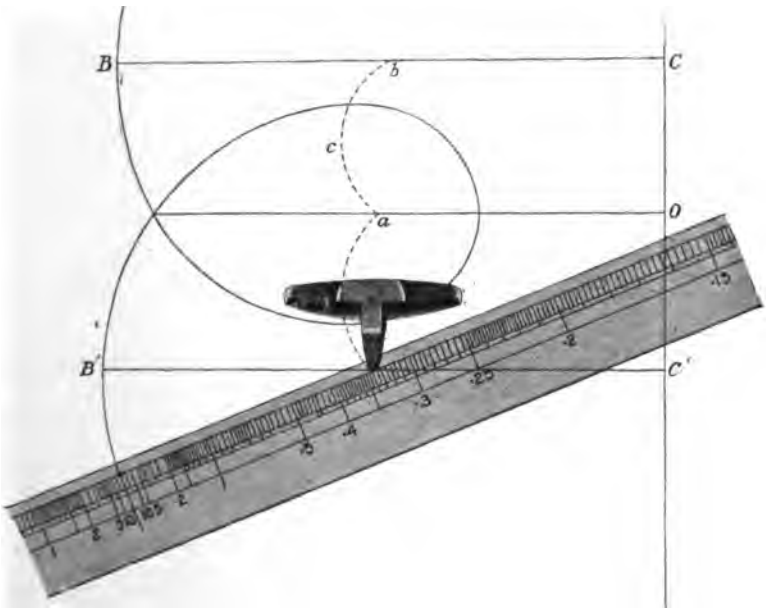
Thus when a curvature $1/r$ is known, the division-mark, found against its numerical value on the scale of reciprocals, is at a distance r from the point marked ∞.

To draw the curves, therefore, Mr Boys proceeds as follows. Using a ruler which has been graduated in this way (see Fig. 75) and carries a pen-point projecting through a small hole at the point ∞ in the line at points on which the divisions are marked

FIG. 75.



he places it in the chosen initial direction on the paper. Then he places a small tripod with needle-points for feet, so that two of the points rest in the paper, the third at that point on the line just referred to which carries the number measuring the curvature. This third point forms a centre round which the rule can turn. The rule is then turned through a small angle, and the pen describes a short circular arc on the paper. The rule is then kept fixed in the new position while the tripod is moved along it and set so that the needle-point on the rule is placed at the division corresponding to the curvature of the next element, which is then described as before, and so on. Of course if no division correspond exactly to the curvature, the proper position for the needle pivot must be found by estimation between two divisions, and this can be done very exactly. This process obviates the continual

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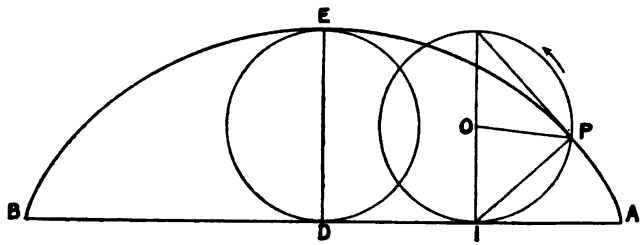
setting of compasses, first on a scale, and then on the paper, and so diminishes to a great extent errors of setting.

The curve drawn in Fig. 75 is the meridian curve of a figure of revolution, of which COC' is the axis. The needle-point is shown resting at the division the scale-number of which is ·34. This is the curvature of the element which is being described by the pen at the point ∞. The curve is called the *nodoid*, and will be discussed under *Capillarity*, where also will be found further details of Mr. Boys' process, and precautions for avoiding inaccuracy. It may be noted here, however, that the two needle-points pressed into the

FIG. 76.



paper leave marks which enable the third point, when the rule is laid aside, to be taken back along its course, and its successive positions marked on the paper. These give the evolute of the nodoid, which is shown by the dotted curve.

106. **The Cycloid.**—Take now, as an example of the preceding discussion of roulettes, the cycloid, which is of considerable dynamical interest. Let O, Fig. 76, be the centre of the generating circle in one of its positions, I the instantaneous axis, P the generating point, $APEB$ the cycloid. If the circle makes more than one turn the curve produced is a succession of cycloids, and clearly has a cusp at each point, as A and B, of meeting of two. Also each cycloid is symmetrical on the two sides of a line DE, at right angles to AB, and of length equal to the diameter, $2a$, of the rolling circle. Let ω be the angular velocity of the rolling circle, then the velocity, u, of I along AB is ωa. The velocity of P is $\omega.IP$ at right angles to IP, and therefore by § 99 the acceleration of P is $\dot{\omega}.IP$ in the direction of motion, $\omega^2.IP$ towards I, and $\omega u(=\omega^2 a)$ in the direction perpendicular to and from AB. Thus if θ be the angle PIA, the total acceleration towards I is $\omega^2.IP - \omega^2 a \sin\theta$, that is $\omega^2 a \sin\theta$, since $IP = 2a\sin\theta$. The acceleration in the direction of motion is $\dot{\omega}.IP + \omega^2 a\cos\theta$, that is $a(2\dot{\omega}\sin\theta + \omega^2\cos\theta)$.

107. **Curvature of Cycloid. Length of Arc and Area of a Cycloid.**—From the former acceleration we can find the radius of curvature of the cycloid at P, by the relation

acceleration = velocity squared × curvature.

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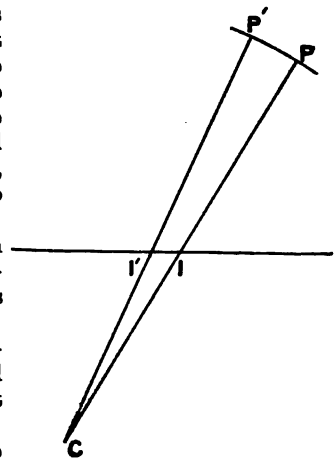
Thus we get

$$\textit{radius of curvature} = 4a\sin\theta.$$

This might be obtained from the following simple consideration, which will often enable the centre of curvature to be found in such cases. In an element of time dt P, Fig. 76′, moves forward a distance $2\omega a\sin\theta.dt$ at right angles to IP, and in the same time I moves along AB a distance ωadt, that is through a distance $\omega a\sin\theta.dt$ at right angles to IP. The normals at the extremities of the element of the curve will therefore meet at the distance $2IP$, that is $4a\sin\theta$ from P. To find C the centre of curvature, therefore, it is only necessary to produce PI to C so that $PI = IC$.

FIG. 76′.



The reader will observe that in general in roulettes the centre of curvature is not at the instantaneous axis, since that is in motion in space. It is the intersection of two consecutive normals to the curve, and these must pass through different points on C_s.

From the radius of curvature we can easily find the length of any arc AP, Fig. 75, of the curve. The rate at which the arc is being generated is $4a\omega\sin\theta$, and the addition made to it when the line CP, Fig. 75, turns through an elementary angle $d\theta$ is $4a\sin\theta d\theta$. But this we have seen in § 104 is the addition made to the quantity $-4a\cos\theta$ by increasing θ by the amount $d\theta$. Thus

$$\text{arc}\,AP = -4a\cos\theta - (-4a\cos 0) = 4a\,(1-\cos\theta).$$

Since at E, the vertex of the curve, $\theta = \pi/2$, and $\cos\theta = 0$, the semi-arc AE of the cycloid has length $4a$, and the whole length is $8a$. The arc EP has the length $4a\cos\theta$.

It is easy to see that the area of a cycloid is $3\pi a^2$, that is three times the area of the generating circle. For the area of a triangle of vertical angle $d\theta$, having its base on the curve and its sides consecutive radii of curvature, is $8a^2\sin^2\theta d\theta$. The smaller triangle, CII', Fig. 76, cut off from this by the element II' of C_s is a quarter of this area, since $CI = IP$. Hence the part of the cycloid contained in the triangle has area $6a^2\sin^2\theta d\theta$. Adding these areas for the whole cycloid we get $3\pi a^2$.

108. **Acceleration of Point moving along Cycloid**—If the circle is rolled with uniform angular velocity, $\dot{\omega} = 0$, and the acceleration along the tangent becomes $\omega^2 a\cos\theta$ from P towards E. Thus the acceleration tends to produce velocity of P towards E, and it is easy

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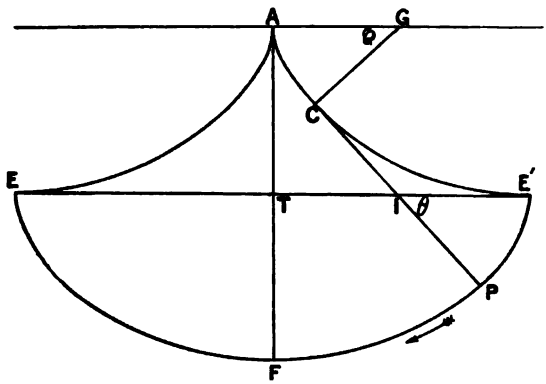
to see that this would also be the case if P were at P' between E and B. Also for the numerical values we have, taking the displacement of P from E,

$$\frac{acceleration}{displacement} = \frac{\omega^2 a \cos\theta}{4a\cos\theta} = \frac{\omega^2}{4}. \tag{84}$$

Thus the motion of the generating point in the cycloidal path, produced by rolling a circle with uniform angular velocity along a straight line, is one of simple harmonic motion along the path,

FIG. 77.



of which half the period is obviously $2\pi/\omega$. This is given also by § 52, which states that $4\pi^2/T^2 = \omega^2/4$.

Let EPE', Fig. 77, be a cycloid upon the base EE', and let C be the centre of curvature for the point P on the curve. C is (§ 104) a point on the evolute of the cycloid. Draw CG at right angles to PC, meeting in G a line parallel to EE' and at a distance $2a$, the diameter of the generating circle of the cycloid, from it. Marking the angles as shown we see that since $CI = 2a\sin\theta$, GC is $2a\sin\phi$. Thus as P moves along the cycloid C traces out an equal cycloid. When P is at the vertex of its curve C is at the cusp A. The radius of curvature of the locus of C is $4a\cos\theta$, and GI is perpendicular to EE'. The evolute of a cycloid is thus an equal cycloid placed as in Fig. 77.

109. **Cycloidal Pendulum.**—If the plane of the diagram, Fig. 77, be the vertical plane and P be a massive particle suspended by a massless cord, ACP, of length $4a$, wrapped round a cycloid from A to C, and unwrapping as P moves in the direction of the arrow, then, as will be seen from § 137, gravity produces an acceleration g in the vertically downward direction, of which the component perpendicular to the tangent is annulled by the action of the cord. The tangential

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component, in the direction of motion, is thus $g\cos\theta$, and we have seen that the displacement of P from the vertex F is $4a\cos\theta$. Thus we have

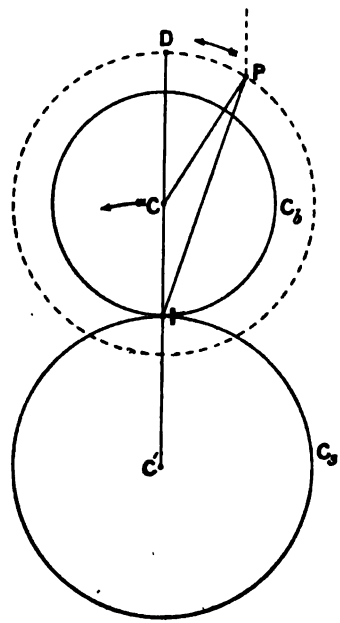
$$\frac{\textit{acceleration}}{\textit{displacement}} = \frac{g\cos\theta}{4a\cos\theta} = \frac{g}{4a}. \tag{85}$$

The period of oscillation is thus given by the equation

$$T = 2\pi\sqrt{\frac{4a}{g}}, \tag{86}$$

and the pendulum oscillates in the period of a simple pendulum of length $4a$ (the length of the semi-cycloid) vibrating in an arc of infinitesimal length. The period of this cycloidal pendulum, it is important to notice, is independent of the amplitude of oscillation, which may be anything from 0 to $4a$.

FIG. 78.



110. **Description of Epitrochoid.**—As a last example of the kinematics of roulettes consider the curve described by a point P in the plane of a circle which is rolled on another of different radius, as in Fig. 78. Let a, b be the radii of the moving and fixed circles, c the distance of P from the centre C of the latter. Then the angle turned through by the rolling circle in time dt, that is $d\theta$, is $ds(1/a + 1/b)$ where ds is the distance described along C_a, and this is the angle (ϕ, say) through which CP has been turned. The velocity of P is $\omega.IP$, and the acceleration of P is $\omega^2.IP$ towards I, $\dot{\omega}.IP$ along the direction of motion at P, and, if $\dot{s}$ is the velocity of I along the fixed circle, $\omega\dot{s}$ in the direction at right angles to the common tangent at I, that is along the dotted line at P. Thus, since $\dot{s} = \omega.ab/(a+b)$, if θ be the angle IP makes with the common tangent, the normal acceleration is $\omega^2\{IP - ab\sin\theta/(a+b)\}$. Now the angle PCD is known, being the supplement of ICP, that is of $s(1/a + 1/b)$, the angle ϕ through which CP has been turned from the position of CP coincident with CI. Also we have $IP = c\sin\phi/\cos\theta$, so that the normal acceleration is $\omega^2\{c\sin\phi/\cos\theta - ab\sin\theta/(a+b)\}$. The reader may verify that when P is on the rolling circle, and b is infinite, we obtain from this the acceleration calculated above for the cycloid.

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The radius of curvature r of the epitrochoid traced out by P is thus given by the equation

$$r = \frac{(velocity)^2}{acceleration} = \frac{2(a+b)c^2\sin^2\phi}{\{2(a+b)c\sin\phi - ab\sin 2\theta\}\cos\theta}, \qquad (87)$$

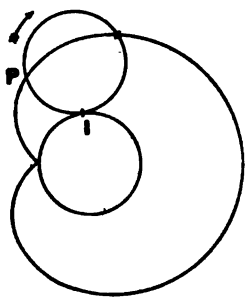
which, when b is infinite, $\phi = 2\theta$, and $c = a$, as in the cycloid, becomes $4a\sin\theta$.

An interesting case is obtained by making $c = a = b$, that is making the rolling circle equal to the fixed circle. We have then $\phi = 2\theta$,

$$r = \tfrac{8}{3}a\sin\theta, \qquad (88)$$

that is $\frac{4}{3}$ of the length IP.

FIG. 79.



The curve described by the carried point P is in this case that called the cardioid. It is shown in Fig. 79.

The length of a cardioid from the cusp round back again to the cusp is $16a$, or twice the length of a cycloid generated by the circle of radius a. Its area, that is the area outside the fixed circle in Fig. 79, is $5\pi a^2$, five times the area of the generating circle.

111. **Equation of Circular Roulette.**—It is worth noticing that the equation of any epitrochoid, or indeed any circular roulette, may be found as follows. The curve may be regarded as generated by the motion of the centre of the moving circle round the centre of the fixed, combined with a motion of the latter round its own centre with an angular velocity which bears a certain relation to that of the centre. Thus in Fig. 78 the angular velocity of C is ω, that of P. The velocity of I due to the motion of C round C' is ωb, and that is exactly neutralised by the relative turning motion of the carried circle round its own centre. Hence if ω' be the angular velocity of the latter motion, we have $\omega' - \omega = \omega b/a$, or $\omega' = \omega(a+b)/a$.

Thus for the co-ordinates of P at any time t reckoned from the instant when P was in the line of centres we have, taking the axis of x along the original line of centres, and y at right angles, and the origin at the centre of the fixed circle,

$$\left.\begin{aligned} x &= (a+b)\cos\omega t + c\cos\left(\frac{a+b}{a}\omega t\right) \\ y &= (a+b)\sin\omega t + c\cos\left(\frac{a+b}{a}\omega t\right). \end{aligned}\right\} \qquad (89)$$

By elimination of t the equation of the curve is obtained. For example let $a = b = c$ and we have

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$$\left.\begin{aligned} x &= 2a\cos\omega t + a\cos 2\omega t \\ y &= 2a\sin\omega t + a\sin 2\omega t, \end{aligned}\right\} \quad (90)$$

from which the properties of the curve can be deduced.

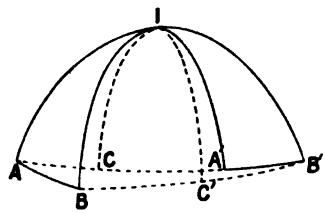
The curve has the form shown in Fig. 78, and is the *cardioid* already described. Its polar equation, the cusp being the origin, may be written

$$\rho = 2a(1 + \cos\theta),$$

as the reader may verify at once, observing that $\theta = \omega t$.

112. **General Displacement of a Rigid Body.**—We consider now and very briefly the general motion of a rigid body. First let one point of the body be fixed, then any displacement of the body may be brought about by the motion of a sphere of which the fixed point is the centre upon a fixed concentric spherical surface.

FIG. 80.



Let two points A, B (Fig. 80) on the surface of the moving sphere be displaced to coincidence with positions A', B' on the fixed surface. The displacement of the body may be effected by turning it round an axis through the fixed point. Draw arcs of great circles of the sphere joining AA', BB', and bisect these in points C, D and draw great circles of the sphere through CD at right angles to AA', BB'. These will intersect in two diametrically opposite points II'. The diameter II' is an axis round which if the body be turned A will be brought to A', and B to B'. The angle of turning is that at I between the tangents to the two great circles which there meet. The reader may exercise himself in realising the full truth of this proposition.

113. **Translation and Rotation in one Plane equivalent to a Rotation.**—Let a body be displaced so that a plane in it moves on a plane fixed in space, first by translation in a direction parallel to the fixed plane, then by rotation round an axis perpendicular to the fixed plane. The resultant of these displacements may be effected by simply turning the body through an equal angle about a parallel axis. For any two points AB in the moving plane come finally to $A'B'$, that is the line AB comes to $A'B'$. This, by § 93, can be brought about by a rotation as specified. If $2s$ be the linear displacement and 2θ the angular, the distances of the second axis from the first are plainly s along, and $s/\tan\theta$ at right angles to the line of motion.

The body may be moving with linear velocity $\dot{s}$, and at the same time turning with angular velocity $\dot{\theta}$ round an axis A at right angles to $\dot{s}$, and moving with the body. The motion is equivalent to one of pure rotation round an axis A' parallel to and at a distance $\dot{s}/\dot{\theta}$ from

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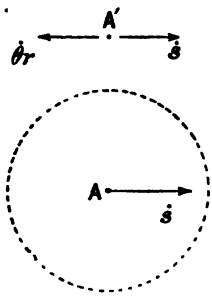
the first measured in a direction perpendicular to $\dot{s}$, as shown in Fig. 81.

114. **Screw Motion of a Rigid Body.**—The most general displacement of a rigid body in space may be brought about by a rotation through a determinate angle about a certain axis, and a displacement without rotation of the whole body parallel to that axis. Let a point A on the body be carried from its initial to its final position A' along the line joining them. Then, as just seen, the body can be turned about an axis through the point till it takes up its final position. The points in the body which lie in any plane at right angles to that axis are turned round in their own plane, and in the preceding translatory motion the successive positions of the plane were parallel to one another, and the direction of motion of each point is inclined at the same angle to the plane. Thus the displacement may be given by translating first the body in the direction perpendicular to this plane, then in a direction parallel to the plane, and finally giving it the motion of rotation just considered. But the last two motions, the motion along the plane and the rotation about an axis at right angles to it, may be combined in a rotation about a parallel axis, found as described in § 93, and therefore the whole displacement may be effected by carrying the body bodily without rotation in a certain direction, and then turning it round an axis parallel to that direction as stated above.

FIG. 81.



The rotation may take place simultaneously with the translatory motion, and so that the ratio v/ω of the translational to the angular velocity is constant throughout.

115. **Pitch of Screw. The Displacement called a Twist.**—The motion just found is that of a nut relatively to the axis of a screw on which it turns. The nut travels along the axis of the screw while rotating about it. Hence the motion may be called a *screw-motion*, and v/ω the *pitch* of the screw. In engineering, however, the pitch of a screw is $2\pi v/\omega$, the distance the nut advances on the screw per turn. The angular velocity is taken as the measure of the motion; the linear velocity is the product of the angular velocity and the pitch. If the pitch be zero the motion is a pure rotation, if the pitch be infinite the motion is pure translation. These, however, are only particular cases of the general one, and need not be treated separately.

For a given displacement of a rigid body there is only one equivalent screw displacement. For the particles of the body which initially lay in the axis of rotation remain in that line in the final position of the body. This could not possibly be the case with any other screw.

In what follows, then, we shall conceive a displacement of a body

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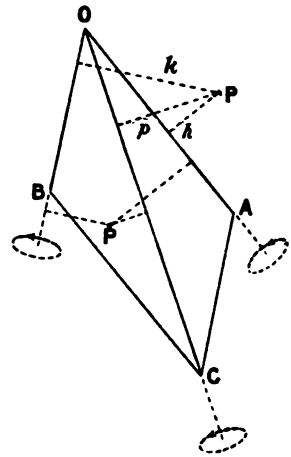
as effected by attaching the body to a nut turning round a very thin cylinder on which a screw-thread of the proper pitch is cut. Such a displacement is called a *twist*. A twist is specified when the axis, the pitch of the screw, and the angle of turning, or *amplitude*, are given.

116. **Composition of Angular Velocities.**—If a body have the point O fixed and be turning with angular velocities ω_1, ω_2 about axes OA, OB inclined at an angle θ, we can show that the motion is equivalent to a single motion of rotation of angular velocity

$$\omega = \sqrt{\omega_1^2 + \omega_2^2 + 2\omega_1\omega_2\cos\theta},$$

about an axis OC inclined to OA at an angle $\cos^{-1}\{(\omega_1 + \omega_2\cos\theta)/\omega\}$.

FIG. 82.



For let the lengths OA, OB, Fig. 82, along the axes be taken proportional to ω_1, ω_2, and construct the step $OC(=\omega)$, which is their resultant, § 17. Its magnitude has the value just stated. OC is the diagonal passing through O of the parallelogram of which adjacent sides are OA, OB. Now take *any* point P in the plane of AOB, and outside that angle. Then it is easy to prove that the sum of the areas of the triangles POA, POB is equal to the area of the triangle POC. But if h, k, p be the distances of P from the lines OA, OB, OC, we have 2 area $POA = \omega_1 h$, 2 area $POB = \omega_2 k$, and the sum of these is the velocity of P due to the two rotations. But 2 area $POC = \omega p$, and therefore ωp is also the velocity of P due to the single rotation about OP. If P be situated within the angle AOB, $\omega_1 h$, $\omega_2 k$ have opposite signs, and ωp is then equal to their algebraic sum.

This discussion shows that angular velocities may be treated as directed quantities and resolved like velocities and accelerations. To represent an angular velocity about any axis graphically take a distance along the axis equal to the angular velocity, and take the positive direction of this line towards that side of a plane, at right angles to the line, from which the rotation in the plane seems to an observer to be in the direction oppposite to that in which the hands of a watch would be seen by him to turn, if its face were at the same time seen by him in the plane. This direction we shall call the left-handed or the counter-clock direction of turning. It is that which the arrows in Fig. 82 are intended to represent.

Thus we see that points in the plane of the axes are rotating round OC, and therefore so must be all the points of the rigid body.

117. **Moments, Composition of Moments in one Plane.**—The products $\omega_1 h$, $\omega_2 k$, ωp may be called moments about the point O of the

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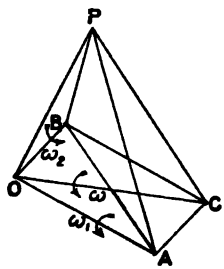
angular velocities ω_1, ω_2, ω which the body has round OA, OB, OC respectively, and we have the theorem, which will be of great service later, that the sum of the moments of any two directed quantities about any point in their plane is equal to the moment of their resultant about the same point. The sum of the moments is zero, if the point be on the resultant, and is not zero unless this is the case. This fact is of great service in finding the direction of the resultant in actual cases.

By continued application of this theorem it can be shown that the sum of the moments of any number of quantities directed along coplanar lines which meet in a point, taken round any point in their plane, is equal to the moment of their resultant about the same point.

FIG. 83.



118. **General Theorem of Composition of Moments.**—This theorem is only a particular case of a more general theorem, namely that the resultant moment, about any point whatever, of quantities directed along lines which meet in a point, is equal to the moment of their resultant about the same point. For let two directed quantities be represented by the lines OA, OB, Fig 83 (in the immediate application here they are angular velocities ω_1, ω_2 of the rigid body). The moments of these quantities about any point P are numerically equal to twice the areas of the triangles POA, POB. The moment of the resultant of OA, OB about P is the moment of OC about P, and its numerical value is twice the area of the triangle POC.

But these moments are themselves directed quantities, and may be represented by straight lines passing through one point and drawn at right angles to the areas which represent the moments. The positive directions of these lines determine the aspect of each area which is to be taken as positive, and are settled as follows. Let the boundary of each area be described by a point which in its course traverses the line representing a quantity (*e.g.*, the line OA) in the positive direction, then the positive direction of the line representing the area is towards that side of the plane from which the moving point is observed to traverse the area in the so-called left-handed or counter-clock direction. Thus if the lines of the circuit $POAP$ are traversed in the directions indicated by the order of the letters, OA is traversed in the positive direction, and (O being the near point of OB) the line representing the area POA is to be drawn towards the observer. Similarly the lines representing POB, POC are respectively to be drawn towards the right from the plane of POB, and towards the side of POC on which A lies. We shall call the lines thus specified the *step-values* of the areas.

119. **Step-Values of Areas.**—An area is thus to be regarded as a directed quantity, and obeys all the laws of directed quantities. Its step-value may be resolved into two components, one at right

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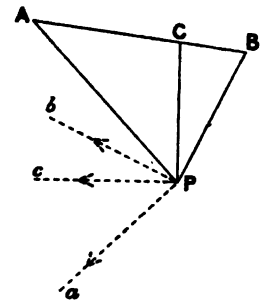
angles to any plane on which the areas are projected, and the other parallel to that plane, and these are the step-values of the projections upon planes perpendicular to them of the corresponding area. If, then, the sum of the projections of any set of areas upon any plane whatever is equal to the projection on that plane of an area S, the sum of the components of the step-values of the areas taken perpendicular to the plane of projection must be equal to the component in the same direction of the step-value of S.

FIG. 84.



Thus if the step-values be drawn in the proper directions from one point, the step-value of S must be the resultant of the steps representing the other areas, otherwise the condition just stated could not be fulfilled.

Now the projection of the parallelogram $OACB$ upon any plane is always a parallelogram, so that on whatever plane the lines of Fig. 83 are projected the result is always a diagram in that plane in which lines are drawn from the projection p to the projections o, a, c, b of the four points O, A, C, B, and these fulfil the condition that the area poc = area poa + area pob. Thus the projection of POC upon any plane whatever is equal to the sum of the projections of POA, POB upon that plane, and if the step-values of the moments of ω_1, ω_2, ω about O be each resolved at O into two components, in and perpendicular to a chosen plane, two projection diagrams will be obtained, for each of which the relation

$$\text{area } poc = \text{area } poa + \text{area } pob$$

holds. Hence, since the theorem is true of the components, the moment of OC, that is of ω, round P is the resultant of the step-values of the moments of ω_1, ω_2 about P.

The step-values obtained by projection on any plane are the moments of ω_1, ω_2, ω round an axis through P perpendicular to that plane (§ 155).

If a rigid body have two angular velocities ω_1, ω_2 about parallel axes A, B, the same theorem holds. This case requires special consideration, though the result might be inferred from the former case.

Let P, Fig. 84, be any point, α, β its distances from A, B, the intersections of the axes with a plane perpendicular to them, and passing through P. Then if $\alpha\omega_1$, $\beta\omega_2$ be taken as steps along AP, BP respectively, $i\alpha\omega_1$, $i\beta\omega_2$ are steps along Pa, Pb, at right angles to AP, BP, and represent the velocities of P due to the angular velocities ω_1, ω_2. The resultant of these is $i(\alpha\omega_1 + \beta\omega_2)$, which is therefore at right angles to the step $\alpha\omega_1 + \beta\omega_2$, the resultant of the steps along AP, BP just specified. But this, as we have seen, § 20, is the step joining P with C, the point which divides AB so that $AC/CB = \omega_2/\omega_1$, and is $(\omega_1 + \omega_2)CP$. Therefore we have $i(\alpha\omega_1 + \beta\omega_2) = i(\omega_1 + \omega_2)CP$.

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Thus the point P turns round C with angular velocity $\omega_1 + \omega_2$, and the numerical value of its linear velocity along Pc is $(\omega_1 + \omega_2)CP$.

120. **Composition of Screw-Motions and Twists.**—We now compound two screw-motions given about rectangular axes meeting in a point. Let the directions be along OX, OY of Fig. 85, and the amounts and pitches of the motions be ω_1, ω_2, p_1, p_2. The translational velocities of the two motions are $\omega_1 p_1$, $\omega_2 p_2$ respectively along OX, OY. If θ be the angle the direction of the resultant angular velocity makes with OX, we have for that resultant $\omega_1 \cos\theta + \omega_2 \sin\theta$ along OP. Also $\omega_1 \sin\theta - \omega_2 \cos\theta = 0$, so that $\tan\theta = \omega_2/\omega_1$, which determines the direction of OP. The linear velocities resolved along and at right angles to this direction give us

$$\omega_1 p_1 \cos\theta + \omega_2 p_2 \sin\theta, \quad \omega_1 p_1 \sin\theta - \omega_2 p_2 \cos\theta.$$

The former of these and the angular velocity $\omega (= \sqrt{\omega_1^2 + \omega_2^2})$ give a screw-motion about OP, of pitch $(\omega_1 p_1 \cos\theta + \omega_2 p_2 \sin\theta)/\omega$. Hence

FIG. 85.

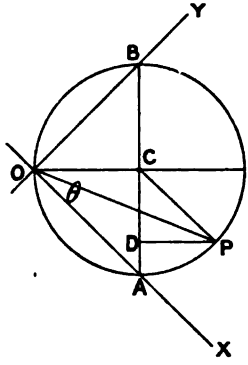
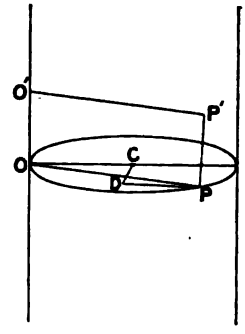


FIG. 86.



since $\omega_1 = \omega\cos\theta$, $\omega_2 = \omega\sin\theta$, we have, denoting this pitch by p,

$$p = p_1 \cos^2\theta + p_2 \sin^2\theta. \tag{91}$$

Now the angular velocity ω about OP, and the velocity

$$\omega_1 p_1 \sin\theta - \omega_2 p \cos\theta$$

perpendicular to it are (§ 113) equivalent to a rotation about an axis parallel to OP, and above the plane of the axes OX, OY, at a distance given by the relation $\omega z = \omega_1 p_1 \sin\theta - \omega_2 p_2 \cos\theta$, or, since $\omega = \omega_1/\cos\theta = \omega_2/\sin\theta$,

$$z = \tfrac{1}{2}(p_1 - p_2)\sin 2\theta. \tag{92}$$

Thus the resultant is a screw-motion of angular velocity $\sqrt{\omega_1^2 + \omega_2^2}$ about a line parallel to OP at a distance $\frac{1}{2}(p_1 - p_2)\sin 2\theta$ above the plane of OX, OY.

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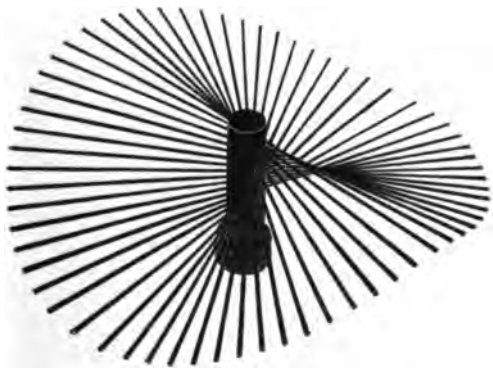
The same result holds for two screw-displacements or twists of amplitudes represented by ω_1, ω_2.

The line may be constructed as follows. Take two points A, B (Fig. 85) on OX, OY equidistant from O, such that $AB = p_1 - p_2$, and join AB. On AOB describe a circle, and join the centre C (the middle point of AB) with P. The angle ACP is 2θ, and $CP = \frac{1}{2}(p_1 - p_2)$. Thus $PD = \frac{1}{2}(p_1 - p_2)\sin 2\theta$. Now (Fig. 86) draw from P at right angles to the plane of the circle a line PP' of length equal to PD, and from P' draw a line parallel to PO. This latter will meet the axis OZ, which is perpendicular to the plane of XY, in a point O'. The line $O'P'$ is the axis of the resultant twist, that is the axis of the screw which represents the given motion.

121. **The Cylindroid.**—If we suppose θ to vary uniformly by variation of ω_1, ω_2, while p_1, p_2, on which θ does not depend, are kept constant, we get a succession of positions of the axis $O'P'$ of the

FIG. 87.



resultant twist. Thus as θ varies uniformly, OP turns round O, and $O'P'$ turns round the axis OO', and moves along OO' with a simple harmonic motion of which two periods are contained in one rotation. Thus $O'P'$ lies on a surface the equation of which is to be found as follows. Let the co-ordinates of P with reference to OX, OY be x, y, and that of P' be z. Then we have

$$y = x\tan\theta$$
$$z = \tfrac{1}{2}(p_1 - p_2)\sin 2\theta.$$

This gives at once $y(\cos^2\theta)/z = 2x/(p_1 - p_2)$. But $\cos^2\theta = x^2/(x^2 + y^2)$, so that we find

$$2z(x^2 + y^2) = (p_1 - p_2)xy \tag{93}$$

This surface is called the *cylindroid*. Fig. 87 shows the successive positions of the generator $O'P'$, and gives an idea of the form

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of the surface. The cut (which is taken from Sir Robert Ball's *Theory of Screws*) represents a model; the axis from which the generators spring ought to be made infinitely thin, and the generators prolonged to infinity. The axis OO' (Fig. 86) is called the *directrix* of the surface.

122. **The Pitch-Conic.**—The pitch of any screw on the cylindroid is given by equation (91). If we construct the curve (a conic section) the equation of which is

$$p_1x^2 + p_2y^2 = H, \tag{94}$$

where H is any constant quantity and $x = r\cos\theta$, $y = r\sin\theta$, we see that the pitch p of the screw for a generator making an angle θ with OX is H/r^2. Thus the pitch is inversely proportional to the square of the radius vector, in the given direction, of the conic represented by (92).

If the pitches are of the same sign (94) is an ellipse, if of opposite signs a hyperbola. The curve may be called the *pitch-conic*. If it is a hyperbola there are two directions, the asymptotes of the hyperbola, for which the radius vector is infinite. These are the directions for which $\tan^2\theta = -p_1/p_2$. For these the pitch given by the conic is zero, and the corresponding motion is one of pure rotation.

123. **Any Two Screws lie on a Cylindroid. Equilibrant of given Twists about any Two Screws.**—It can now be proved that a cylindroid can be described so as to contain any two screws, that is to say a cylindroid two of whose generators represent the axes of the screws, and the distribution of pitch on which is such that these generators have also the pitches of the given screws.

If $\theta, \theta', p, p', z, z'$ be the quantities specifying the two screws, the differences $\theta - \theta'$, $p - p'$, $z - z'$ (A, ϖ, h, say), and p, p' are known. In terms of these the reader may find the values of the pitches p_1, p_2 for a cylindroid on which the two given screws lie, and the values of θ and z. It is easy to verify that

$$p_1 - p_2 = \sqrt{\varpi^2 + h^2}/\sin A, \quad p_1 + p_2 = p + p' - h\cot A,$$
$$\theta = \tfrac{1}{2}\{A - \tan^{-1}(\varpi/h)\}, \quad z = \tfrac{1}{2}(h - \varpi\cot A).$$

Let a body receive twists of amplitudes α, β, γ about three screws on a cylindroid, and let the axes make angles θ, ϕ, ψ with OX. The angular displacements of the body about the axes OX, OY are $\alpha\cos\theta + \beta\cos\phi + \gamma\cos\psi$, $\alpha\sin\theta + \beta\sin\phi + \gamma\sin\psi$; the linear displacements along the axes are these expressions multiplied respectively by the pitches p_1, p_2 of the twists about the axes. Each of the four displacements will vanish identically if the relations

$$\frac{\alpha}{\sin(\phi - \psi)} = \frac{\beta}{\sin(\psi - \theta)} = \frac{\gamma}{\sin(\theta - \phi)} \tag{95}$$

hold, since this converts each of the angular displacements into

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a constant multiplied by a determinant of which two rows are identical. Hence if these relations are fulfilled the resultant of the three twists is zero, and the body has not been displaced.

There are many other very interesting results in this theory, but there is no space to deal with it here. It will be referred to again in connection with the dynamics of a rigid body, but the reader should consult the essay* by Sir Robert Ball, to whom the development of this mode of treating rigid dynamics is mainly due.

The subject of the degrees of freedom of a rigid body and their limitation by constraints belongs properly to kinematics, and can be very elegantly treated by the theory of screws. It will be more convenient, however, to defer it for treatment in connection with the theory of equilibrium of a rigid system, and we shall return to it under *Statics*.

124. **Dimensions of Velocity and Acceleration.**—Since the numerical measure of velocity is the number of units of displacement divided by the number of units of time in which it is effected, we may, as indicated in § 7, write for any actual velocity the equation

$$velocity = v[LT^{-1}],$$

where v is the numerical value of the velocity in terms of a unit of length L and a unit of time T. If the units be changed from L, T to L', T', where $L = \lambda L'$, $T = \tau T'$, we have for the same velocity the equation

$$velocity = v\lambda\tau^{-1}[L'T'^{-1}] = v'[L'T'^{-1}].$$

The new numerical value is $v' = v\lambda\tau^{-1}$, so that the change ratio is $\lambda\tau^{-1}$.

For example, let L be 1 mile, T be 1 hour, and it be required to find the change-ratio v'/v when the units are changed to 1 foot and 1 second. This ratio is 5280/3600 or 22/15. Thus the numerical value v' of a velocity in terms of the latter units is 22/15 of the numerical value v in terms of the former units.

In the same way the dimensional formula of acceleration can be found. The numerical value of an acceleration is the number of units of velocity generated in a given interval of time divided by the number of units of time in the interval. The unit of acceleration is an acceleration in which the unit of velocity is generated in the unit of time, that is it is expressed by the formula $[LT^{-2}]$ where L is, as before, the unit of length and T the unit of time. Hence for any given acceleration of which the numerical value is a we may write the equation

$$acceleration = a[LT^{-2}].$$

Thus for the change of units indicated above from L, T to L', T'

* *Theory of Screws.* Dublin: Hodges, Foster and Co. Fig. 86 is taken from this work.

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$$\textit{acceleration} = \alpha\lambda\tau^{-2}[L'T'^{-2}].$$

If L, L' be a mile and a foot, and T, T' an hour and a second, we have

$$\textit{acceleration} = \alpha\frac{5280}{3600^2}[L'T'^{-2}] = \alpha'[L'T'^{-2}].$$

The change-ratio is thus 11/27000, so that $\alpha' = \alpha.11/27000$.

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CHAPTER III.

DYNAMICS.

125. **Actions between Bodies. Action and Reaction.**—We now come to the consideration of dynamics proper. It is matter of common observation that bodies act on one another so that the relative motions of the different parts of a system are continually undergoing change. A stone thrown into the air is attracted towards the earth, and describes, relatively to the earth, a curve, which we have already shortly considered. A stone let fall from a height travels vertically downwards with uniformly increasing speed; and we are led to the conclusion that this acceleration is due to the presence of the earth. As we shall see presently, the action is mutual. The earth, we say, attracts the stone, the stone attracts the earth.

Between the two bodies there is said to exist a *stress*, that is, a mutual action, which has two aspects, the equal and opposite actions on the two bodies. Thus when a stretched spiral spring has two masses attached to its ends there exists a stress in the spring, and the bodies are pulled by the spring in opposite directions, and if the acceleration of the matter of the spring itself is neglected or is zero, the pulls are equal in a sense which we shall explain later.

It is very important to keep this double aspect of the action between every pair of bodies continually in view. Without it the theory of energy, on which we shall enter presently, becomes in view of the relativity of motion unintelligible. When we speak below of a stress as applied to a body we consider one only of its aspects.

There is a common idea that there is some action exerted on a moving body without which the body could not go on moving. Thus Kepler imagined that the earth was held between rays projecting from the sun and rotating so as to carry the earth round its orbit. Again, the body is frequently regarded as having the property of putting forth a certain "force" when it meets an obstacle, as, for example, when a falling body strikes the earth.

These ideas are generally more or less vaguely expressed; but so far as they have any definite meaning they are erroneous. A change of motion in any portion of matter is always brought about by the action of some other portion of matter.

When a shell explodes, its different parts are driven in different directions by the action upon them of the gaseous products suddenly

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generated by the rapid combustion of the charge; the well-polished curling-stone, made to slide along well-swept smooth ice, moves a much longer way without coming to rest than when impeded by obstacles or by the friction of a rough surface; the skater knows well that when he has got up speed he cannot change his direction of motion without striking to one side or the other, and that the distance he can run without striking depends on the smoothness of the surface, and the amount of any resistance he experiences.

126. **Newton's First Law of Motion.**—The real truth of the matter was first expressed by Newton in his *Philosophiae Naturalis Principia Mathematica*, in his First Law of Motion.

Corpus omne perseverare in statu suo quiescendi vel movendi uniformiter in directum, nisi quatenus illud a viribus impressis cogitur statum suum mutare.

Every body continues in its state of rest or of uniform motion in a straight line, except in so far as it is compelled to change that state by forces impressed upon it.

He adds the following brief explanation of the meaning of the law:

Projectiles continue in their motions except in so far as they are retarded by the resisting action of the air and are impelled downwards by the action of gravity. The different parts of a wheel are in consequence of their connections continually made to deviate from motions in right lines, and the wheel does not cease to rotate, except in so far as it is retarded by the air. Planets and comets, however, preserve for a very long time their translational and orbital motions since their bodies meet with less resistance in the spaces in which they move.

It is affirmed in this law that a body's motion is changed only by the action of other bodies. Also it follows from the statement that if a body can be found which there is sufficient reason for believing is not acted on by other bodies, that body describes equal distances in equal times, and may be made to measure time. Also the measurements of time given by different such bodies will be consistent. This is the dynamical method of reckoning time referred to above (§ 8) by which the uniformity of the earth's rotatory motion is to be tested, and by which an attempt has already been made to measure the rate of change of the period of rotation. We shall endeavour to give some account of this comparison later.

127. **Relativity of Motion.**—It is necessary to notice here that motion is relative, and that to observe and measure the motion of a system it is necessary to set up a set of axes of reference. These axes may be set up on any body whatever, and be at rest relatively to that body. Then the motions referred to it take no account of changes (relatively to some other reference system) which are common to the axes and the body the motion of which is under consideration. But, though we might consider accelerations relative to these axes, and obtain a corresponding system of impressed forces, the realisation of the method of reckoning time would require reference to axes fixed on a body unacted on by other bodies, and so we

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are brought face to face with the difficulty of finding such a body. Without it we cannot say what we mean by a straight line or by uniform motion.

128. **Comparison of Abstract Dynamics with Experience. Definition of a Particle.**—The way out of these difficulties lies in regarding dynamics as founded on ideal conceptions just as geometry is, and depending for its application to practical problems on certain assumptions to be justified by the results of experience. In fact, abstract dynamics is a purely ideal science, geometric in a somewhat extended sense caused by the introduction of certain notions not ordinarily employed in purely geometrical processes. So long as we confine ourselves to the ideal as we do in geometry there are about it only difficulties of the same kind as we have in geometrical conceptions, and these we do not here discuss. The special fundamental difficulties arise when the science is applied to the interpretation of nature, and any solutions obtained by its aid must depend on hypotheses to be justified by the comparison of results with observation.

It is not, however, desirable to discuss dynamical principles, and build up the science, as a body of pure abstractions: on the contrary as soon as the fundamental conceptions are laid down, we may and ought to have recourse to experience and observation at every step of our progress. Abstract dynamics should go hand in hand with practical dynamics, that is, should have continual reference to and be illustrated as far as possible by examples from physics and engineering. Without this it is made a mere peg on which to hang examples of pure geometry and analysis, instead of the proper theoretical basis for the great experimental, observational, and mathematical science which we may call physical dynamics.

We shall often in what follows employ the term *particle*. A particle may be ideally regarded as a mass concentrated at a geometrical point, though of course we do not in practice have to deal with a single particle fulfilling this definition. A particle may in practice be regarded as a body, every dimension of which is very small in comparison with the distance of the body from any other.

129. **Measurement of Time.**—First, then, we have to take some standard for the measurement of time. In this we are guided by the idea derived from the law of motion stated above, that any body in relative motion, which there is observational evidence for concluding is not changed by the action of other bodies, may be taken as timekeeper. In taking the earth's rotation as our standard for the measurement of time, we have recourse really to a joint result of this idea and the equality of action and reaction already referred to, as we shall show in what follows. Or we may adopt some other time-measurer either suggested by this idea or chosen arbitrarily, and its suitability for physical use will be a matter merely of *a posteriori* verification.

Having agreed upon or chosen some way of specifying equal intervals of time, the next step is to set up a system of axes of reference. The velocities and accelerations of different particles

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(§ 125) relatively to these axes are considered. The particles have any accelerations relatively to them which may be assigned or which are deducible from data given, and so from the configuration at any given epoch that at any other, that is, to speak shortly, the motion can be found. If the particles do not change their configuration relatively to one another, limitations are imposed on the motion, the particles constitute a *rigid* body. Thus we have the science of pure kinematics with which the preceding chapters have been occupied.

130. **Passage from one Reference System to another.**—Now we may suppose our reference system, which we may call A, to be in motion relatively to another reference system B, and the motion of the particles referred to that other system will be compounded for any instant of the motion which the particles would have with respect to B, if they were rigidly connected with A, in the positions they have at that instant, and of the motions which the particles at the same instant have with respect to A. There is no difficulty, if the motion of A with respect to B is specified, in finding the former part of the motion of each body. It will vary, it is to be remembered, with the changing positions of the particles in consequence of their motions with respect to A. Examples of this process are afforded by the uniplanar motion of a rigid body discussed above, *e.g.*, the motion of the tracing point of a cycloid in the rolling circle and in the sheet on which the curve is traced.

We can push the reference further back, and so pass from reference system to reference system whenever it is necessary to do so. There is no reason to believe that by any such process we should ever reach axes which can be regarded as absolutely fixed, but it is the process by which corrections suggested by experience are to be introduced, as will be seen below.

It is found, then, by observation that we can stop at some reference system fixed on actual bodies, it may be the first, A, which is naturally suggested to us by the circumstances of the case. To a certain extent we can consider the effect of referring one chosen reference system to other reference systems naturally suggested, and be sure that the additional motions necessary for the parts of our system are imperceptible or negligible.

Thus we refer the motion of a projectile to three axes fixed in the earth, say, one vertically upwards, one drawn north, the other west, and consider the motion. We find that the results of considering this set of axes as fixed coincide only approximately with experience, and we have to correct them on account of the earth's rotation relatively to the fixed stars. It may be that there are other corrections which more nearly perfectly exact observations might disclose, but there is no evidence for their existence.

131. **Inertia or Mass.**—So far the ideas have been purely kinematical; we must now introduce the notion of *inertia* or *mass*. We find that with reference to our naturally chosen system of axes of reference, different bodies have, when placed in what we are justified by experience in regarding as *the same circumstances*,

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different accelerations, and hence we get the idea of the inertias of bodies. In estimating similarity of circumstances we assume the constancy of the physical properties of bodies, such as the constancy of the matter of a body, of the elastic properties of a spring, and the like. Thus if we take a given spiral spring and apply it with the same stretch in it to the same body, we find the same momentum given to the body in the same time. Since we find the results consistent with those obtained with different bodies and springs, the possibility of *pari passu* variations of physical properties producing a balance of results need not be considered.

132. **Comparison of Inertias of Bodies.**—Thus applying similar circumstances (which we here merely typify by a spring with a given stretch) to different particles, we find the accelerations produced to be different, and so are led to a comparison of the inertias of the bodies. For example, suppose a spring with a given stretch in it to be applied for a second to each of a number of particles, and let the accelerations produced be a_1, a_2, a_3, &c. Then if we take quantities inversely proportional to a_1, a_2, a_3, ..., say μ/a_1, μ/a_2, ..., μ_1/a_2, ..., and multiply each of these by the proper acceleration, we obtain, of course, the same product, μ, in each case. This we take as a measure of the stress in the spring, which on the hypothesis stated above is the same at each application, at the point at least at which the spring acts on the body, and which we regard as the action producing the change of motion. If one of the particles have the chosen unit of inertia, we may take for the *numerical value of* μ that of the acceleration a of this body. Then another particle in which the acceleration produced in the same circumstances is a_1 will give for the stress the numerical value $a/a_1.a_1$. The quantity a/a_1 measures the inertia of this body, and we denote it by m_1. For other particles the accelerations of which are a_2, a_3, ... the inertia-values are $m_2 = a/a_2$, $m_3 = a/a_3$, and so on.

We thus obtain a comparison of the inertias of different particles by taking them to be inversely proportional to the accelerations produced in the particles when placed individually under the action of the same stress. It is found that precisely the same result is obtained if the comparison is effected by applying any other constant stress to each of the system of particles.

It is not necessary to limit the statement made here to particles. Bodies of finite size will serve equally well if we suppose that in every case considered every particle of the body has produced in it the same acceleration. The action of the spring which is applied where the spring is attached to the body is transmitted through the connections of the particles so that every one is accelerated to the same extent. The conditions of this transmission of action from particle to particle of a body are given by the Third Law of Motion which will be considered presently. That the inertia of a body is not affected by the distribution of its matter may, however, be regarded a result of experiment, as we shall see later.

133. **Comparison of Stresses.**—On the other hand, when we compare the motion-producing effects of different stresses, § 125, (typified

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by the action of different stretches in the same spring or in different springs) we may use as follows any system of bodies for which the inertias have been compared as described. Let us suppose that by these stresses accelerations $a'_1, a'_2, \ldots$ are produced in the particles the accelerations of which were formerly $a_1, a_2, \ldots$; then we say that the stresses applied to the bodies have the measures $m_1a'_1$, $m_2a'_2, \ldots$. Thus by means of the comparison of inertias obtained by applying the same stress to each of a number of different particles, we obtain a comparison of stresses, according to which the stresses applied are proportional to the accelerations they produce in equal masses. For if the inertias $m_1, m_2, \ldots$ were all found equal by the previous comparison the stresses would be proportional to a'_1, $a'_2, \ldots$. The products m_1a_1, $m_2a_2, \ldots$ are also called *mass-accelerations*.

134. **Unit of Weight.**—The quantities of matter in bodies are compared by the method of weighing with a balance (see chapter on *Measures and Instruments* below), and bodies which equilibrate one another in the scales of a just balance *in vacuo* are said to have the same *weight*. The word weight is used in two senses, in the sense of quantity of matter in a body, and sometimes, though perhaps more rarely in popular language, as the downward force of gravity on a body in certain specified circumstances. It seems impossible to discard the former use of the term even in scientific speech, and therefore we shall use the word generally in this sense, and in the latter sense speak of the *gravity* of a body.

The *imperial standard pound avoirdupois* is the legal standard of weight in this country. It is a piece of platinum marked "P.S. 1844, 1 lb.," preserved in the Exchequer Office, and is constituted the "legal and genuine standard of weight" by the Act of Parliament referred to in § 3 above; and it is provided in the same Act that if the standard is lost or destroyed it is to be replaced by its certain specified copies kept in various repositories.

The comparison of weights will be discussed in the chapter referred to, but any multiple of the pound is got by forming or obtaining the requisite number of pieces of matter which all equilibrate the standard pound, or an accurate copy of it. These are found all to equilibrate one another when placed in the scales; and the required multiple is a piece of matter which placed in one scale balances the whole number of pounds placed in the other. A submultiple is obtained in a similar way. The required number of pieces of matter all of the same weight is obtained, and if they together equilibrate a pound, each is the submultiple required.

For scientific purposes the unit generally adopted is the *gramme*. It is defined as 1/1000 of the standard *kilogramme* which is preserved at Paris. One pound is 453·6 grammes nearly.

135. **Unit of Inertia and Unit of Mass. Density.**—We may take the inertia of a body as a measure of the quantity of matter in the body or, as it is called, the body's mass, and we shall see (§ 14) that the comparison of masses thus obtained must agree with that carried out by the method of weighing. The body of unit inertia

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may or may not, however, have the unit of weight; this depends on the system of units adopted. According to the system here chosen we may either define the unit stress as that which produces unit acceleration in the body of unit inertia, taking as the unit of inertia that of the unit of weight arbitrarily chosen, or we may previously define the unit of stress and then choose as the unit of inertia that in which the unit of stress produces unit acceleration. Thus if we take a body which has the same inertia as a pound as having the unit of inertia, the unit of stress will be that which gives it an acceleration of say one foot per second per second. (This unit is called a *poundal.*) If we take as unit stress that exerted on a pound of matter by gravity, the inertia of the body to which this stress would give an acceleration of one foot per second per second would be g pounds, if g denote the acceleration in foot-second units (about 32·198 in England) produced by gravity in a falling body. Thus, while we compare the masses of bodies by their inertias, the unit of weight and the unit of inertia may be quite distinct; for example, the former may be 1 pound and the latter 32·198 pounds, or the former 1 gramme and the latter 981·4 grammes. Very frequently, however, the two units are taken so as to coincide; *e.g.*, as both 1 pound or both 1 gramme.

We can now define the *density* at any point of a body as the weight there per unit of volume. If dm be the weight of a small volume $d\varpi$ including the point, the density, ρ say, is the limiting value of the ratio $dm/d\varpi$ when $d\varpi$ is taken infinitely small.

136. **Impressed Force. Newton's Second Law of Motion.**—We have now seen that change of motion of a body is a result of stress acting between that body and others, or arising from the existence of the body in the presence of others. To stress, thus regarded as producing change of motion of a body, Newton gave the name *impressed force.* We only know force by its effects in producing change of motion, and by the system just discussed it is measured by the product of the measures of the inertia and the acceleration. This system is really laid down in Newton's Second Law of Motion.

Mutationem motus proportionalem esse vi motrici impressae, et fieri secundum lineam rectam qua vis illa imprimitur.

Change of motion is proportional to the moving force impressed, and takes place in the direction in which that force acts.

Newton adds the following explanation, which must of course be taken along with the law:

If a force generate any motion, a double force will generate a double motion, a triple a triple motion, whether they are applied simultaneously and at once, or gradually and successively. And this motion, if the body was already moving, is either added to the previous motion if in the same direction, or subtracted from it if directly opposed, or is compounded with the previous motion if the two are inclined at an angle.

137. **Discussion of Second Law of Motion.**—This law is sometimes regarded as merely stating an identity, inasmuch as we only know force by the change of motion it produces. This, however,

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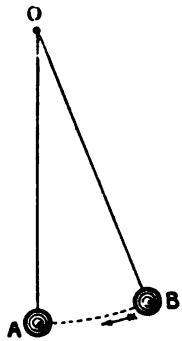
seems too hasty a conclusion. From the point of view here adopted the law contains the theoretical methods of comparing masses and forces explained above. Moreover a further very important interpretation of it is furnished by the explanatory remarks translated above. It is that when a body is placed under different stresses each is to be regarded as having its full effect in changing the motion of the body irrespective of the others. For example, when a simple pendulum consisting of a heavy particle suspended by a thread is deflected as in Fig. 88 and then left to itself, the acceleration the bob receives in consequence of gravity is to be supposed to exist co-ordinately with that produced by the stress in the cord. When the pendulum hangs vertically the stretch in the cord may be taken as producing an acceleration equal and opposite to that which gravity produces downwards, that is the action of the stress in the cord in producing acceleration is directed along the cord. When the pendulum is let go after being deflected the stress in the cord is to be taken as producing, as in the former case, an acceleration along the cord, and since the acceleration is at that instant along the tangent to the circle described from O as centre and passing through AB, the action of the cord simply annuls the component in the direction OB of the acceleration due to gravity. Thus if α be the angle AOB and m the inertia of the bob of the pendulum, we have for the components of the mass-acceleration of the bob $mg\cos\alpha$ along OB, and $mg\sin\alpha$ along the tangent towards A. The former is taken off by the stress in the cord, and thus the mass-acceleration towards O from B produced by the cord is $mg\cos\alpha$. The component $mg\sin\alpha$ sets the bob into motion along the circle.

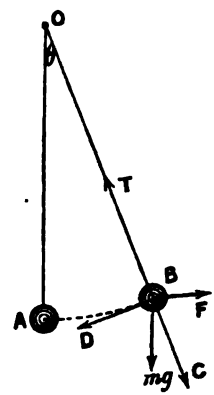
FIG. 88.



When the bob has received a velocity v it has an acceleration v^2/r towards O if $OB = r$. Thus if θ be the angle OB now makes with OA the mass-acceleration towards O applied by the cord is

$$R = mg\cos\theta + m\frac{v^2}{r}. \qquad (1)$$

FIG. 89.



138. **Equilibrium of Deflected Pendulum. Motion of Simple Pendulum.**—Again let the pendulum be held at rest in the deflected position, just referred to above, by a horizontal stress applied to the bob, as in Fig. 89. The cord, as before, applies a stress towards O. In the mode of regarding the effects of the stresses just explained the vertical stress due

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to gravity gives a mass-acceleration mg vertically downwards, and now this is neutralised by the upward mass-acceleration due to the stress in the cord. The horizontal mass-acceleration due to the latter is neutralised by the horizontal force F. Thus if we write T for the mass-acceleration due to the cord we have

$$T = mg/\cos\theta, \quad F = T\sin\theta = mg\tan\theta. \tag{2}$$

These are called the equations of equilibrium of the bob. They merely assert that the mass-accelerations for which the single symbols F, T are put on the left hand, have the values on the right, that is that the stresses acting alone would produce in the bob the mass-accelerations stated. The idea that F, T on the left denote forces in the sense of causes of motion, and that the expressions on the right are effects, is a fallacy. Of course we may speak, and have spoken above, of stresses as causing motion of bodies, but the point here insisted on is that only the effects of stresses inferred by such considerations as those given above can enter into dynamical equations.

If the force F do not exist, and the pendulum move in the plane OAB, the particle B has (§ 137), at each instant of its motion, rate of change of momentum $mg\sin\theta$ along the tangent to the circular path, where θ is the angular deflection of the thread from the vertical at the instant. If θ be always small and be taken in radian measure, $mg\sin\theta$ is approximately $mg\theta$. The acceleration is therefore $g\theta$, is always towards the equilibrium position A of the bob, and is along a path that may be taken as a horizontal line through A. The linear displacement of B from A along this line is $l\theta$, if l denote the length of the thread. Hence numerically

$$\frac{\textit{acceleration}}{\textit{displacement}} = \frac{g}{l},$$

and by § 53 the motion is simple harmonic in a period given by

$$T = 2\pi\sqrt{\frac{l}{g}}. \tag{3}$$

This equation, however, holds only for small arcs of vibration. The complete theory will be given under *Measures and Instruments*.

139. **Meaning of Applied Forces on a Single Particle. Unit of Force.**—Again in such equations as

$$m\ddot{x} = X, \quad m\ddot{y} = Y, \quad m\ddot{z} = Z, \tag{4}$$

the symbols X, Y, Z are generally said to denote applied forces acting on the particle m in the directions of x, y, z, and the quantities on the left the rates of change of momentum which they produce, and the former are frequently spoken of as if they were entities quite distinct from the latter. This last idea seems also quite fallacious. The quantities X, Y, Z are merely put temporarily for values of $m\ddot{x}$, $m\ddot{y}$, $m\ddot{z}$, which are to be found from the given circumstances of the motion, that is from the data and relations stated (see § 140).

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There is no end served, except sometimes brevity of expression, in referring to X, Y, Z as causes and $m\ddot{x}$, $m\ddot{y}$, $m\ddot{z}$ as effects.

In the centimetre-gramme-second system of units, now adopted for scientific purposes, the unit of force is the *dyne*, that is the rate of change of momentum of one gramme moving with an acceleration of one centimetre per second per second. Other units are the poundal (§ 135), the force of gravity on a body of one pound mass, and the force of gravity on a body of one gramme. The latter two are respectively 32·198 poundals and 981·4 dynes (nearly) in England.

140. **Dynamical Example — Spiral Spring stretched by a Weight.**—That the language of cause and effect may conduce to brevity may be seen by an example. Let a particle be hung by a spiral spring (the inertia of which may be neglected) which the stress exerted on the particle by gravity keeps in equilibrium when stretched by a length s. Then we know (1) that by the interpretation of the Second Law of Motion given above, the stress in the spring just gives to the particle an upward mass acceleration mg, and therefore if it acted alone would give the particle an actual upward acceleration g. Also, since experiment shows that different weights hung on the spring elongate it by amounts proportional to them, we know (2) that when the spring is stretched by an amount $s+x$ the elastic reaction would produce an upward acceleration $g(s+x)/s$. Hence at the elongation $s+x$ the resultant upward acceleration of the particle will be gx/s, and if $\ddot{x}$ represent downward acceleration, we get the equation of motion

$$m\ddot{x} = -mg\frac{x}{s}$$

which is ready for solution.

We greatly abbreviate the above statements by saying that in the first case the upward force exerted by the spring is mg, and in the second (from the experimental result) $mg(s+x)/s$. The resultant upward force on the particle is therefore mgx/s, or $-mgx/s$ in the downward direction. Then $-mgx/s$ being put for X in the equation of motion gives the equation stated above.

Thus though we may use, and do use constantly, the language of cause and effect in this connection, it ought to be remembered that when matters have been reduced to the solution of a dynamical problem, we have a purely mathematical process to carry out in which we render explicit only that which is already implicitly involved in our equations.

This in no way does away with the consideration of stresses as physical realities, it only states what is substantially involved in the application of dynamics to physical problems. The *objectivity* of force, about which much has been written, does not concern us.

To complete the solution of the problem stated above we notice that we have $-\ddot{x}/x = g/s$. The motion is therefore, § 53, simple harmonic, and of period given by the equation $4\pi^2/T^2 = g/s$, or

$$T = 2\pi\sqrt{\frac{s}{g}}. \qquad (5)$$

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The spring therefore, § 138, vibrates in the period of a pendulum of length equal to the distance through which the spring is drawn out when in equilibrium with the particle attached.

141. **Relativity of Force.**—It is to be observed that since the measure of force in Newton's second law, namely $m\ddot{x}$, is relative, the forces considered must also be relative. Now it is sometimes stated that as our idea of force (that is, it must be remembered, our notion of the existence of a stress in a body, our muscles for example) is derived from sensation, force is not relative. Our muscular sense certainly informs us that a stress in a body as distinguished from a mass-acceleration exists, but in no case as to what in any absolute sense are the forces acting on the body considered. The force we feel does not depend on our point of view; it is made known to us by a sensation aroused by the displacement of certain parts of the muscular or nervous system relatively to others, or perhaps relative motions of different parts of muscular or nervous tissue; the force we regard as acting on the body certainly does depend on the acceleration the body is regarded as having. An acceleration which we observe is a perfectly real thing, but the acceleration of a particle is altogether dependent for its value on the system with reference to which it is reckoned. If different parts of the body of a man have different accelerations, as when he is seated in a swiftly moving train and the brakes are suddenly applied, he is conscious of the existence of a stress which may indeed be severe; he is, however, unconscious of the equal acceleration which every particle of his body has towards the sun or towards the centre of the circle in which he is moving in consequence of the earth's rotation. Hence the force acting on any particle of his body depends upon the point of view from which it is regarded. If we could exhaust the totality of the stresses acting on a particle as a part of the universe of matter, we might be entitled to speak of the absolute acceleration of the particle, but that totality is perhaps indeterminate.

142. **Composition and Resolution of Forces.**—It is evident that the theorems of resolution and composition of velocity and acceleration, and of directed quantities generally, which have been given above, hold also for forces. Thus we get the theorems of the triangle or parallelogram of forces, of the polygon of forces, and conversely of the replacement of forces given in magnitude and direction by components in specified directions.

The triangle of forces states merely that if two forces which act on a particle be represented in magnitude and direction by the two sides AB, BC of a triangle (Fig. 5, § 17 above), their resultant, mass-acceleration of the particle, is represented in the same way by the third side AC.

The parallelogram of forces is really the same proposition in a slightly different form. From A (Fig. 5) a line AD is drawn in the same sense and of the same length as BC, and therefore represents equally with BC the force in that direction. The resultant AC is then the diagonal through A of the parallelogram $ABCD$.

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The polygon of forces is the proposition stated for steps in § 17. Any number of forces acting on a particle being represented by the sides OA, AB, ..., NP of an unclosed polygon, their resultant is represented by the remaining side OP. The sequence of directions, from O to A, from A to B, and so on, is generally expressed by saying that the sides of OA, AB, ... are *taken in order*. The resultant, it must be carefully observed, is taken from the initial point O to the final P. If P coincide with O the resultant is, of course, zero.

It is to be noticed that the proposition is true for forces not in one plane. Each successive pair of sides of the polygon are, of course, in one plane, but no three need be coplanar.

143. **Forces on Particle describing any Path.**—Since the acceleration of a particle moving in any path is resolvable into two components, one along the tangent to the path and one towards the centre of curvature, the particle is acted on by corresponding forces in these directions, equal in each case to the acceleration multiplied by the measure, m, of the inertia of the particle. Thus, § 55, the forces are $m\ddot{s}$, $m\dot{s}^2/R$, where $\dot{s}$, $\ddot{s}$ are the velocity and acceleration along the tangent, and R is the radius of curvature.

There is no acceleration at right angles to the osculating plane at any point, and the forces are therefore at every point confined to that plane. This follows from the fact that the motion along *two* consecutive elements of the path is in the osculating plane; but it is easy to prove it analytically by finding from the direction cosines $\dot{x}/\dot{s}$, $\dot{y}/\dot{s}$, $\dot{z}/\dot{s}$ of the tangent at P those of the tangent at an adjacent point, and hence those of the normal to the osculating plane. It will be found that $\ddot{x}$, $\ddot{y}$, $\ddot{z}$ resolved parallel to that normal will give a zero result. The same calculation will verify that the force towards the centre of curvature is $m\dot{s}^2/R$.

144. **Bodies have Masses proportional to their Gravities.**—The dynamical method explained above for the comparison of masses is very difficult to carry out with any accuracy; but the comparison can be effected with the utmost exactness by the method of weighing, since as was shown first by Galileo, and again proved in a more refined manner by Newton, the masses of bodies are proportional to the attractions between them and the earth. Galileo's experiment is well known. It had been asserted, and it was strongly held by the schoolmen of the day, that of two different weights let fall together from the same point, the greater would reach the ground first. Galileo dissented from this view, and affirmed that they would reach the ground together. To put the matter to the test in the presence of his opponents, he took a number of pieces of matter of very different weights, and let them fall together from the top of the leaning tower of Pisa to the pavement beneath. The result was a triumphant confirmation of Galileo's prediction.

Since besides reaching the ground at the same instant the bodies kept pace during their descent, the acceleration at each instant was the same for all. Denoting by G_1, G_2, ... the gravitational forces exerted on the bodies by the earth, reckoned relatively to a frame fixed

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with reference to the fixed stars (§ 130) that is the rates of change of momentum which the bodies experienced at any instant, by m_1, m_2, ... the masses of the bodies as obtained by the dynamical comparison, and by f the common downward acceleration at any instant, we have

$$G_1 = m_1 f, \quad G_2 = m_2 f, \quad \ldots;$$

that is, G_1, G_2, ... were proportional to the masses m_1, m_2, ... But by the assumptions made above as to similarity of circumstances, and the independence of forces, G_1, G_2, ... were very nearly (see § 277) those forces which were annulled by the action of equal and opposite forces when the bodies were at rest relatively to the earth. Thus bodies which in the scales of a just balance equilibrate one another, have equal inertias or masses, and the inertias of different bodies can be compared by the process of weighing.

Newton proved the same thing by showing that pendulums of the same length consisting of different masses suspended by threads of the same length vibrated in the same period. Some account of his experiments is given in the *Scholium Generale* at the end of Sectio VI. of the Second Book of the *Principia*. It appears that he suspended a round wooden box by a thread 11 feet long, which was attached by a ring at the upper end to a hook. The concave surface of the hook in contact with the ring was sharpened to an edge. The pendulum was made to oscillate in the vertical plane at right angles to the plane of the hook, so that the points of contact of the ring and hook were nearly invariable. The box was filled with different metals, and the wire carefully regulated to have the same length in each case. Newton found that the periods were the same, and that further, the rate of falling off of the oscillation corresponded precisely to the mass of the pendulum. In two sets of experiments made—(1) with the empty box, (2) with the box filled so that the total weight was 78 times that of the box, he found that in (2), with the same initial amplitude, 77 oscillations passed before the range of motion was reduced by the amount of reduction which took place in one oscillation when the box was empty. This showed that the action of the air on the moving mass was the same in both sets of experiments, and therefore depended only on the external surface of the body.

The inference from the equality of period is stated clearly in Prop. xxiv. Cor. 1 of the same book, namely that the quantities of matter are as the weights of the bobs. This equality of periods can be easily verified by the reader if he will hang a number of balls, say a ball of wood, a ball of iron, and a ball of lead, of about the same size, by threads of equal length from points in a horizontal line. Thus he will obtain a set of equal pendulums hanging side by side with their bobs in a line. These pendulums should be all deflected to the same extent by pushing the bobs to one side by a flat strip of wood laid against them, and then left to themselves by a quick withdrawal of the piece of wood. It will then be seen that for many oscillations the pendulums keep pace very exactly;

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though after a large number of periods one or more will begin to lag on account of greater disturbance of its motion by the air.

Let, as before, the actions of gravity on the bobs be G_1, G_2, ..., and the inclination of all the threads to the vertical at any instant be θ. The components of G_1, G_2, ... along the line of motion are (§ 137) $G_1 \sin\theta$, $G_2 \sin\theta$, ... Let the common acceleration be a, and the masses of the bobs according to the dynamical comparison be m_1, m_2, ... Then neglecting the motion of the pendulums with the earth we have

$$G_1 \sin\theta = m_1 a, \quad G_2 \sin\theta = m_2 a, \ \ldots$$

that is again G_1, G_2, ... are proportional to m_1, m_2, ...

In the Third Book of the *Principia* (*De Mundi Systemate*), Prop. vi. Theor. vi., Newton describes pendulum experiments which by the result just described prove that all bodies of equal gravities have equal inertias, or, as he expressed it, contain equal quantities of matter. The following is a translation of the passage:

"That the descents of heavy bodies to the earth (with correction for the retardation arising from the very slight resistance of the air) take place in equal times has long been a known result of observation. It is possible to observe very accurately this equality of time in pendulums. I have tried the experiment with gold, silver, lead, glass, sand, common salt, wood, water, and wheat. For the comparison I used two equal round wooden boxes. One of these I filled with wood, and in the other I suspended as exactly as I could at the centre of oscillation, the same weight of gold. The boxes were hung by equal threads eleven feet long, and constituted pendulums altogether equal as regards weight, figure, and resistance of the air; and these when placed in proximity vibrated backwards and forwards in equal times. Accordingly the quantity of matter in the gold was to the quantity of matter in the wood as the action of the moving force on the gold was to the action of the moving force on the wood, that is as the gravity of the gold was to the gravity of the wood; and so in the other cases. In these experiments it was quite possible to detect a difference of mass, in bodies of the same gravity, less than one thousandth part of the total mass."

145. **Newton's Third Law of Motion. Discussion of Action and Reaction.**—We come now to the Third Law of Motion, which was stated by Newton in the following terms:

Actioni contrariam semper et aequalem reactionem, sive corporum duorum actiones in se mutuo semper esse aequales et in partes contrarias dirigi.

To every action there is an equal and contrary reaction, or the mutual actions of two bodies are equal and opposite.

To this he added the following explanatory statement:

Whatever presses or draws another body is pressed or drawn to the same extent by that body. If one presses a stone with the finger, the finger is pressed by the stone. If a horse pulls on a stone by a rope, the horse is pulled equally towards the stone, for the rope being stretched will urge the horse towards the stone and the stone towards the horse; and

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to the extent in which the forward motion of the one is aided the forward motion of the other is impeded. If any body impinging on another body has changed the motion of the latter, the impinging body has sustained the same change of its own motion by the action of the other body; the changes of motion (not of velocities) produced by these actions are equal provided the bodies are not otherwise impeded. Inasmuch as the motions are equally changed, the changes of velocity made in opposite directions are reciprocally proportional to the masses of the bodies. This law holds even in attractions, as will be proved in the scholium which follows.

The explanation of the Third Law of Motion quoted above from Newton has frequently been misunderstood, and the law has been stated, by many of those peculiar people who have been called *paradoxers*, to be obviously untrue. Because action and reaction are equal and opposite in the case (to take Newton's illustration) of a horse pulling a stone, these critics of dynamical processes imagine that neither the horse nor the stone can get into motion. The confusion arises from regarding the action, which is a forward force on the stone, as being cancelled by what is (if, for a moment, we neglect the mass of the rope or chain between the two bodies) the equal and opposite force which acts, and this is what is overlooked, not upon the stone, but upon the horse, and therefore does not affect the motion of the stone.

Besides that applied by the rope there may be other actions on the stone, and others again on the horse, and the motion of each body is changed by the actions on that body and those actions alone. Thus there are two groups of actions, one group applied to the stone and the other to the horse, and all that is asserted in the equality of action and reaction in this illustration is that that particular action of the first group, which is the force exerted on the stone by the horse, is equal to that action of the second group which is the force exerted on the horse by the stone.

The action and reaction which are stated in Newton's law to be equal and opposite are always merely the opposite aspects of a stress existing between two bodies; one aspect is a force on one body, the other aspect is a force on the other body. These forces affect different bodies, and the changes of motion due to them can only be regarded as balancing one another for a system which includes both the bodies concerned. It is remarkable that the persons who have difficulty in understanding this simple matter have no such difficulty in apprehending correctly a perfectly analogous but *financial* transaction. Let a man borrow from one of these people a sum say of £100. A transaction takes place between the parties, and that transaction has two aspects, one which affects the lender and another which affects the borrower. The borrower has received £100, the lender has parted with £100. Suppose the borrower were to say, "My receipt of the £100 is exactly cancelled by your parting with it." One can easily imagine the nature of the lender's reply. He would say, "With the receipt of the money I have

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nothing to do; the phase of the transaction which affects me is the parting with the money, and that can only be cancelled by the return of the money (together with the consideration for its use mentioned in our agreement);" a view in which, he would hint, he would be likely to be supported by the county court! It is astonishing how much the clearness with which people reason depends upon the matter of the reasoning, not to speak of personal considerations.

It is, however, very important to observe that action and reaction are most properly regarded as applied at the same place, though not to the same thing. Across any cross-section of the rope, in Newton's illustration, a stress acts; one aspect of this stress is a forward force on the part of the cord immediately behind the cross-section, the other a backward force on the part of the cord just in front of the cross-section. An excellent example is the action and reaction between two links of a chain, which are exerted across the surface of contact between the links, the action being a force on one link, the reaction a force on the other link. Here, as in all other cases, the action and reaction do not cancel each other, simply because they are applied to what are here regarded as entirely different things. Of course, if we are considering the motion which a system consisting of different parts may have *as a whole*, the actions and reactions between these parts do cancel one another.

It will make this matter clearer perhaps if we consider the action and reaction across a cross-section—(1) of a rod or wire under longitudinal pull (a *tie* as it is called in engineering), for example a vertical wire supporting a weight at its lower end; (2) of a beam under end-thrust (that is a *strut*), or a vertical pillar supporting a load. In the former case take any cross-section A, that is a plane which allows us to divide in imagination the matter on one side from that on the other side of the plane. Across A is exerted by the matter on either side of A on the matter on the other side a pulling force, and these two forces are equal and opposite. Thus in the vertical wire the part above A pulls across A *on the part below* A, and the part below exerts an equal and opposite pull *on the part above* A. These two pulls do not cancel one another since they are exerted on different portions of matter.

If we take two cross-sections A and B of the wire, the part of the wire between A and B is pulled upward at its upper end by the part above the upper section, and is pulled downward at its lower end by the part below. Thus the forces applied to the portion of wire, at its ends, by the remainder tend to increase the length AB, and it is said to be under tension or traction, that is the action of the forces is to produce a stretch in the wire. The same thing holds of course for any wire or bar under pull applied to its end.

In the case of the strut or pillar the portion on either side of a cross-section pushes against the portion on the other side, and the pushes thus applied are equal and opposite, and just as before must not be regarded as cancelling one another. The part between two

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cross-sections AB is pushed against at its ends by the portion of the pillar beyond in each case, so that the portion AB is acted on by opposite forces applied to its ends tending to shorten it. The part is said to be under end-thrust or under compression.

In the discussions of physical theories given in this book an attempt will be made to emphasise the view that every action upon a body is due to the existence of stress in matter in contact with the body; and that therefore if it be found that the presence of one body in the field, so to speak, of another gives rise to a force on the latter, this action can only have place in virtue of a state of stress set up in a medium or intermediary between the two bodies, or in the bodies themselves if they are in direct contact. Further, if any *interface* (geometrical, not physical, separating surface) be taken in the medium or intermediary between the bodies, then whatever action is exerted in virtue of the stress across any area of that interface by the matter situated on one side a on the matter which lies on the other side b, an equal and opposite action is exerted by the matter situated on the side b on the matter which is on the side a.

Another view of action and reaction, also due to Newton, will be exemplified by the theory of *Work and Energy* discussed in chapter iv. In that the *action of an agent* (a body or system of bodies), involved in any force applied by the agent to another body or system, is measured by the product of the force applied by the agent, and the velocity of its point of application to the other body or system resolved in the direction of the force. The reaction is the product of the equal and opposite force, applied by the second body or system to the agent, and the velocity of its point of application to the latter similarly resolved.

The meaning of this statement will be clear in all cases if we adopt the view that where the force is regarded as acting on one body, *there* and there only what is properly called the reaction is applied to the other body, the view in fact that all mutual action between two bodies takes place across an interface between the two bodies at which they are in contact, or taken in the medium by which they are connected. Then the surface has the same velocity for both forces exerted across it, and as these are equal and opposite the product of any element of the one force by the velocity of its point of application is equal and opposite to the product of the corresponding reaction by the same velocity.

But the product of a force by the velocity of its point of application is called the rate of working or *activity* of the force, and thus the activity of one aspect of a stress exerted across an interface is exactly equal and opposite to the activity of the other aspect of the stress. Here again there is no cancelling of one activity by the other. They are two aspects of one transaction which affect different bodies, just as when money is paid over there are two persons affected: one pays, the other receives, but the money passes over. In many cases of equal and opposite activities it will be possible to note the passage of what will be called *kinetic energy* (§ 214) from one

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body to the other, in others though there is undoubtedly redistribution of energy the fact of transference is not so clear.

As an example we take the case of two billiard balls, one at rest, the other moving (without spin) along the lines joining the centres. They strike, and are both compressed at the place of contact. As will be proved in the articles dealing with the dynamics of collision (chap. xv.) the interface of contact of the two balls moves *throughout the collision* with half the speed of approach which the balls had before collision began. Also throughout the collision, in consequence of the tendency to relief of the state of elastic strain set up in the two balls, the ball impinged on has its velocity increased, the one impinging has its velocity diminished, and is finally left at rest. The former finally moves off with only a very little less than the velocity which the latter had before impact, the latter remains at rest.

Here there have been equal and opposite activities throughout the collision, and the kinetic energy of the impinging ball has as a consequence been transferred to the other. In fact the whole process of transference and redistribution of energy by the mutual actions of bodies affords illustration of the Third Law of Motion; in the laws of that process is summed up the whole of physical science. The theory of work and energy will be considered in chapter iv.

The real difficulty in connection with this law of motion arises in connection with the choice of the system of reference axes. None of the mutually acting bodies can be taken as part of the reference frame; otherwise that body would have to be regarded as without acceleration, and therefore either as possessing infinite inertia, or as unacted on by force. Neither of these suppositions is admissible: the action of the body (*e.g.*, its attraction) on other bodies must be finite, and if the force on it is zero the law of action and reaction does not hold for it. See also § 195.

146. **Dynamical Example. Atwood's Machine.**—As an example we may take here the case of a thin flexible string of very small mass passed over a smooth horizontal fixed peg so that the two parts not on the peg are kept stretched in vertical lines by two weights hung at the extremities. We shall suppose the length of the string to be invariable.

Let m, m' be the masses of the two bodies, and let m be greater than m'. The string exerts an upward pull on each mass, and if there is no friction at the peg this must be the same throughout the string. For consider the string on one side. If the inertia of the string is, as we here suppose it to be, negligible in comparison with m and m', the upward force on the upper end is to this approximation equal to the downward force on the lower end. Any difference which exists is only that which is required for the acceleration of the part of the cord considered. We shall here assume, what will be justified under *Statics*, that the stretching force is the same in the parts of the string on the two sides of the peg.

Putting T for the mass-acceleration due to the force applied to

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either mass by the string, and a for the numerical value of the downward acceleration of m and the upward acceleration of m', we have

$$ma = mg - T, \quad m'a = T - m'g.$$

These give, according as a or T is eliminated,

$$T = \frac{2mm'}{m+m'}g, \quad a = \frac{m-m'}{m+m'}g. \tag{6}$$

If the masses were equal a should be zero, and the value of T would take the value mg, which it is clear without calculation it would have.

The arrangement here discussed is an elementary form of *Atwood's machine* for the illustration of the laws of motion. In this machine the cord passes over the grooved rim of a vertical wheel, the axle of which rests on anti-friction rollers (§ 230). The inertia of the wheel prevents the stretching force in the cord from being the same on the two sides of the wheel, so that the simple theory given for the string on the peg is not applicable. The effect, however, of the inertia of the wheel on the acceleration may be taken into account by adding an inertia-equivalent μ for the wheelwork to $m+m'$ in the expression for a, so that the second equation of (6) stands

$$a = \frac{m-m'}{m+m'+\mu}g. \tag{6'}$$

The meaning of the quantity μ will be fully understood by the reader when he has read the sections which follow on rotational motion. The results obtained by means of Atwood's machine may, however, be here simply stated.

Two equal masses are attached to the string in the first instance. These may consist of two cylindrical boxes containing shot so that the total mass is capable of easy adjustment. The difference $m-m'$ is produced by laying on top of one of the boxes a cross-bar of this mass. This bar projects beyond the box, and is removed by a ring which can be adjusted at any point R on a vertical scale along which the descending weight passes. The heavier box is placed at a higher point S on the scale, and supported by a catch which is removed either by hand or by completing the circuit of an electromagnet. Let the top of the box be at S, and let the catch be removed at an instant marked by a time-registering apparatus (for example an electric chronograph or a "fly-back" stop-watch). The masses get gradually into motion, and the instant at which the heavier weight has descended so that its top has been brought on a level with the ring is registered by the chronograph. The additional weight is removed by the ring, and that the further motion is uniform is verified by noticing that the times taken by the weight to descend further distances are proportional to those distances. The velocity acquired is given of course

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from any such further distance and the time taken to describe it. Thus it is verified experimentally that the velocity acquired is proportional to the time of fall from S to R.

Also by taking different distances s_1, s_2, ... between S and R, and noting the times t_1, t_2, ... taken to describe them, it is verified that

$$s_1 = ct_1^2, \quad s_2 = ct_2^2, \ \ldots,$$

where c is a constant. This constant by (6′) above is $\frac{1}{2}a$. (See § 90.)

Again, by using different additional weights $m - m'$ and keeping the total weight $m + m'$ attached to the string constant, and again by keeping $m - m'$ constant and varying $m + m'$, the value corresponding to that given above for a can be verified.

The machine cannot, however, be used as even a fairly exact means of determining the value of the acceleration produced by gravity. The best method of doing this is by careful observation of the period of oscillation of a pendulum of known length, in the manner shortly described under *Gravitation*. Very fairly approximate results can, however, be obtained by means of a simple pendulum made with a pellet of lead and a thin thread hung in an ordinary room, and observed with the eye placed in a vertical plane through the point of support, and perpendicular to the plane of oscillation. This vertical plane may be that of the edges of a couple of books set upon end on a table. The pendulum is set oscillating through an arc not exceeding 10° in total range, and the observer with his eye in this plane gives by a sharp tap a signal at the instant when the pendulum passes across the plane, and another notes the time as given by the seconds-hand of his watch. The pendulum goes on vibrating and the instants of the completion of the 10th, the 20th, the 30th, &c. oscillations are noted. From these the period T can be got with very considerable accuracy, and, from this and l, g can be found by (3).

147. **Centre of Inertia or Centroid.**—We can now deduce some conclusions of great importance from the laws of motion. Most of these refer to what is called the *centre of inertia* of a system of particles, and it is therefore necessary first to specify the meaning of this term.

The centre of inertia of a system of particles may be defined as a point the distance $\bar{x}$ of which from any plane whatever is given by the equation

$$\bar{x} = \frac{m_1x_1 + m_2x_2 + m_3x_3 + \ldots}{m_1 + m_2 + m_3 + \ldots}, \tag{7}$$

where m_1, m_2, m_3, ... are the masses of the particles, and x_1, x_2, x_3, ..., their respective distances from a chosen plane of reference.

The centre of inertia is thus coincident with the mean point of a system of points of multiplicities, m_1, m_2, m_3, ..., as explained at § 19 above.

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To *define* the position of the point we may take three planes mutually at right angles to one another, one from which the distances of the particles are $x_1, x_2, x_3, \ldots$, a second from which the distances are $y_1, y_2, y_3, \ldots$, and a third from which the distances are $z_1, z_2, z_3, \ldots$. The distance $\bar{x}$ of the centre of inertia from the first is given by (7), its distances $\bar{y}, \bar{z}$ from the others are given by

$$\bar{y} = \frac{m_1 y_1 + m_2 y_2 + m_3 y_3 + \ldots}{m_1 + m_2 + m_3 + \ldots}$$

$$\bar{z} = \frac{m_1 z_1 + m_2 z_2 + m_3 z_3 + \ldots}{m_1 + m_2 + m_3 + \ldots}.$$

We may abbreviate these equations somewhat by the notation

$$\bar{x} = \frac{\Sigma mx}{\Sigma m}, \quad \bar{y} = \frac{\Sigma my}{\Sigma m}, \quad \bar{z} = \frac{\Sigma mz}{\Sigma m}, \tag{8}$$

where, for example, Σm denotes the sum $m_1 + m_2 + \ldots$ of the masses, Σmx, the sum of all products of the form $m_1 x_1$, $m_2 x_2$, ..., and so on.

148. **Momentum of Material System relatively to Centroid is Zero.**—In what follows we shall for brevity generally use the term *centroid* for centre of inertia.

Let x', y', z', be the co-ordinates of the point x, y, z relatively to axes through the centroid parallel to the axes of x, y, z, then we have $x = \bar{x} + x'$, $y = \bar{y} + y'$, $z = \bar{z} + z'$, so that

$$\left.\begin{array}{lll} \dot{x} = \dot{\bar{x}} + \dot{x}', & \dot{y} = \dot{\bar{y}} + \dot{y}', & \dot{z} = \dot{\bar{z}} + \dot{z}', \\ \ddot{x} = \ddot{\bar{x}} + \ddot{x}', & \ddot{y} = \ddot{\bar{y}} + \ddot{y}', & \ddot{z} = \ddot{\bar{z}} + \ddot{z}' \end{array}\right\} \tag{9}$$

Also by (8)

$$\Sigma mx' = 0, \quad \Sigma my' = 0, \quad \Sigma mz' = 0, \tag{10}$$

so that

$$\left.\begin{array}{lll} \Sigma m\dot{x}' = 0, & \Sigma m\dot{y}' = 0, & \Sigma m\dot{z}' = 0 \\ \Sigma m\ddot{x}' = 0, & \Sigma m\ddot{y}' = 0, & \Sigma m\ddot{z}' = 0 \end{array}\right\} \tag{11}$$

149. **Theorem of Grouping of Particles. Determination of Centroids.**—As proved at § 20 above, the particles may be divided up into groups, the centre of inertia of each group found, and the group replaced by a particle the mass of which is equal to the sum of the masses of the particles of the group. The centre of inertia of the new system is the same as the centre of inertia of the original system.

This theorem is of great use in the determination of the centroids of bodies, for it is generally convenient to proceed according to some particular plan of dividing up the body into groups of particles. For example, a thin triangular plate of mass per unit of area the same at every point may be divided into narrow

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strips of uniform breadth parallel to one of the sides. The centroid of each strip is, of course, at its centre, and so the triangle may be replaced by a group of particles situated at the centres of these strips, each having a mass equal to that of the strip which it replaces. As these particles lie on the straight line joining the centre of the side parallel to which the strips are taken with the opposite vertex, the centroid of the plate lies on this line. But the triangle might have been divided into narrow strips parallel to either of the other two sides; it therefore lies on the line joining the centre of either of the two sides with the opposite vertex, and therefore is at the point of crossing of these lines. We infer the well-known result of geometry that the three lines joining the vertices of a triangle with the middle points of opposite sides meet in a point. If D be the point of bisection of the side BC, which is opposite A, the reader will see that the centroid of the plate is on AD at G, where $DG = \frac{1}{3}DA$.

Again, consider a tetrahedron, that is a four-sided solid having each side a triangle. This may be divided up into groups of particles forming thin plates parallel to one of the faces, the centroid of each can be found as in the last example, and the series of plates can be replaced by particles at their centroids, as already described. These lie in the line joining the centroid of the base with the vertex; hence the centroid of the tetrahedron lies on this line. In the same way it can be shown that the centroid lies on the line joining the centroid of either of the other sides with the opposite vertex; hence it lies where those two lines intersect. We can infer the other well-known result of geometry that the lines joining the centroids of the faces of a tetrahedron to the opposite vertices meet in a point: that point is the centroid of the tretrahedron.

The reader may prove that this point is at a distance from any vertex of the tetrahedron of three quarters of the length of the line joining the vertex with the centroid of the opposite face. This is easily done by the properties of similar triangles.

Again, in determining the centroid, say of a uniform hemispherical shell, we may divide the shell up into a series of circular rings all parallel to the circular base of the hemisphere. The centroid of each ring will be at its centre, and the shell can thus be replaced by a series of particles of masses equal to those of the corresponding rings, distributed along the line joining the centre of the circular base with the middle point of the curved surface. This calculation can be carried out completely by the aid of the integral calculus, and it would be found that the centroid is at the middle point of this line. This result might be arrived at without calculation from the known fact that the curved surface of the portion of a sphere intercepted between two parallel planes is precisely equal to the portion, intercepted between those two planes, of the curved surface of a cylinder circumscribed to the sphere, with its axis perpendicular to the planes.

Again, the centroid of a hemisphere of uniform density might be

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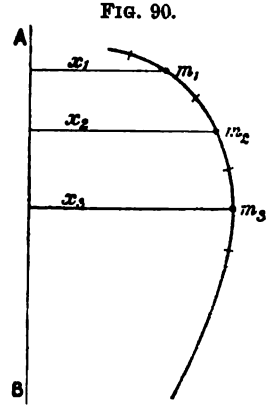
found by dividing it into a series of thin circular plates all parallel to the plane base of the hemisphere; thus the sphere would be replaced by a series of particles as in the last case, and the centroid would be found by calculation to be at a point on the line joining the centre of the circular base at a distance of three-eighths of the radius from that point.

150. **Method by Integration.**—The determination of centres of inertia is to be carried out, except in a few easy and obvious particular cases, by the methods of the integral calculus. A body of continuous mass and known distribution of matter is divided up in imagination into a large number of parts or elements, so small that every point of each may be supposed to be at the same distance from a chosen plane of reference, and that the element may be supposed throughout of uniform density equal to the density at any interior point of it. Then if $d\varpi$ be the volume of one of these elements of the system, ρ its density, and x its distance from a plane of reference, we have

$$\bar{x} = \frac{\int \rho x d\varpi}{\int \rho d\varpi}, \tag{12}$$

where the numerator stands for the sum taken for the body, or system of bodies, of the products of the form $\rho x d\varpi$ (that is the mass $\rho d\varpi$ of the element multiplied by x, the distance of the element from the plane of reference), and the denominator stands for the sum of the products of the form $\rho d\varpi$, that is the total mass of the body or system.

FIG. 90.



151. **Theorems of Pappus.**—The following theorems enable the centre of inertia to be found with great ease for a large number of commonly occurring cases.

A uniform linear distribution of matter in one plane, that is a distribution along a curve in which the amount of matter per unit length of the curve is the same at every point, is turned through any angle about an axis in the plane of the curve. The area of the surface of revolution traced out is equal to the length of the curve multiplied by the length of the path described by the centre of inertia of the linear distribution. This may be proved in the following manner. Let the curve be divided into successive short elements ds_1, ds_2, ..., and let these be replaced by particles m_1, m_2, m_3, ... of masses equal to the masses in the elements ds_1, ds_2, ds_3, ... at their centres. Let x_1, x_2, x_3, ... be the distances of these particles from the axis AB, say as shown in Fig. 90, then the centre of inertia

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of these particles, which is also the centre of inertia of the curve, is at a distance $\bar{x}$ from the axis given by

$$\bar{x} = \frac{\Sigma(mx)}{\Sigma m}. \tag{13}$$

Let the curve be turned through an angle θ about the axis AB; the arc described by m_1 is $x_1\theta$ and so on, while that described by the centre of inertia is $\bar{x}\theta$. But by (69)

$$\bar{x} = \frac{\Sigma(mx\theta)}{\theta\Sigma m}.$$

But if σ be the mass per unit of length in the curve we have $m_1 = \sigma ds_1$, $m_2 = \sigma ds_2$, ... &c., and so $\Sigma m = \sigma s$, where s is the length of the curve, and $\Sigma mx\theta = \sigma\Sigma x\theta ds$. Hence the equation is

$$s\bar{x}\theta = \Sigma(x\theta ds), \tag{14}$$

but $x\theta ds$ is the length of the path $x\theta$ described by a particle multiplied into the distance along the curve between the particle and the next, that is the area of the zone of the surface of revolution between the two paths. The sum of these on the right of (14) is the area of the surface, and the expression on the left is the length of the curve s multiplied into the length $\bar{x}\theta$ of the path described by the centre of inertia.

To this there is a companion theorem which the reader can easily prove in a similar manner. If an area be enclosed by a plane curve, the volume of the solid swept out by the rotation of this area through any angle, about any axis in its own plane, is equal to the product of the revolved area by the length of the path described by the centroid of the area regarded as a uniform laminar distribution of matter, that is as a thin plate having everywhere the same amount of matter per unit of area.

These are known as the theorems of Pappus of Alexandria, a Greek geometer who flourished towards the end of the fourth century of our era. They are also sometimes attributed to Paul Guldin, a French mathematician of the beginning of the seventeenth century.

152. **Examples of the Theorems of Pappus.**—As an example of the use of these theorems we may find the centroid of an arc AB of a circle uniformly loaded with matter, and subtending an angle α at the centre O, as in Fig. 91.

It is clear from symmetry that the centre of inertia lies on the line joining O with the middle point C of the arc. Let the arc be turned through a complete revolution about an axis in the plane of the arc at right angles to OC; it sweeps out a zone of a spherical surface of radius r. Then if I be the centroid, and OI be denoted by x, $2\pi x$ is the length of the path described by I. If r be the radius the length of the arc is $r\alpha$, and the area swept out is $r\alpha \times 2\pi x$.

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But the area of the zone of the spherical surface is equal to the length of the chord AB multiplied into $2\pi r$, that is $2r\sin\frac{1}{2}\alpha \times 2\pi r$. Hence equating these two values we obtain

$$x = \frac{2r}{\alpha}\sin\frac{\alpha}{2}. \tag{15}$$

If the arc be a complete semi-circle we have

$$x = \frac{2r}{\pi}. \tag{16}$$

Again, let it be required to determine the position of the centre of inertia of a thin uniform lamina in the form of a segment ACB

FIG. 91.

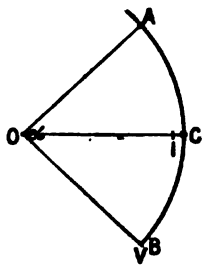
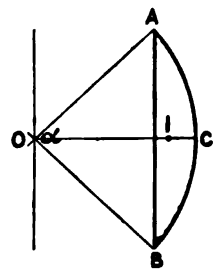


FIG. 92.



of a circle (Fig. 92), the circular boundary of which subtends an angle α at the centre O. Let the segment be revolved through a complete turn about an axis through the centre parallel to the chord AB. The centre of inertia is at some point I on the line OC which bisects the segment. The path described by I is $2\pi x$. The area of the segment is $\frac{1}{2}\alpha r^2 - \frac{1}{2}r^2\sin\alpha$. Hence the volume swept out is $\pi r^2\alpha x - \pi r^2 x\sin\alpha$. But this volume is the volume swept out by the sector $OACB$ *minus* the volume swept out by the triangle OAB, that is $\frac{4}{3}\pi r^3\sin\frac{1}{2}\alpha(1-\cos^2\frac{1}{2}\alpha)$. Hence we have

$$x = \frac{4}{3}r\frac{\sin^3\frac{1}{2}\alpha}{\alpha - \sin\alpha}. \tag{17}$$

If the segment be the complete semi-circle we have $\alpha = \pi$, and

$$x = \frac{4r}{3\pi}. \tag{18}$$

153. **Total Momentum of System of Particles.**—Returning to equation (8), let us denote by $\dot{x}_1, \dot{x}_2, \ldots$ the velocities of the particles $m_1, m_2, \ldots$ parallel to the axis of x, by $y_1, y_2, \ldots$ their velocities parallel to the axis of y, and by $\dot{z}_1, \dot{z}_2, \ldots$ their velocities parallel to the

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axis of z; then it follows from these equations that the components $\dot{\bar{x}}$, $\dot{\bar{y}}$, $\dot{\bar{z}}$ of the velocity of the centroid are given by

$$\dot{\bar{x}} = \frac{\Sigma m\dot{x}}{\Sigma m}, \quad \dot{\bar{y}} = \frac{\Sigma m\dot{y}}{\Sigma m}, \quad \dot{\bar{z}} = \frac{\Sigma m\dot{z}}{\Sigma m}. \tag{19}$$

To prove this consider a specimen particle of mass m and distance x from the plane of reference. In a small interval of time τ this distance will have changed to $x + \dot{x}\tau$, and $\bar{x}$ will have changed to $\bar{x} + \dot{x}\tau$. Thus by the equations for the position of the centroid we have

$$\bar{x} + \dot{\bar{x}}\tau = \frac{\Sigma m(x + \dot{x}\tau)}{\Sigma m}.$$

Subtracting from this the equation $\bar{x} = \Sigma mx/\Sigma m$ we get $\dot{\bar{x}} = \Sigma m\dot{x}/\Sigma m$, and so on for the others.

By throwing the equations thus found into the form

$$\dot{\bar{x}}\Sigma m = \Sigma m\dot{x}, \quad \dot{\bar{y}}\Sigma m = \Sigma m\dot{y}, \quad \dot{\bar{z}}\Sigma m = \Sigma m\dot{z}, \tag{20}$$

we see that the sum of the momenta of the system of particles in any direction is equal to the momentum which the system would have if it were collected into a single particle at the centroid, and moved with the velocity of the centroid. Further, it follows that whatever the motions of the separate particles may be, if the sum of their momenta is zero in every direction the centroid is at rest.

154. **Rate of Change of Momentum of System of Particles.**—In precisely the same way we obtain, denoting the accelerations of the particles parallel to x, y, z by the symbols $\ddot{x}_1, \ddot{x}_2, \ldots, \ddot{y}_1, \ddot{y}_2, \ldots, \ddot{z}_1, \ddot{z}_2, \ldots$, and the components of acceleration of the centroid by $\ddot{\bar{x}}$, $\ddot{\bar{y}}$, $\ddot{\bar{z}}$, the equations

$$\ddot{\bar{x}} = \frac{\Sigma m\ddot{x}}{\Sigma m}, \quad \ddot{\bar{y}} = \frac{\Sigma m\ddot{y}}{\Sigma m}, \quad \ddot{\bar{z}} = \frac{\Sigma m\ddot{z}}{\Sigma m}. \tag{21}$$

or as we may write them

$$\ddot{\bar{x}}\Sigma m = \Sigma m\ddot{x}, \quad \ddot{\bar{y}}\Sigma m = \Sigma m\ddot{y}, \quad \ddot{\bar{z}}\Sigma m = \Sigma m\ddot{z}. \tag{22}$$

in the latter form these equations assert that the sum of the rates of change of momentum of the system of particles in any direction is equal to that which the system would have if it consisted of a single particle coinciding with the centroid of the system.

155. **Moments of Directed Quantities.**—The moment, about an axis, of a directed quantity acting in a specified line is defined as follows. The directed quantity is resolved at any point in its line of action into two components, one parallel to the axis, and another at right angles to the axis, in the plane of the former and the quantity itself. Thus let AB, Fig. 93, be the axis and P the directed quantity, and make the resolution at the point, O, which is at the

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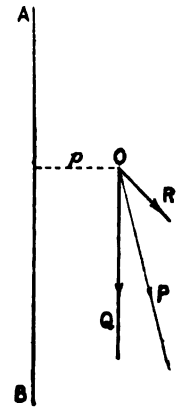
shortest distance from AB. P is resolved at O into Q parallel to AB, and R perpendicular to AB and in the plane of Q and P. R is, in general, not in the plane of AB and O. Let p be the perpendicular distance between AB and the line of action of R, then the moment of P round AB is Rp. This is taken positive or negative by an observer looking along the axis from A towards O, according as R seems to turn the arm p as shown in the diagram, or in the reverse direction.

It may be represented graphically by a distance taken along AB, and towards or from the observer according as the moment is positive or negative.

FIG. 93.



The directed quantity may be anything whatever; most frequently it is a force, and then the definition above gives the moment of a force about an axis. If the quantity is the momentum of a particle, we have the moment of momentum of the particle about the axis, and so on.

The moment round P (Fig. 83) of each of the components of the directed quantity ω in § 118, which are obtained by projecting the diagram on any plane, is the moment of that component round an axis drawn through P perpendicular to that plane. It is there shown that the moment of ω round any axis whatever is equal to the sum of the moments of its components about the same axis. We shall now show that it is equal to the sum of the moments of the quantity about axes of x, y, z passing through a point of the axis, and exhibit the values of these moments.

Let X, Y, Z be the components of the directed quantity, whether force, moment of momentum, or other physical magnitude. If P be the resultant the cosines of the angles it makes with the axes of x, y, z are X/P, Y/P, Z/P respectively. Let the axis AB pass through the origin, and λ, μ, ν be the cosines of the angles it makes with the axes of x, y, z. Let also the co-ordinates of the point O be z, y, z. It is obvious that a plane can be drawn through the axis parallel to P: let ψ be the angle between a line drawn in this plane parallel to P and the direction of the axis. The length p of a perpendicular let fall from any point (x, y, z) on the line of P to this plane is shown in treatises on analytical geometry to be

$$\{x(Y\nu - Z\mu) + y(Z\lambda - X\nu) + z(X\mu - Y\lambda)\}/P\sin\psi.$$

By the definition the moment sought is the product of this perpendicular into the component $P\sin\psi$ of P perpendicular to AB, that is *moment* $= Pp\sin\psi$, and

$$Pp\sin\psi = x(Y\nu - Z\mu) + y(Z\lambda - X\nu) + z(X\mu - Y\lambda). \qquad (23)$$

If we represent this moment graphically by a length laid off as specified above, along the line AB, it resolves itself into three components $Zy - Yz$, $Xz - Zx$, $Yx - Xy$, along the axes of x, y, z, which

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are the moments of P round these axes. The reader may verify that if these component moments be denoted by H_1, H_2, H_3, and H denote $Pp\sin\psi$, then $H=\sqrt{\{H_1^2+H_2^2+H_3^2\}}$.

For example the components of momentum of a particle are $m\dot{x}$, $m\dot{y}$, $m\dot{z}$, and the components H_1, H_2, H_3 of moment of momentum are respectively

$$m(\dot{z}y-\dot{y}z),\quad m(\dot{x}z-\dot{z}x),\quad m(\dot{y}x-\dot{x}y).$$

Now consider any system of particles. The quantities

$$\Sigma m(\dot{z}y-\dot{y}z),\ \Sigma m(\dot{x}z-\dot{z}x),\ \Sigma m(\dot{y}z-\dot{x}y)$$

are the sums of the moments of momentum of the particles of the system about the axes of x, y, z respectively. For any particle m at the point the co-ordinates of which are x, y, z has components of velocity $\dot{x}$, $\dot{y}$, $\dot{z}$, and, with the definition given above, it is easy to show that the moment of any quantity round an axis is equal to the sum of the moments of its components about the axis. The first of these components by the definition has no moment round the axis of x, the moments of the others are (for axes of x, y, z directed as shown in Fig. 3), $-m\dot{y}z$, and $m\dot{z}y$, so that the moment of momentum of the particle is $m(\dot{z}y-\dot{y}z)$; and so for the other axes. It is to be noticed that the time-rates of change of these sums are

$$\Sigma m(\ddot{z}y-\ddot{y}z),\ \Sigma m(\ddot{x}z-\ddot{z}x),\ \Sigma m(\ddot{y}x-\ddot{x}y),$$

and these will appear below in equations of motion, which are of great importance.

If, as at § 348 above, $x=\bar{x}+x'$, $y=\bar{y}+y'$, $z=\bar{z}+z'$ so that x', y', z' are the co-ordinates of a particle relatively to parallel axes through the centroid, the moment of momentum of the system about the axis of x is

$$\Sigma m\{(\dot{\bar{z}}+\dot{z}')(\bar{y}+y')-(\dot{\bar{y}}+\dot{y}')(\bar{z}+z')\}.$$

But by the properties of the centroid we know that $\Sigma m\dot{y}z=0$, $\Sigma m\dot{y}'\bar{z}=0$, &c., so that this moment of momentum becomes

$$\Sigma m(\dot{z}'y'-\dot{y}'z')+\Sigma m(\dot{\bar{z}}\bar{y}-\dot{\bar{y}}\bar{z}).$$

This is the sum of the moment of momentum of the system round an axis through the centroid parallel to the axis of x, and the moment of momentum round the axis of x of the whole system supposed collected at the centroid, and moving with the velocity of that point.

Since the axis of x may be taken in any direction this is true of all axes. By properly choosing the axes of co-ordinates the moment of momentum, or its rate of change, about any axis can be calculated by the expression just given. Or if H_1, H_2, H_3 denote the components of moment of momentum about the axes of x, y, z, that is $\Sigma m(\dot{y}z-\dot{z}y)$, &c., the moment of momentum about an axis through the origin the direction cosines of which are l, m, n is, as the reader may verify, $lH_1+mH_2+nH_3$.

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156. **Equations of Motion of System of Particles. External and Internal Forces. Conservation of Momentum.**—Now consider any system of particles whatever, whether or not forming a rigid body. Let the particles of the system be denoted by $m_1, m_2, m_3, \ldots$, and their co-ordinates by $x_1, y_1, z_1, x_2, y_2, z_2, \ldots$, so that $\ddot{x}_1, \ddot{y}_1, \ddot{z}_1, \ddot{x}_2, \ddot{y}_2, \ddot{z}_2, \ldots$ are the component accelerations of the particles. These accelerations are due to the components of actions on the particles in the directions of the axes. Each action is the resultant of the forces applied to the particles from without the system, and the forces due to the other particles of the system. Thus on the particle m_1, in the direction of x, there acts a force $X_1 + X'_1$, in which X_1 denotes the rate of change of momentum along x which the external action on m_1 would produce, and X'_1 denotes the same thing for the internal action on m_1. Thus we get

$$m_1\ddot{x}_1 = X_1 + X'_1.$$

Similarly equations are obtained for the other directions and the other particles, so that we have

$$\left.\begin{array}{lll} m_1\ddot{x}_1 = X_1 + X'_1, & m_1\ddot{y}_1 = Y_1 + Y'_1, & m_1\ddot{z}_1 = Z_1 + Z'_1 \\ m_2\ddot{x}_2 = X_2 + X'_2, & m_2\ddot{y}_2 = Y_2 + Y'_2, & m_2\ddot{z}_2 = Z_2 + Z'_2 \\ \cdot \quad \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot \end{array}\right\} \qquad (24)$$

there being a group of three equations for each particle. We shall call the forces $X_1, Y_1, Z_1, \ldots$ the *external applied forces*, the forces $X'_1, Y'_1, Z'_1, \ldots$ the *internal forces*. If we add together first all the x-equations, then all the y-equations, then all the z-equations we obtain

$$\Sigma m\ddot{x} = \Sigma X, \quad \Sigma m\ddot{y} = \Sigma Y, \quad \Sigma m\ddot{z} = \Sigma Z, \qquad (25)$$

in which the quantities on the right are the sums of the *external* applied forces only. The sums of the internal forces vanish by the third law of motion, since the "internal force" on any particle, m_1 say, is the resultant of the forces exerted on that particle by the other particles of the system. Now each force exerted by a particle m_2 on m_1 is accompanied by an equal and opposite force exerted on m_2 by m_1, and this comes into the account when the total action on m_2 is considered. Thus when the forces are added as above, the sums of the internal forces on the particles vanish identically.

The force X, Y, Z is here supposed applied to a particle at the point x, y, z, and so for the other forces. Of course, to many of the particles of the system no external force may be directly applied; for each of these X, Y, Z is zero. Again, in some cases the force X, Y, Z may not be directly applied to any particle of the system considered, but applied to some framework or link connected with the system, the motion of which is to be found; for example a force may be regarded as applied at any point of a cord by which pull is applied to the system. In fact, as the reader ought to notice, all the theorems given below as to the action of forces are independent of the point in the line of action of a force which is taken as the point

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at which the force is applied. This result, which is often assumed as the basis of statics, follows naturally from the dynamical method given by Newton.

By § 153 above we see that equations (24) may be written

$$\ddot{\bar{x}}\Sigma m = \Sigma X, \quad \ddot{\bar{y}}\Sigma m = \Sigma Y, \quad \ddot{\bar{z}}\Sigma m = \Sigma Z, \tag{26}$$

which state that the acceleration of the centroid of the system is the same as that which a particle of mass Σm, equal to the total mass of the system, would have if it were acted on by the external forces applied to it without change of magnitude or direction.

Thus the internal forces have no effect whatever in altering the momentum of a system of particles. If the external forces be zero for any direction, the momentum of the system in that direction does not change.

157. **Equations of Moments.**—Now multiply the first x-equation of (24) by y_1, the second x-equation by y_2, ..., and add the resulting equations together; then multiply the y-equations by x_1, x_2, ... respectively, and add the resulting equations; finally, subtract the first sum from the second. We thus get

$$\Sigma m(\ddot{y}x - \ddot{x}y) = \Sigma\{(Y + Y')x - (X + X')y\}.$$

Treating in the same way the remaining equations we obtain two other equations of the same form. It is to be observed that for the same reason as stated above we must have identically $\Sigma Y'x = 0$, $\Sigma X'x = 0$, and so on. Hence the equations obtained finally are

$$\left.\begin{aligned} \Sigma m(\ddot{z}y - \ddot{y}z) &= \Sigma(Zy - Yz) \\ \Sigma m(\ddot{x}z - \ddot{z}x) &= \Sigma(Xz - Zx) \\ \Sigma m(\ddot{y}x - \ddot{x}y) &= \Sigma(Yx - Xy). \end{aligned}\right\} \tag{27}$$

The quantities on the right of (27) are, by § 155, the moments of the external applied forces round the axes x, y, z, which pass through the origin of co-ordinates, and the quantities on the left are the moments of the rates of change of momentum of the particles about the same axis, or since

$$\ddot{y}x - \ddot{x}y = \frac{d}{dt}(\dot{y}x - \dot{x}y), \quad \ldots\ldots, \quad \ldots\ldots,$$

the quantities on the left are (§ 155 above) the rates of change of the moments of momentum of the system about the axes. Thus the equations assert that the rate of change of the moment of momentum of the system of particles (or as it is sometimes put, the sum of the moments of the *effective forces*) about any axis is equal to the sums of the moments of the external applied forces about the same axis. Since the origin may be taken anywhere, and the axis in any direction, the theorem thus holds for any axis whatever.

158. **Equations of Moments. Properties of the Centroid.**—By § 148 equations (27) can be written in the form

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$$\left.\begin{array}{l}\Sigma m\{(\ddot{\bar{z}}+\ddot{z}')(\bar{y}+y')-(\ddot{\bar{y}}+\ddot{y}')(\bar{z}+z')\}=\Sigma\{Z(\bar{y}+y')-Y(\bar{z}+z')\}\\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\end{array}\right\}(28)$$

But since these equations are true for all axes they must be true for the axes just supposed drawn through the centre of inertia, and hence the equations just written split into two sets:—

(1) The set

$$\Sigma m\{(\ddot{\bar{z}}+\ddot{z}')y'-(\ddot{\bar{y}}+\ddot{y}')z'\}=\Sigma(Zy'-Yz')$$
$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$
$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

which, since $\Sigma m\ddot{\bar{y}}x'=\ddot{\bar{y}}\Sigma mx'$, and $\Sigma mx'=0$, ..., may be written

$$\left.\begin{array}{l}\Sigma m(\ddot{z}'y'-\ddot{y}'z')=\Sigma(Zy'-Yz')\\ \Sigma m(\ddot{x}'z'-\ddot{z}'x')=\Sigma(Xz'-Zx')\\ \Sigma m(\ddot{y}'x'-\ddot{x}'y')=\Sigma(Yx'-Xy').\end{array}\right\}\quad(29)$$

(2) The set

$$\left.\begin{array}{l}\Sigma m(\ddot{\bar{z}}\bar{y}-\ddot{\bar{y}}\bar{z})=\Sigma(Z\bar{y}-Y\bar{z})\\ \Sigma m(\ddot{\bar{x}}\bar{z}-\ddot{\bar{z}}\bar{x})=\Sigma(X\bar{z}-Z\bar{x})\\ \Sigma m(\ddot{\bar{y}}\bar{x}-\ddot{\bar{x}}\bar{y})=\Sigma(Y\bar{x}-X\bar{y}).\end{array}\right\}\quad(30)$$

Equations (29) state that the sum of the moments, round any axis through the centroid of the system, of the rates of change of momentum of the particles of the system relatively to the centroid, is equal to the sum of the moments of the applied forces round the same axes. It is to be noticed that the accelerations of the centre of inertia do not enter into the result.

Equations (30) state that the sum of the moments, taken round any axis, of the so-called effective forces on the particles of the system, supposed transferred to the centroid without change of magnitude or direction, is equal to the sum of the moments, round the same axis, of the external applied forces.

Let now $r_z=\sqrt{x^2+y^2}$ be the length of a perpendicular let fall from any particle on the axis of z, and let ψ be the angle this perpendicular makes with the axis of x. We have

$$x=r_z\cos\psi,\quad y=r_z\sin\psi.$$

Then we obtain by a double differentiation with respect to t and reduction

$$\ddot{y}x-\ddot{x}y=2r_z\dot{r}_z\dot{\psi}+r_z^2\ddot{\psi}.$$

If ϕ, χ be the angles the perpendiculars from the particle on the axes of x and y make similarly with the axes of y and z, and r_x, r_y be the lengths of these perpendiculars, we have

$$\Sigma m(\ddot{z}y-\ddot{y}z)=\Sigma m(2r_x\dot{r}_x+r_x^2\ddot{\phi})$$

with two similar equations.

But since $Yx-Xy=r(Y\cos\psi-X\sin\psi)=r_zR_z$ where R_z is the force on the particle at right angles to r_z and to the axis of z, and similarly for the others, we see that equations (27) may be written

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$$\left.\begin{aligned}\Sigma m(r_x^2\ddot{\phi}+2r_x\dot{r}_x\dot{\phi})&=\Sigma R_x r_x\\ \Sigma m(r_y^2\ddot{\chi}+2r_y\dot{r}_y\dot{\chi})&=\Sigma R_y r_y\\ \Sigma m(r_z^2\ddot{\psi}+2r_z\dot{r}_z\dot{\psi})&=\Sigma R_z r_z\end{aligned}\right\}\qquad(31)$$

That these results hold for parallel axes through the centroid, can of course be deduced from equations (29). They are not sufficient to determine the motion of a non-rigid system: it is necessary for this purpose to know either all the forces on each particle, or, what is really the same thing, the geometrical relations connecting the positions of the particles.

The reader will notice carefully that if there be no moment of forces round an axis there can be no rate of change of moment of momentum about that axis. This simple result is of great use in explaining the behaviour of rotating bodies such as tops, gyrostats, &c.

159. **Equations of Moments for a Rigid Body. Conservation of Moment of Momentum.**—Let now the system of particles be a rigid system, and take axes through the centroid. Then if ρ_x, ρ_y, ρ_z be the distances of any particle from the axes of x', y', z' through the centroid, these will replace r_x, r_y, r_z. But since the system is rigid the centroid is fixed relatively to it, so that $\dot{\rho}_x$, $\dot{\rho}_y$, $\dot{\rho}_z$ are zero for every particle. Also $\dot{\phi}$, $\dot{\chi}$, $\dot{\psi}$ and $\ddot{\phi}$, $\ddot{\chi}$, $\ddot{\psi}$ must be the same for every particle at any given instant. Hence we obtain

$$\ddot{\phi}\Sigma m\rho_x^2=\Sigma R_x\rho_x,\quad \ddot{\chi}\Sigma m\rho_y^2=\Sigma R_y\rho_y,\quad \ddot{\psi}\Sigma m\rho_z^2=\Sigma R_z\rho_z.$$

If the system be rigid and turn round a fixed axis through the origin, r is zero for every particle, and the quantities $\ddot{\phi}$, $\ddot{\chi}$, $\ddot{\psi}$ are the same for each particle at any given instant. Equations (31) become

$$\ddot{\phi}\Sigma mr_x^2=\Sigma R_x r_x,\quad \ddot{\chi}\Sigma mr_y^2=\Sigma R_y r_y,\quad \ddot{\psi}\Sigma mr_z^2=\Sigma R_z r_z.\qquad(32)$$

Since the axis is fixed its direction cosines are (§ 24 above) $\dot{\phi}/\omega$, $\dot{\chi}/\omega$, $\dot{\psi}/\pi$, where ω is the angular velocity of the body about the axis, *viz.*, $\sqrt{\dot{\phi}^2+\dot{\chi}^2+\dot{\psi}^2}$. The angular accelerations $\ddot{\phi}$, $\ddot{\chi}$, $\ddot{\psi}$ must clearly be proportional to the component angular velocities, that is the direction cosines of the axis are also $\ddot{\phi}/\dot{\omega}$, $\ddot{\chi}/\dot{\omega}$, $\ddot{\psi}/\dot{\omega}$. From this we could show that the rate of change of the moment of momentum of the system about the axis is equal to the sum of the moments of the external applied forces about the axis; but the same result is obtained at once by taking the axis as coincident with that of x, say. Then $\dot{\chi}$, $\dot{\psi}$, $\ddot{\chi}$ $\ddot{\psi}$ are zero, and we have the single equation

$$\ddot{\phi}\Sigma mr^2=\Sigma Rr,\qquad(33)$$

in which the suffixes are dropped as no longer necessary.

The conclusions more than once referred to above, that the internal forces of the system have no effect in changing the momentum of the system in any direction, or its moment of momentum about any axis, should be most carefully noticed. Thus the total momentum in any direction of the bodies of the solar system remains unchanged by their

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mutual actions, as does also the total moment of momentum round any axis. These can only be changed by the actions of bodies outside the system.

Again, if any of the bodies of the system increase or diminish in bulk under internal forces only (it may be taken as certain that they are all contracting, and that those that are not gaseous are also cooling), their period of rotation will in the former case increase, in the latter diminish, inasmuch as their moments of momentum about their axes of rotation must remain constant. Thus the cooling of the earth and its consequent contraction must tend to diminish the length of the day.

160. **Moment of Momentum as Rate of Description of Area. Axis of Maximum Moment of Momentum.**—The moment of momentum of a system round any axis through the origin may be represented in the following manner as a sum of rates of description of area. Let a plane through the origin and perpendicular to the axis be taken. Then let a particle of the system in an element of time dt describe an element of path $\dot{s}dt$, and let this elementary displacement be projected on the plane referred to, and its extremities joined to the origin. These joining lines are of very approximately the same length, ρ say. The area included between the lines and the projection of the element of path is $\frac{1}{2}\rho^2\dot{\theta}dt$, where $\dot{\theta}$ is the angular velocity of the particle round the axis, and, if m is the mass of the particle, $\frac{1}{2}m\rho^2\dot{\theta}$ is clearly the moment of momentum of the particle about the axis. The total moment of momentum of the system about the axis is therefore $\frac{1}{2}\Sigma m\rho^2\dot{\theta}$, that is it is half the sum of the rates of description of the areas thus taken on the plane perpendicular to the axis.

Now take this sum of rates round each of the rectangular axes of x, y, z through the origin, and denote the sums obtained by H_1, H_2, H_3. Then these resolved in the usual way give a resultant H about an axis making with the axes of co-ordinates angles the direction cosines of which are H_1/H, H_2/H, H_3/H. From the fact, as shown above, § 118, that areas may be represented by steps, it follows that H is the sum of rates obtained directly for the plane at right angles to the axis thus defined. The reader may, however, supply a separate analytical proof (see § 155).

The sum of the projections of these rates on any plane, the direction cosines of the normal to which are l, m, n, is $lH_1 + mH_2 + nH_3$ or $H\cos\phi$, if ϕ be the angle between the direction of H and the line l, m, n. Thus H is the maximum value of the sum of areas described per unit of time, that is of the moment of momentum.

161. **Conservation of Moment of Momentum. Invariable Plane. Invariable Line.**—Let now the system be self-contained, that is subject to no external forces. The forces are then only actions and reactions between the particles, and the sum of the moments of all the forces between parts of the system is zero for every axis. The moment of momentum of the system about any axis therefore remains

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unaltered, and H, the maximum moment for the origin (which may be any point), is constant both in magnitude and direction. The plane through the origin at right angles to the axis of H is called the *invariable plane* at the origin. It remains throughout all changes of the configuration of the system unaltered in position. The invariability of H has been called by writers on physical astronomy the *conservation of areas*.

The line through O at right angles to the invariable plane is called the *invariable line at O*.

162. **Invariable Plane for Different Points.**—The invariable plane is not the same for different points of space. To find it for any point taken as origin, let l, m, n be the direction cosines of $\dot{s}dt$ for the representative particle. We shall put m' for the mass of this particle to prevent confusion with the direction cosine m. Its projection on any plane is $\dot{s}dt\sin\psi$, if ψ be the angle between the line l, m, n, and the normal to the plane. Now let p be the length of the common perpendicular to the axis of H and the line of $\dot{s}$; the product $p\dot{s}\sin\psi dt$ is $r^2\dot{\theta}dt$. But if x, y, z be the co-ordinates of the particle describing $\dot{s}dt$, and λ, μ, ν be the cosines of the normal to the invariable plane, p is the length of the common perpendicular between the line l, m, n passing through x, y, z, and the line λ, μ, ν passing through the origin. The length of this line is easily found to be

$$\{x(m\nu - n\mu) + y(n\lambda - l\nu) + z(l\mu - m\lambda)\}/\sin\psi.$$

Hence (since $\dot{s}l = \dot{x}$, $\dot{s}m = \dot{y}$, $\dot{s}n = \dot{z}$)

$$p\dot{s}\sin\psi = x(\dot{y}\nu - \dot{z}\mu) + y(\dot{z}\lambda - \dot{x}\nu) + z(\dot{x}\mu - \dot{y}\lambda).$$

Now let $\lambda = 1$, $\mu = \nu = 0$, then we obtain

$$\Sigma(m'p\dot{s}\sin\psi) = H_1 = \Sigma m'(y\dot{z} - z\dot{y});$$

and similarly $H_2 = \Sigma m'(z\dot{x} - x\dot{z})$, $H_3 = \Sigma m'(x\dot{y} - y\dot{x})$, expressions already obtained in § 155.

Again if for λ, μ, ν be put H_1/H, H_2/H, H_3/H where H is the resultant of H_1, H_2, H_3, that is if the axis be taken through the origin in the direction of this resultant, we get for $\Sigma(m'p\dot{s}\sin\psi)$ the value H, which verifies that the sum of the projections of areas on the plane through the origin at right angles to the line $(H_1, H_2, H_3)/H$ is the resultant of the components specified.

If for x, y, z be put $\bar{x} + x', \bar{y} + y', \bar{z} + z'$, $\Sigma m(y\dot{z} - z\dot{y})$ or H_1 becomes $\Sigma m(\bar{y}\dot{\bar{z}} - \bar{z}\dot{\bar{y}}) + \Sigma m(y'\dot{z}' - z'\dot{y}')$, and so for the other components H_2, H_3, that is the moment of momentum of the system about the axis at the origin is equal to the moment of momentum of the whole mass supposed collected at the centroid, together with the moment of momentum of the system about a parallel axis through the centroid, a result also already obtained above.

For different positions of the origin the values of x', y', z' and their rates of variation will be different, and so the values of H_1, H_2, H_3 will also be different. Thus by changing the origin to the point whose co-ordinates relatively to the former origin are f, g, h we alter H_1 to $\Sigma\{m(y-g)\dot{z} - (z-f)\dot{y}\}$, which is the former value of

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H_1 *minus* the value of H_1 for an axis at the former origin and a particle of mass Σm at f, g, h moving with velocity $\dot{x}, \dot{y}, \dot{z}$. The direction of the invariable plane thus varies from point to point, as does also the value of H.

It is evident that if x', y', z' be proportional to $\bar{x}, \bar{y}, \bar{z}$, that is if the point lie on the line described by the centroid of the system (supposed self-contained, so that the centroid moves along a straight line with uniform speed), the invariable plane is parallel to the invariable plane for the centroid, and that for all points on a straight line passing through any fixed point f, g, h the directions of the invariable plane will be the same provided $x-f$, $y-g$, $z-h$ be proportional to $\bar{x}, \bar{y}, \bar{z}$, that is if the line be parallel to the path of the centroid.

163. **Invariable Plane of the Solar System.**—If we suppose the solar system to be self-contained, that is if we regard the attractions of the fixed stars on the sun and the planets to be negligible, the plane at right angles to the axis of resultant moment of momentum will be invariable in direction. The position of the invariable plane through the centroid of the system will not sustain any change of position in consequence of the mutual actions of the different parts of the system.

The position of this invariable plane cannot be fully determined owing to want of knowledge of the data necessary for the calculation of the moments of momentum of the bodies composing the solar system. But if the various bodies be regarded as particles coinciding with the centroids, and the orbital motions only are taken into account, the moment of momentum can be approximately estimated. If the bodies were spherical, and each were of density symmetrical about the centre, their motions of rotation would not be altered by the forces of attraction, since, as will be shown under *Attraction*, each body would attract and be attracted as if its whole mass were collected at its centroid. The rotational motions would thus throughout all changes of configuration contribute a constant part of the moment of momentum. Thus an invariable plane could be found for the remainder, and this has been called the *astronomical invariable plane*. The maximum moment of momentum is obtained by projecting the orbits of the planets upon the plane, multiplying the mass of each planet by half the area which its radius-vector describes in unit of time on the projected orbit, and adding all the products together.

Laplace has calculated the position of this plane for two epochs, 1750 and 1950, two hundred years apart, and has given* the following values for its inclination to the plane of the ecliptic and the longitude of the ascending node in 1750, $1^\circ{\cdot}7689$, $114^\circ{\cdot}3979$; in 1950, $1^\circ{\cdot}7689$, $114^\circ{\cdot}3934$. This result is illustrated in Fig. 94, which shows the two planes (EE the plane of the ecliptic, II the invariable plane) with some exaggeration of the angle, $1^\circ{\cdot}7689$, between them. The line NN' (supposed here to pass through the sun at S) in which the

* *Mécanique Céleste*, t. iii. p. 163.

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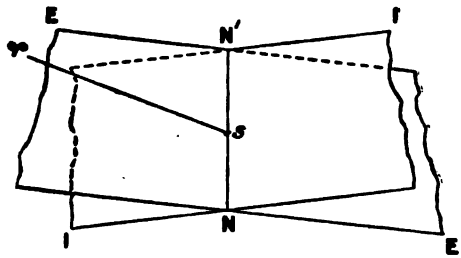
two intersect is called the *line of nodes*. If a planet were to move in an orbit in the invariable plane, in the direction in which the planets move round the sun, it would pass from the "lower" or southern side of the plane of the ecliptic on the side N of the diagram, and from the northern side on the side N'. The former is therefore called the *ascending node*, the latter the *descending node*. The angle between a line drawn from S to the first point of Aries denoted by ♈, and the line SN, that is the angle ♈SN is the longi-

FIG. 94.



tude of the ascending node, and according to Laplace's statement is about 114°·4. This is only one of several extremely important results established by Laplace and Lagrange regarding the stability of the solar system. For these the reader must refer to works on Physical Astronomy, for example Tisserand's *Mécanique Céleste*.

164. **Moment of Inertia.**—The quantities Σmr_x^2, Σmr_y^2, Σmr_z^2, and $\Sigma m\rho_x^2$, $\Sigma m\rho_y^2$, $\Sigma m\rho_z^2$, on the left in equations (32) and (33), are what are called the moments of inertia of the system about the axes specified. It will be convenient to consider here the moment of inertia of a system of particles in some little detail. Let, then, m_1, m_2, m_3, ... be the masses of the particles, r_1, r_2, r_3, ... their distances from the axis, then the sum $m_1r_1^2 + m_2r_2^2 + \ldots$, or as it may be written shortly, Σmr^2, is called the moment of inertia of the system about the given axis. This quantity is of great importance in the discussion of rotational motion of bodies.

165. **Moment of Inertia of any System. Property of the Centroid. Radius of Gyration.**—The following theorem is of continual use. The moment of inertia of a system of particles about any axis is equal to the moment of inertia of the system about a parallel axis through its centre of inertia, *plus* the moment of inertia about the given axis of a single particle of mass equal to the total mass of the system, and placed at the centre of inertia. To prove this, let r (Fig. 95) be the distance of a particle m from the given axis, h the distance of the centre of inertia from the axis, and r' the distance of the particle from a parallel axis through the centre of inertia. Thus let P be the position of the particle supposed in the plane of the paper, A the projection of the axis on the plane of the paper, and G the projection of the centre of inertia on the same plane. Then

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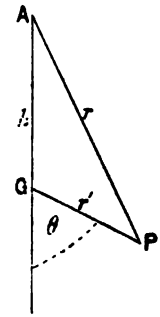
AP is r, AG is h, and GP is r'. If θ be the angle between AG produced and GP, we have by trigonometry

$$r^2 = h^2 + r'^2 + 2hr'\cos\theta.$$

But $r'\cos\theta$ is equal to the distance of the particle from a plane through the centre of inertia at right angles to the line AG. Let it be denoted by x, then $r^2 = h^2 + r^2 + 2hx$, and therefore

FIG. 95.



$$mr^2 = mh^2 + mr'^2 + 2hmx.$$

Forming this equation for each of the particles and adding, we obtain for the system

$$\Sigma mr^2 = h^2\Sigma m + \Sigma mr'^2 + 2h\Sigma mx.$$

But Σmx is the sum of the product of each mass into its distance from a plane passing through the centre of inertia, and must therefore be zero; hence we have the result

$$\Sigma mr^2 = h^2\Sigma m + \Sigma mr'^2, \qquad (34)$$

which is the theorem stated above.

Let M denote Σm, the mass of the system, then it is possible to find a quantity k such that $Mk^2 = \Sigma mr'^2$. Hence we obtain

$$\Sigma mr^2 = M(h^2 + k^2), \qquad (35)$$

which also expresses the theorem. The quantity k is called the *radius of gyration* of the system about the parallel axis through the centre of inertia.

166. **Examples of Moment of Inertia.**—The use of the moment of inertia is illustrated by the fact that the moment of momentum of a system turning as a rigid body about an axis, with angular velocity ω, is equal to the product $\omega\Sigma mr^2$, that is to the angular velocity multiplied by the moment of inertia. To prove this observe that the moment of momentum of a system of particles turning about an axis is equal to Σmrv, where v is the linear velocity of the particle of mass m, whose distance from the axis is r. But since the particles of the system have a common angular velocity ω, $v = \omega r$, and so

$$\Sigma mvr = \omega\Sigma mr^2. \qquad (36)$$

The *kinetic energy* of a system of particles is the sum of the values of $\frac{1}{2}mv^2$ for all the particles of the system, where m is the mass of a particle and v its velocity with respect to the chosen system of reference. In the case of a rigid system rotating about a fixed axis $v^2 = \omega^2 r^2$ where ω is the velocity common to all the particles of the system. Thus, since the distances r are constant, we have

$$\tfrac{1}{2}\Sigma mv^2 = \tfrac{1}{2}\omega^2\Sigma mr^2. \qquad (37)$$

Kinetic energy will be more fully discussed below.

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The calculation of the moment of inertia for different bodies can in general be carried out only by the methods of the integral calculus. The following rule suffices for a large number of practical cases.

The moment of inertia about an axis A of symmetry through the centroid, of a body of uniform density in the form of a rectangular prism, an elliptic plate, or an ellipsoid, is equal to the mass of the body multiplied by the sum of the squares of the semi-axes perpendicular to A, and divided by 3, 4, or 5 according to the form of the body.

For example, the moment of inertia of a bar of rectangular section of length $2l$, and breadth $2a$, about an axis through the centre of inertia at right angles to the faces of these dimensions, is

$$Mk^2 = \tfrac{1}{3}M(l^2 + a^2). \tag{38}$$

In the case of a uniform elliptic plate, about an axis through the centre of the plate at right angles to its plane, of semi-axes of lengths a and b,

$$Mk^2 = \tfrac{1}{4}M(a^2 + b^2). \tag{39}$$

If the moment of inertia be about an axis in the plane of the disk coinciding say with the semi-axes of length b, we have

$$Mk^2 = \tfrac{1}{4}Ma^2. \tag{40}$$

In the case of an ellipsoid of uniform density about an axis coinciding with one of the three principal axes

$$Mk^2 = \tfrac{1}{5}M(a^2 + b^2), \tag{41}$$

where a, b are the lengths of the semi-axes at right angles to that about which the moment of inertia is taken.

For a sphere, which of course is a particular case of the ellipsoid with all the axes equal,

$$Mk^2 = \tfrac{2}{5}Ma^2, \tag{42}$$

where a is the radius.

167. **Poinsot's Momental Ellipsoid.**—Let the direction cosines of an axis through the origin O (Fig. 96) be α, β, γ, and consider the moment of inertia round this axis of a particle of mass m at the point P, the co-ordinates of which are x, y, z. The square of the distance OP is $x^2 + y^2 + z^2$, and the projection OM of this distance on the axis is $\alpha x + \beta y + \gamma z$. The square of the distance PM of the particle from the axis is therefore $x^2 + y^2 + z^2 - (\alpha x + \beta y + \gamma z)^2$. Hence the moment of inertia of any system of particles about the axis is

$$\Sigma m(x^2 + y^2 + z^2) - \Sigma m(\alpha x + \beta y + \gamma z)^2.$$

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Expanding this, and remembering that $\alpha^2+\beta^2+\gamma^2=1$, we write it

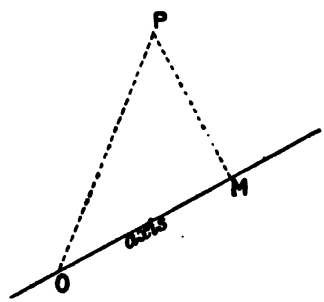
$$\alpha^2\Sigma m(y^2+z^2)+\beta^2\Sigma m(z^2+x^2)+\gamma^2\Sigma m(x^2+y^2)$$
$$-2\beta\gamma\Sigma myz-2\gamma\alpha\Sigma mzx-2\alpha\beta\Sigma mxy.$$

Now if $A=\Sigma m(y^2+z^2)$, $B=\Sigma m(z^2+x^2)$, $C=\Sigma m(x^2+y^2)$, $D=\Sigma myz$, $E=\Sigma mzx$, $F=\Sigma mxy$, this expression becomes

$$A\alpha^2+B\beta^2+C\gamma^2-2D\beta\gamma-2E\gamma\alpha-2F\alpha\beta.$$

It is clear that the moment of inertia of the system about the axis of x is A, about the axis of y is B, about the axis of z is C

FIG. 96.



The quantities D, E, F are called *products of inertia* about the axes of x, y, z respectively.

Now a surface of the second degree, or *quadric surface* as it is called, the co-ordinates of any point on the surface of which are ξ, η, ζ, may have the equation

$$A\xi^2+B\eta^2+C\zeta^2-2D\eta\zeta-2E\zeta\xi-2F\xi\eta=k^2, \qquad (43)$$

where k is a constant. If $\rho^2=\xi^2+\eta^2+\zeta^2$, the square of the distance from the origin of a point ξ, η, ζ on the surface, and α, β, γ be the direction cosines of ρ, then, since $\alpha=\xi/\rho$, $\beta=\eta/\rho$, $\gamma=\zeta/\rho$, the equation of the surface may be written

$$A\alpha^2+B\beta^2+C\gamma^2-2D\beta\gamma-2E\gamma\alpha-2F\alpha\beta=\frac{k^2}{\rho^2}. \qquad (44)$$

Hence, by the expression found above, the moment of inertia of the system of particles about any axis through the origin in the direction α, β, γ is inversely proportional to the square of the radius-vector of the quadric surface (43) drawn in that direction; or the radius of gyration about any axis is inversely proportional to the radius-vector. Thus the quadric surface represents graphically by its radii-vectores the values of the radius of gyration of the system about the axis through the origin.

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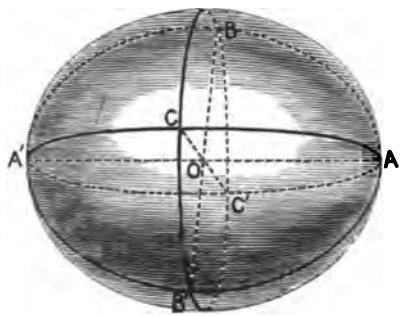
The value of the moment of inertia about any axis is clearly positive; hence the left side of (43) is a positive quantity. The surface, as will be seen presently, is an ellipsoid surrounding the origin, and may of course in particular cases be a sphere, or a prolate or oblate ellipsoid of revolution.

Fig. 97 represents an ellipsoid of which OA, OB, OC are the principal axes, that is the three lines mutually at right angles which, as is shown in § 160, can be drawn to meet the surface perpendicularly at A, B, C. Each line meets the surface thus in two points, so

FIG. 97.



that there are the three pairs, AA', BB', CC', and the lines all meet and are bisected in a point O, which is therefore called the centre of the surface. The axes OX, OY, OZ to which the surface is referred in (43) are any other three mutually rectangular directions through O.

168. **Principal Axes of Momental Ellipsoid.**—It is shown in treatises on solid geometry that the direction cosines of a line perpendicular to a quadric surface at the extremities of a radius-vector drawn in the direction α, β, γ are, if the surface be represented by (43), proportional to the quantities

$$A\alpha - F\beta - E\gamma, \quad -F\alpha + B\beta - D\gamma, \quad -E\alpha - D\beta + C\gamma.$$

If these values coincide with α, β, γ the radius-vector meets the surface at right angles. The conditions necessary and sufficient for this are

$$\left.\begin{aligned} A\alpha - F\beta - E\gamma &= \kappa\alpha \\ -F\alpha + B\beta - D\gamma &= \kappa\beta \\ -E\alpha - D\beta + C\gamma &= \kappa\gamma \end{aligned}\right\} \qquad (45)$$

where κ is a constant. Elimination of α, β, γ will give a cubic equation

$$\begin{vmatrix} A-\kappa, & -F, & -F \\ -F, & B-\kappa, & -D \\ -E, & -D, & C-\kappa \end{vmatrix} = 0 \qquad (46)$$

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for the determination of κ, all the roots of which can be shown to be real and positive. Any one of these roots used for κ in (45) will enable α, β, γ to be found, and thus (unless the cubic has equal roots) three directions are obtained in which lines drawn from the origin meet the surface at right angles. It can easily be shown that these directions are mutually perpendicular.

For let a^2, b^2, c^2 be the roots, supposed all unequal, of the cubic, and α_1, β_1, γ_1, α_2, β_2, γ_2 be the values of the direction cosines given by any two of them, a^2, b^2, say. Then substitute a^2 for κ in (45) and α_1, β_1, γ_1 for α, β, γ, and multiply the first equation by α_2, the second by β_2, the third by γ_2. This gives three equations involving $\alpha_1\alpha_2$, $\beta_1\beta_2$, $\gamma_1\gamma_2$, and other products. In like manner putting b^2 for κ, α_2, β_2, γ_2 for α, β, γ, and multiplying the equations obtained by α_1, β_1, γ_1 respectively, we get other three equations. The sum of the last three subtracted from the sum of the first three gives the relation $(b^2-a^2)(\alpha_1\alpha_2+\beta_1\beta_2+\gamma_1\gamma_2)=0$, or $\alpha_1\alpha_2+\beta_1\beta_2+\gamma_1\gamma_2=0$, that is the directions α_1, β_1, γ_1, α_2, β_2, γ_2 are at right angles to one another. Similarly it may be shown that each of the other two pairs which can be formed from the three directions are at right angles to one another.

When there are equal roots this discussion fails; indeed the directions cannot be determined from equations (45). If there are two equal roots, (45) give one determinate direction; the other two are indeterminate. The solution is, however, that all lines at right angles to the determinate direction and passing through the origin meet the surface at right angles, that is the surface is one of revolution about the determinate axis. If all the roots are equal, all lines drawn through the origin meet the surface at right angles, and the surface is a sphere.

Now let the axis of ξ coincide with the direction of which the cosines α_1, β_1, γ_1 are given by substitution of a^2 for κ in (45). On this supposition $\alpha_1=1$, $\beta_1=0$, $\gamma_1=0$. Then (45) give $A=a,^2$ $F=0$, $E=0$. The equation of the surface then becomes

$$a^2\xi^2+B\eta^2+C\zeta^2-2D\eta\zeta=k^2.$$

If also the axis of η coincide with the line α_2, β_2, γ_2 we obtain $B=b^2$ and $D=0$. The remaining axis will coincide with the line α_3, β_3, γ_3, and (45) will give the additional result $C=c^2$. Thus the equation referred to the three axes passing through the origin and meeting the surface normally (the *principal axes*, as they are called) has the simple form

$$a^2\xi^2+b^2\eta^2+c^2\zeta^2=k^2, \tag{47}$$

where ξ, η, ζ are the co-ordinates of any point referred to the principal axes.

Every plane section of this surface is an ellipse (except those parallel to two planes in certain positions through the axis of intermediate length, which are circles), and the surface is therefore called an ellipsoid. It has three unequal axes, the squares of the semi-lengths

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of which are k^2/a^2, k^2/b^2, k^2/c^2, as is obvious from (47). The quantities a^2, b^2, c^2 are the moments of inertia about the principal axes.

The fact that the moments of inertia of a material system about axes in different directions through any point can be represented thus by the radii of an ellipsoid was discovered by Poinsot, and the ellipsoid just discussed is generally called *Poinsot's momental ellipsoid.*

Any diametral section is called the *momental ellipse* for that section.

It is possible, of course, to draw round any point an ellipsoid the equation of which is

$$\frac{\xi^2}{a^2}+\frac{\eta^2}{b^2}+\frac{\zeta^2}{c^2}=\frac{1}{k^2}. \tag{48}$$

This is called the reciprocal ellipsoid of the former.

Another and somewhat more recondite theorem connects the reciprocal ellipsoid drawn for the centre of inertia with the directions of the principal axes for any other point; but for this the reader is referred to general treatises on dynamics. [See Thomson and Tait's *Natural Philosophy*, vol. i. Part i. § 283.]

169. **Rigid Body turning round Fixed Horizontal Axis.**—We now consider some examples of the principles set forth above.

(1) A rigid body is under the action of gravity, and can turn freely round a fixed horizontal axis: it is required to determine the motion.

Taking equations (32), and supposing that the axis is the axis of x, we have (33), namely

$$\ddot{\phi}.\Sigma mr^2=\Sigma Rr.$$

Here ϕ is the angle which a line drawn at right angles to the axis to any point of the body makes with another perpendicular drawn from the same point of the axis and fixed in space. The acceleration $\ddot{\phi}$ is the same for all such lines drawn from the axis to points in the body. The quantity on the right is the sum of the moments of the applied forces round the axis.

The external applied forces acting on the body are the forces of gravity on the particles composing the body, and the forces applied to the body at the axis. The latter have no moment round the axis, and hence it is only necessary to take account in ΣRr of the forces of gravity. These are forces proportional to the masses of the particles, and act vertically downwards.

If then ϕ be the angle between a line drawn perpendicular to the axis to a particle P of mass m and a fixed perpendicular to the axis, as in Fig. 98, we have for R, the force of gravity on m, the value mg. The moment Rr is thus $mgr\sin\phi$, and ΣRr is $\Sigma mgr\sin\phi$. But $r\sin\phi$ is the distance y, say, of P from a vertical plane containing the axis, and therefore $\Sigma mr\sin\phi=\bar{y}\Sigma m$, or, if h be the length of a perpendicular let fall from the axis to the centroid, and θ the angle this perpendicular makes with the vertical, $\Sigma mr\sin\phi=h\sin\theta\Sigma m$. If then M denote the whole mass of the body $\Sigma Rr=Mgh\sin\theta$.

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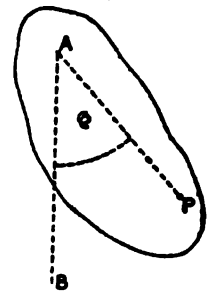
When θ is thus taken $Mgh\sin\theta$ always acts in the direction to diminish θ, and so the equation must be written in the form

$$\ddot{\theta}\Sigma mr^2 + Mgh\sin\theta = 0. \tag{49}$$

As we have seen above, $\Sigma mr^2 = M(h^2 + k^2)$ where k is the radius of gyration of the body about a parallel axis through the centroid, and therefore the equation just found may be written

FIG. 98.



$$\ddot{\theta}(h^2 + k^2) + gh\sin\theta = 0. \tag{50}$$

170. **Theory of Compound Pendulum.**—If θ be always small (50) becomes

$$-\frac{\ddot{\theta}}{\theta} = g\frac{h}{h^2 + k^2}. \tag{51}$$

When both numerator and denominator of the ratio on the left are multiplied by h, it becomes the ratio of the linear acceleration of the centroid along the circle in which it moves to its linear displacement along the same circle from the undisturbed position. The motion is therefore (§ 53) simple harmonic in the circular path, and the period, T, is given by the relation $4\pi^2/T^2 = gh/(h^2 + k^2)$. Comparing this with the equation (§ 138) for the period of a simple pendulum of length l, namely $4\pi^2/T^2 = g/l$, we see that the length of a simple pendulum which would oscillate in the same period as that in which the rigid body oscillates about the fixed axis, or, as it is called, the length of the equivalent simple pendulum, is given by the equation

$$l = \frac{h^2 + k^2}{h}. \tag{52}$$

A rigid body thus oscillating about a fixed horizontal axis is called a *compound pendulum.*

Writing the equation for l in the form

$$h^2 - hl + k^2 = 0,$$

and solving for h we find

$$h = \tfrac{1}{2}l \pm \tfrac{1}{2}\sqrt{l^2 - 4k^2}, \tag{53}$$

so that for a given value of l there are in general two values of h.

The value of h for the body is of course always real, and k is a constant; therefore values of l which give $l^2 < 4k^2$, that is which give imaginary values of h, must be excluded. The smallest possible value of l, therefore, for the body is $l = 2k$. The value of h is then k, and the pendulum has its minimum period for the given direction of the axis relatively to the body. The reader will verify at once that if h be very great (the rigid body being then supposed con-

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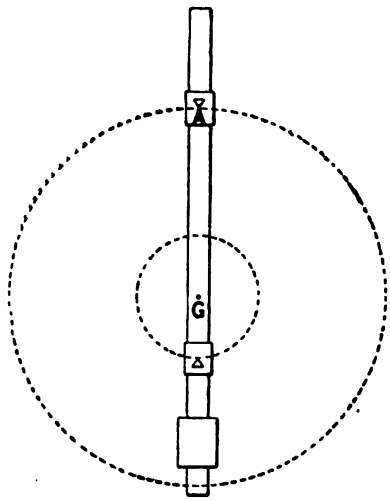
nected with the axis by a massless rigid framework) l is also very great, and that if h be very small l is again very great. But l is finite when h is, and so the existence of a minimum value of l is obvious.

In Fig. 99 let the plane of the paper represent the vertical plane in which the centroid G of the body moves, and A be the intersection of that plane with the axis. From G as centre, with the values of h

FIG. 99.



given by (53) as radii, describe two circles in the plane of the paper. Then if the body oscillate about a parallel axis through any point of either of these circles the period will have the value $2\pi\sqrt{l/g}$.

The value of l, it is to be observed, is the sum of these two radii, that is the sum of the two roots of the quadratic equation. The length of the equivalent simple pendulum is thus the length of a line drawn from a point in one of these circles through the centroid to meet the other circle beyond the centroid. The extremity of such a line at which the axis is situated is sometimes called "the centre of oscillation," and the other extremity "the centre of suspension," and the interchangeability of these two points is referred to as the "convertibility of the centres of oscillation and suspension." This is not, however, a proper mode of describing the result obtained, even if the names were above criticism, as there are obviously an infinite number of "centres of suspension" for which the period has the same value, and a corresponding number of "centres of oscillation."

171. **Determination of Acceleration due to Gravity by Compound Pendulum.**—In a form of pendulum invented apparently

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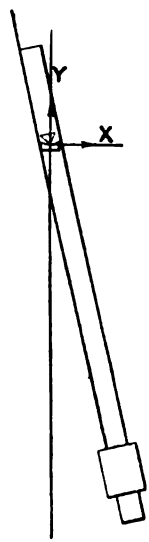
independently by de Prony, Bohnenberger, and Kater,* the result here obtained is taken advantage of for determinations of the value of g made at different positions on the earth's surface by means of pendulum observations. What is called a compound pendulum, consisting of a rigid rod carrying sliding weights which can be clamped in different positions on the rod, and adjustable knife-edges by which the pendulum can be suspended, is carried about from place to place, and made to oscillate wherever g is to be determined. The two knife-edges are so adjusted on opposite sides of the centroid of the pendulum that the period of oscillation about each is the same. The distance between them is then the length of the equivalent simple pendulum, unless both be at the same distance from the centroid. In the latter case the distance is twice one of the roots of the equation, not the sum of the roots (except in the particular case in which it is the minimum value of l), and does not give the value of g.

The mistake of taking twice one of the roots of the equation instead of the sum of the roots is not infrequent. This may very readily happen if the centroid is too nearly half-way between the knife-edges for their two positions, that is when the roots are nearly equal. In general, however, when the mistake is made it is announced by the outrageous value of g obtained.

FIG. 100.



If, as in some forms of the pendulum, the knife-edges are fixed while the distribution of matter in the pendulum is adjustable by sliding weights, a knife-edge should be between one of the sliding weights and the centroid.

The action of the air on the pendulum cannot be considered here. Trouble is caused by its not being quite the same for vibrations about one knife-edge as for vibrations about the other. This is obviated, however, in a form of compound pendulum due to Bessel and constructed by Repsold, in which the weights are moved for adjustment along the interior of a hollow tube, so that the external figure of the pendulum is the same for suspension about either knife-edge.

172. **Forces applied to Compound Pendulum by Supports.**—As another example of these equations we may determine the forces applied by the supports to a body turning about a fixed horizontal axis; for example the forces applied to the knife-edges of a compound pendulum by the plates on which they rest. Let the distribution of matter in the body be supposed to be symmetrical about the plane through the centroid perpendicular to the axis, and forces of gravity to be the only applied forces. Then

* See *Memoires Relatifs à la Physique, publiés par la Société Francaise de Physique*, tome iv.

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all the forces applied to the body at the axis will be reducible to a single force in that plane. Let X be the horizontal component of these forces, Y the vertical component, supposed to act towards the right and upwards respectively, as in Fig. 100. By the theorem stated in § 152 the acceleration of the centroid is the same as if all the forces acting there were transferred to that point, and the body were a particle there situated of mass equal to the total mass of the body. The acceleration has components $h\ddot{\theta}$ along the circle in which the centroid is constrained to move, and towards the lowest point, and $h\dot{\theta}^2$ towards the axis. Resolving the forces along the circle and towards the centre we have

$$\left.\begin{aligned} Mh\ddot{\theta} &= X\cos\theta + Y\sin\theta - Mg\sin\theta \\ Mh\dot{\theta}^2 &= Y\cos\theta - X\sin\theta - Mg\cos\theta \end{aligned}\right\} \quad (54)$$

But it has been seen above (50) that

$$(h^2+k^2)\ddot{\theta} + gh\sin\theta = 0,$$

so that the first of these equations becomes

$$-\frac{Mgh^2}{h^2+k^2}\sin\theta = X\cos\theta + Y\sin\theta - Mg\sin\theta.$$

This gives

$$X\cos\theta + Y\sin\theta = Mg\frac{k^2}{h^2+k^2}\sin\theta. \quad (55)$$

The quantity on the left is the component of force on the knife-edge towards the right perpendicular to the length of the pendulum.

But (50), multiplied by θ, is the time-rate of variation of

$$\tfrac{1}{2}(h^2+k^2)\theta^2 - gh\cos\theta$$

which therefore is constant. If θ_1 be the angle which the pendulum made with the vertical at the extremity of its range, that is when $\theta = 0$, this constant value is $-gh\cos\theta_1$. Hence

$$\tfrac{1}{2}(h^2+k^2)\dot{\theta}^2 = gh(\cos\theta - \cos\theta_1), \quad (56)$$

Hence the second equation of (54) becomes

$$\frac{2Mgh^2}{h^2+k^2}(\cos\theta - \cos\theta_1) = Y\cos\theta - X\sin\theta - Mg\cos\theta.$$

This leads to

$$Y\cos\theta - X\sin\theta = \frac{Mg}{h^2+k^2}\{(3h^2+k^2)\cos\theta - 2h^2\cos\theta_1\}. \quad (57)$$

The quantity on the left is the force on the knife-edges upward parallel to h. From (55) and (57) the reader may at once find X, Y.

173. **Material System under the Action of Parallel Forces.**—Consider a system of particles to which a system of parallel forces P_1, P_2, P_3, ... is applied at points a_1, a_2, ..., and let the cosines of the angles which the direction of these forces makes with the axes of

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x, y, z be α, β, γ, then the x-components of the forces are $P_1\alpha, P_2\alpha, \ldots$, the y-components are $P_1\beta, P_2\beta, \ldots$, and the z-components $P_1\gamma, P_2\gamma, \ldots$. If these be the only forces the motion of the centroid of the body is the same as if the component forces $(P_1 + P_2 + \ldots)\alpha$, $(P_1 + P_2 + \ldots)\beta$, $(P_1 + P_2 + \ldots)\gamma$ acted on the system collected into a particle at that point. That is the equations of motion of the centroid are

$$\left.\begin{aligned} M\ddot{x} &= (P_1 + P_2 + \ldots)\alpha \\ M\ddot{y} &= (P_1 + P_2 + \ldots)\beta \\ M\ddot{z} &= (P_1 + P_2 + \ldots)\gamma \end{aligned}\right\} \qquad (58)$$

These give, what we see without them must be the case,

$$M\ddot{s} = P_1 + P_2 + \ldots \qquad (59)$$

where $\ddot{s}$ is the acceleration of the centroid in the direction of the forces.

174. **Moments of Parallel Forces. Centre of Parallel Forces.**—By equations (27), if x, y, z be the co-ordinates relatively to the centroid of the system of a particle of mass m, we have the equations

$$\left.\begin{aligned} \Sigma m(\ddot{z}y - \ddot{y}z) &= \Sigma P(\gamma y - \beta z) \\ \Sigma m(\ddot{x}z - \ddot{z}x) &= \Sigma P(\alpha z - \gamma x) \\ \Sigma m(\ddot{y}x - \ddot{x}y) &= \Sigma P(\beta x - \alpha y) \end{aligned}\right\} \qquad (60)$$

which express the fact that the rates of change of moment of momentum of the system of particles about the axes of x, y, z through the centroid are equal to the moments of the system of external applied forces about these axes. Now consider one of the quantities on the right of those equations, say $\Sigma P\beta x$. Since β is a constant this may be written $\beta\Sigma Px$. If, then, we determine a quantity ξ such that

$$\xi\Sigma P = \Sigma Px$$

we can replace ΣPx by $\xi\Sigma P$. Similarly ΣPy, ΣPz can be replaced by $\eta\Sigma P$, $\zeta\Sigma P$. Thus the equations written above become

$$\left.\begin{aligned} \Sigma m(\ddot{z}y - \ddot{y}z) &= (\eta\gamma - \zeta\beta)\Sigma P \\ \Sigma m(\ddot{x}z - \ddot{z}x) &= (\zeta\alpha - \xi\gamma)\Sigma P \\ \Sigma m(\ddot{y}x - \ddot{x}y) &= (\xi\beta - \eta\alpha)\Sigma P \end{aligned}\right\} \qquad (61)$$

The point whose co-ordinates are ξ, η, ζ coincides with the centroid of a system of particles of masses proportional to $P_1, P_2, \ldots$ placed at the points where these forces are regarded as applied to the system. The point is called the *centre of parallel forces*. The equations indicate that the moments of the forces round the axes will not be affected by replacing the system of parallel forces by a single force in the same direction equal to their sum passing through the centre of parallel forces.

It is also to be observed that the same results are obtained by taking the point of application of each of the parallel forces *anywhere on its line of action*. For any such alteration in the positions

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of the points of application can only shift the position of the centre of parallel forces along a line parallel to the forces, which does not affect the result of their replacement by a single force in the same direction equal to their sum and passing through the centre of parallel forces. This is the so-called principle of transmissibility of force along its line of action. Such a force made to replace the system of parallel forces is called the resultant of the system of parallel forces. Thus by (61) the resultant of two parallel forces P, Q

FIG. 101.

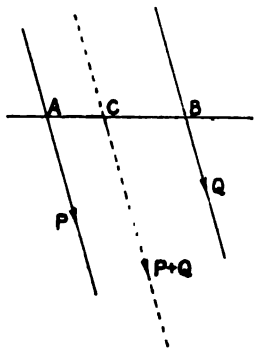
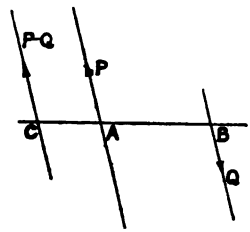


FIG. 102.



is a force $P+Q$ in the same direction, cutting any line AB (Fig. 101) drawn across the lines of action of the two forces from a point A in the line of action of one to a point B in the line of action of the other, so that the segments AC, CB are inversely proportional to the forces P, Q in the adjacent lines of action, that is $P/Q = CB/AC$.

If the forces are in opposite directions, as in Fig. 102, the resultant is the algebraic sum, and acts in the direction of the greater force. Its line of action cuts the line AB produced in a point C beyond the line of action of the greater force, and the same relation holds as before.

175. **Couples.**—We can now consider the case of two equal and opposite forces in different lines applied to a system of particles. Such a system of forces is called a *couple*. It is represented by Fig. 101 if $P=Q$. In this case, however, C is infinitely remote from A, since $P=Q$ gives $AC=BC$. By the theorems proved above the application of a couple to a body does not affect the motion of the centroid, since the forces transferred to that point give zero force on the whole.

Again we must have in this case also by (31)

$$\Sigma m(2r\dot{r}\dot{\theta}+r^2\ddot{\theta})=\Sigma Pr,$$

where r on the left is the distance of a particle m from an axis at right angles to the plane in which the forces P are applied, and on

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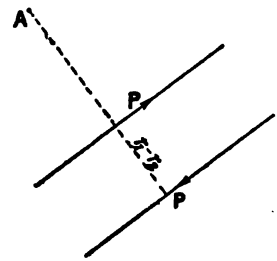
the right r is the distance of the line of a force P from the axis. We have then $\Sigma Pr = P(r_1 - r_2)$ where r_1, r_2 are the distances of the two forces of the couple from the axis. Thus the sum of the moments of the forces round the axis is equal to the product of either force into the distance between the forces, and is therefore independent of the absolute distance of the forces from the axis, that is the moment of the forces is the same for all parallel axes.

The sum of the moments is of course, according to the directions taken as positive for the rotational motions considered above, positive when the forces and distances are as shown in Fig. 103.

FIG. 103.



The product $P(r_1 - r_2)$ is called the moment of the couple. It is graphically represented by a line of length numerically equal to the moment drawn at right angles to the plane in which the forces act, towards that side of the plane from which the forces seem to an observer to be as shown in Fig. 103. This line, when given in magnitude and direction, is called the *axis* of the couple. It will be observed that the axis is completely defined by its magnitude and direction. It has no determinate direction.

It is clear by equations (25) and (29) or (31) that it is necessary for the equilibrium of any set of forces that (1) the sum of the components of the forces in every direction should vanish, (2) the sum of the moments of the forces round any axis whatever should also be zero. These conditions will not be sufficient for equilibrium in the case of a system of particles (whether acting on one another or not) which do not form a rigid system, as in that case, though the expressions on the right of equations (31) vanish, they only do so because the expressions of the form

$$\Sigma m(2r\dot{r}\dot{\theta} + r^2\ddot{\theta})$$

all vanish. In this case it is necessary to know the internal as well as the applied forces wherever there is deviation from rigidity. If the system be rigid the conditions stated above are both necessary and sufficient.

It is clear that a couple acting on a rigid system can only be equilibrated by a couple of equal and opposite moment in the same plane or a parallel plane. For only then will the sum of the moments of the forces round any axis that may be chosen be zero. A couple, therefore, given in any plane may be transferred without change of effect to a parallel plane if its moment and direction are left unaltered. Also two couples are equivalent if their planes are parallel and their moments are equal and in the same direction, that is if the same axis will represent either.

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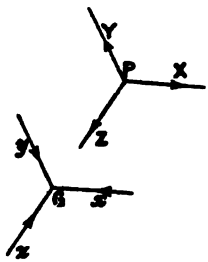
176. **Poinsot's Mode of Resolution of Forces.**—Equations (29) may be expressed by means of couples as follows. Let (X, Y, Z) represent the actual force applied to a particle m at the point $P(x, y, z)$ as before. Then introduce at the centroid two forces equal and opposite to (X, Y, Z). The three forces thus obtained form a force equal to and in the same direction as the given force, but applied at the centroid, and a couple the moments of which in the planes of yz, zx, xy are $Zy' - Yz'$, $Xz' - Zx'$, $Yx' - Xy'$ where x', y', z' are the co-ordinates of m relatively to the centroid. These couples are the moments of the given force about axes of x, y, z drawn through the centroid. This process gone through for all the forces of the system gives

FIG. 104.



$$\Sigma(Zy' - Yz'),\ \Sigma(Xz' - Zx'),\ \Sigma(Yx' - Xy')$$

for the moments of the couples in the planes of yz, zx, xy. These sums of moments of couples are the sums of the moments of the external applied forces about axes of x, y, z passing through the centroid. Thus we may regard the quantities on the right of equations (29) which give the angular accelerations of the body as the sums of moments of the couples obtained in this way.*

It is to be observed that the couples give the rate of change of the moments of momentum about the axes through the centroid, the forces at the centroid equal and parallel to the external applied forces give the motion of the centroid, so that if the body is rigid the motion is completely determined.

177. **Poinsot's Central Axis. System of Forces called a Wrench.**—The same method may be employed to transfer the forces to any point the co-ordinates of which are ξ, η, ζ. The components ΣX, ΣY, ΣZ are not changed, but the moments of the couples the axes of which are parallel to the axes of co-ordinates become

$$\Sigma\{Z(y-\eta) - Y(z-\zeta)\},\ \ldots$$

or if L, M, N denote the couples when the forces are transferred to the origin, and P, Q, R denote ΣX, ΣY, ΣZ,

$$L - Q\zeta + R\eta,\ M - R\xi + P\zeta,\ N - P\eta + Q\xi.$$

Now let ξ, η, ζ be so chosen that the axis of the resultant of these couples is parallel to the resultant force; then we have

$$\frac{L - Q\zeta + R\eta}{P} = \frac{M - R\xi + P\zeta}{Q} = \frac{N - P\eta + Q\xi}{R}. \qquad (62)$$

* This mode of resolution has been used with great effect by Poinsot, *Théorie nouvelle de la Rotation des Corps* (*Liouville's Journal*, 16, 1851). See also *On the Percussion of Bodies*, by M. Poinsot (translated from *Comptes Rendu*, vol. 48, by T. A. Hirst), *Phil. Mag.* 15, 18, 19 (1858, 1859, 1860).

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These are the equations of a straight line. If F be put for the resultant of P, Q, R, and G for that of L, M, N, these equations can be transformed to

$$QRF\xi - Q(MF - QG) = PRF\eta + P(LF - PG) = PQF\zeta,$$

which show that the line passes through the fixed point of which the co-ordinates are

$$\frac{MF - QG}{RF}, \; -\frac{LF - PG}{RF}, \; 0.$$

The line thus found was called by Poinsot the *central axis* of the system of forces. The action of the forces reduces to that of a single force F along the central axis, and to a single couple of moment G the plane of which is at right angles to the central axis. The system is then said to constitute a *wrench of pitch* G/F.

By the equations of the central axis (62) the component couples given above reduce to L, M, N, so that the moment of the resultant couple is G.

It is possible to reduce the system to a couple and a single force in a straight line parallel to the central axis by introducing in that straight line two opposite forces, each equal to F. One of these with the F along the central axis will give a couple in the plane of the two parallel straight lines, the other force will be the single force required. But the couple now obtained must be compounded with G to give the resultant couple, which is therefore greater than G. The mode of resolution adopted therefore gives the minimum value which G can have with the choice of axes made.

It will be observed that the moments L, M, N of the component couples depend upon the position of the origin as well as on the directions of the axes, and that the central axis has only one direction and one position in space.

The reduction of a system of forces to a wrench, and other modes of reduction will be referred to in chap. v.

178. **Impulsive Forces.**—We now consider what are called impulsive forces. In many circumstances—for example, in cases of collision—a very great force acts on a body or system for a very short interval of time, and a finite change of motion of the body is produced. As, however, the interval of time is very short, and the velocity of every part of the system is always finite, the system is only slightly changed in configuration during the interval. If the interval be vanishingly small the change of configuration is zero.

A good example of impulse is a blow from a hammer. Experience shows that the amount of the impulsive force in such a case is very great, as an effect in crushing a stone or altering the shape of a piece of hot iron is produced, which could not be effected by steadily applied force unless the force were very great. The momentum of the hammer-head is not itself of very great amount, but in the impact the velocity is diminished with great rapidity, that is $m\dot{v}$ for the

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hammer-head is numerically very great, and therefore may exceed the resistance of the material.

Again, when a portion of a plate of wood or iron is struck, and a hole is punched in it (*e.g.*, in the well-known experiment of firing a candle through a deal board), a large change of momentum is locally and suddenly impressed on the portion of material struck before the part has moved so far from its initial position as to develop by strain of the body sensible stress, resisting the motion of the part affected on the one hand and displacing the rest of the material on the other. In consequence of the inertia of the latter, however, resistance is offered up to the limit fixed by the strength of the material.

Another example of the same sort is that of a coin laid on a card which covers the mouth of a glass. When the card is flicked out by a sharp blow from a paper-knife, the coin falls into the glass. The coin does not go with the card when the latter is suddenly set into motion, inasmuch as the utmost force the card can exert on the coin, that due to friction, is less than the force required to overcome the coin's inertia sufficiently quickly.

The time integral of an impulsive force F, that is $\int_0^\tau F dt$, taken over the very short interval τ during which it acts, is called the impulse of the force, and of course measures the change of momentum produced by the impulsive force in the interval. We shall denote the component impulses in the directions x, y, z applied at any point by P, Q, R. Let the change of velocity of a particle m at x, y, z be from $\dot{x}_0$ to $\dot{x}$, then the change of momentum is $m(\dot{x}-\dot{x}_0)$: the changes of momentum in the other directions are similarly $m(\dot{y}-\dot{y}_0)$, $m(\dot{z}-\dot{z}_0)$. Instead of equations (25) we have

$$\Sigma m(\dot{x}-\dot{x}_0)=\Sigma P,\ \Sigma m(\dot{y}-\dot{y}_0)=\Sigma Q,\ \Sigma m(\dot{z}-\dot{z}_0)=\Sigma R. \qquad (63)$$

From these follow of course as before

$$\Sigma m(\bar{\dot{x}}-\bar{\dot{x}}_0)=\Sigma P,\ \Sigma m(\bar{\dot{y}}-\bar{\dot{y}}_0)=\Sigma Q,\ \Sigma m(\bar{\dot{z}}-\bar{\dot{z}}_0)=\Sigma R. \qquad (63')$$

The equations of moments (27) become

$$\left.\begin{aligned}\Sigma m\{(\dot{z}-\dot{z}_0)y-(\dot{y}-\dot{y}_0)z\}&=\Sigma(Ry-Qz)\\ \Sigma m\{(\dot{x}-\dot{x}_0)z-(\dot{z}-\dot{z}_0)x\}&=\Sigma(Pz-Rx)\\ \Sigma m\{(\dot{y}-\dot{y}_0)x-(\dot{x}-\dot{x}_0)y\}&=\Sigma(Qx-Py)\end{aligned}\right\} \qquad (64)$$

These state that the changes in the moments of momentum of the system

$$\Sigma m(\dot{z}y-\dot{y}z),\ \Sigma m(\dot{x}z-\dot{z}x),\ \Sigma m(\dot{y}z-\dot{x}y)$$

about the axes, produced in the short interval during which the impulses act, are equal to the sum of the moments of the impulses about the axes.

As before the last three equations split into two sets; the equations

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$$\left.\begin{aligned}\Sigma m\{(\dot{z}'-\dot{z}'_0)y'-(\dot{y}'-\dot{y}'_0)z'\}&=\Sigma(Ry'-Qz')\\ \Sigma m\{(\dot{x}'-\dot{x}'_0)z'-(\dot{z}'-\dot{z}'_0)x'\}&=\Sigma(Pz'-Rx')\\ \Sigma m\{(\dot{y}'-\dot{y}'_0)x'-(\dot{x}'-\dot{x}'_0)y'\}&=\Sigma(Qx'-Py')\end{aligned}\right\}\qquad(65)$$

and another set which can be written down from (30) above, but which we shall hardly have occasion to use.

Equations (65) state that the changes of moment of momentum about axes of co-ordinates through the centroid are equal to the sum of the moments of the applied impulses about the respective axes. These changes are independent of the changes of motion of the centroid, which are given by (63′).

For a rigid body the equations (65) become

$$\left.\begin{aligned}\Sigma mr_x^2(\dot{\phi}-\dot{\phi}_0)&=\Sigma S_x r_x\\ \Sigma mr_y^2(\dot{\chi}-\dot{\chi}_0)&=\Sigma S_y r_y\\ \Sigma mr_z^2(\dot{\psi}-\dot{\psi}_0)&=\Sigma S_z r_z\end{aligned}\right\}\qquad(66)$$

where (if ϕ, χ, ψ be the angles specified in § 158)

$$S_x=(R\cos\phi-Q\sin\phi)r_x,\quad S_y=(P\cos\chi-R\sin\chi)r_y,$$
$$S_z=(Q\cos\psi-P\sin\psi)r_z,$$

that is the impulses perpendicular respectively to the axes of x, y, z.

The reader will observe that the second terms in brackets on the left in (31) yield no terms for the equations of impulses. The time-integrals of these terms vanish since each integral must be of the form $2[r\dot{r}\dot{\phi}]\tau$, where square brackets indicate the mean value of the quantity enclosed, and τ is the time-interval. The multiplier of τ is finite, and τ is indefinitely small; hence the terms are insensible.

179. **Robins' Ballistic Pendulum.**—The following is a good example of impulsive forces. A rigid pendulum with a massive bob is free to turn about a horizontal knife-edge, and the position of its centroid, and its moment of inertia about an axis through the centroid parallel to the knife-edge have been determined. A bullet of mass m moving horizontally in the plane through the centroid at right angles to the knife-edge, is received and retained by the bob, and the angular deflection of the pendulum produced is observed. It is required to determine the velocity of the bullet.

Let M be the mass of the pendulum including the bullet, h be the distance of the centroid from the axis, k the radius of gyration of the pendulum about the axis through the centroid parallel to the knife-edge, z the distance of the line of motion of the bullet from the knife-edge, and θ_1 the total angle through which the pendulum is turned. The moment of momentum of the bullet before impact is mvz about the knife-edge, and this, by the principle just established, must be equal to the moment of momentum of the pendulum and bullet about the same axis after impact. Thus we obtain, if $\dot{\theta}_0$ be the angular velocity just after impact,

$$mvz=M(h^2+k^2)\dot{\theta}_0.\qquad(67)$$

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But the pendulum in swinging round has its angular velocity diminished by the action of gravity. In fact we have

$$-\ddot{\theta}=g\frac{h}{h^2+k^2}\sin\theta.$$

But this gives us

$$-\dot{\theta}\ddot{\theta}=g\frac{h\dot{\theta}}{h^2+k^2}\sin\theta.$$

or multiplying by an infinitesimal element of time dt

$$-\dot{\theta}d\dot{\theta}=g\frac{h}{h^2+k^2}\sin\theta d\theta,$$

that is

$$-\tfrac{1}{2}d(\dot{\theta}^2)=g\frac{h}{h^2+k^2}\sin\theta d\theta,$$

which gives

$$\tfrac{1}{2}\dot{\theta}_0{}^2=g\frac{h}{h^2+k^2}(1-\cos\theta_1)=2g\frac{h}{h^2+k^2}\sin^2\frac{\theta_1}{2}.$$

Substituting in (67) and reducing a little we obtain

$$v=2\sqrt{\frac{gh}{h^2+k^2}}\left\{\frac{M}{m}\frac{h^2+k^2}{z}\right\}\sin\frac{\theta_1}{2}. \tag{68}$$

Sometimes the angle θ_1 is measured by a tape attached to a point under the bob, and pulled out by the pendulum against friction applied by a light spring, so that the length of tape drawn out measures the chord of the arc described by its point of attachment as the pendulum swings through the angle θ_1. Let l be the distance of the point of attachment from the knife-edge, and s the length of tape drawn out; then $s/l=2\sin\frac{1}{2}\theta_1$. Thus we obtain

$$v=\sqrt{\frac{gh}{h^2+k^2}}\left\{\frac{M}{m}\frac{h^2+k^2}{z}\right\}\frac{s}{l}. \tag{69}$$

If T be the period of free oscillation of the pendulum about its knife-edges, we have

$$T=2\pi\sqrt{\frac{h^2+k^2}{gh}},$$

so that

$$v=\frac{2\pi}{T}\left\{\frac{M(h^2+k^2)}{mz}\right\}\frac{s}{l}. \tag{70}$$

180. **Example of Impulse: Straight Uniform Rod on Smooth Table.**—As another example of impulsive forces consider a straight uniform rod lying on a smooth horizontal table, and let an impulse be applied at one extremity A in the horizontal direction at right angles to the rod.

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Let the magnitude of the impulse be I, then if m be the mass of the rod the motion of the centroid is given by

$$m\dot{x}_c = I,$$

where $\dot{x}_c$ is the velocity of the centroid at right angles to the rod just after the impulse has been applied, since the rod before the application of the impulse was at rest.

Again, according to (66) the angular velocity of the rod about a vertical axis through the centroid is given by

$$m\frac{a^2}{3}\dot{\theta} = Ia$$

where a is the half length of the rod. This may be written

$$ma\dot{\theta} = 3I.$$

The velocity of the extremity A is therefore $a\dot{\theta} + \dot{x}_c$ or $4I/m$. The point at a distance of $a + \frac{1}{3}a$ from the end A is thus at rest, that is the bar turns round a point distant from A two-thirds of the length of the bar. This point is sometimes called the *centre of percussion*.

We infer that if a vertical pin fixed in the table passed through a hole in the bar at the centre of percussion the pin would experience no impulse, and it is quite easy to prove this formally.

Let the bar be free to turn round a vertical pin at a distance r from the end A and beyond the centroid, and let P_1, P_2 be horizontal impulses at right angles to and along the bar applied by the pin, and let the same impulse as before be applied to A. Then we have for the centroid

$$m\dot{x}_c = P_1 + I, \quad \tfrac{1}{3}ma^2\dot{\theta} = Ia - P_1(r-a). \tag{71}$$

There is no motion of the bar in the direction of its length. The impulse applied to the bar by the pin in consequence of the sudden production of rotation is

$$m(r-a)\int_0^\tau \dot{\theta}^2 dt = P_2,$$

and since $\dot{\theta}^2$ is always finite and τ is infinitely small this time-integral must be zero. Hence $P_2 = 0$.

Now the final value of $\dot{\theta}$ is $\dot{x}_c/(r-a)$, so that for the second of (71) we have

$$ma^2\frac{\dot{x}_c}{r-a} = 3Ia - 3P_1(r-a). \tag{72}$$

Eliminating $\dot{x}_c$ between this equation and the first of (71) we obtain

$$P_1 = -\frac{a(4a-3r)}{3(r-a)^2 + a^2}I. \tag{73}$$

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If $r = 4a/3$ the impulse P_1 is zero, as stated above.

From this it follows that if a uniform rod turning round an axis at right angles to its length, at a distance of two thirds of its length from one end, suddenly impinge at that end on an obstacle there will be no impulse on the axis. Or again if a rod turning round one end strike on an obstacle at a distance of two-thirds of its length from that end there will be no impulse at the axis. This can easily be verified by noticing the absence of jar when a bar of iron held by one end is struck at its centre of percussion against an anvil.

The centre of percussion can be found for a body of any form by a method similar to that indicated above.

181. **Example of Impulse: Body moving on a Smooth Horizontal Table.**—As another example consider a thin disk of any form moving

FIG. 105.

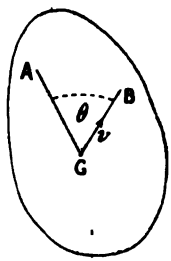
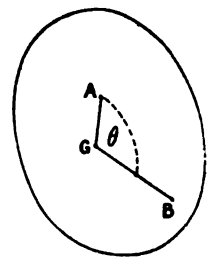


FIG. 106.



without rotation on a smooth horizontal table, and let a point A, Fig. 105, at distance x from the centroid be suddenly fixed.

Let the point be on a line of the body making an angle θ with the direction of motion GB. The velocity of the point A is v in the direction GB, and this is to be annulled by an impulse P applied at A in the opposite direction. A reaction applied by the axis at A annuls the velocity of G along GA. If the velocity of the centroid just after the application of the impulse is v', we have

$$m(v\sin\theta - v) = P\sin\theta, \tag{74}$$

and for the instantaneous motion of the body about the centroid G

$$mk^2\dot{\theta} = Px\sin\theta, \tag{75}$$

where k^2 is the square of the radius of gyration of the body about a vertical axis through G, and $\dot{\theta} = v'/x$. Thus the last equation becomes, with the insertion of the value of P given by (74),

$$v' = \frac{x^2\sin\theta}{k^2 + x^2}v. \tag{76}$$

Thus v' is zero for $\theta = 0$ or $\theta = \pi$, and is $vx^2/(k^2 + x^2)$ for $\theta = \pi/2$.

In the particular case of a uniform disk of radius r moving in its

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own plane we have in the preceding equations to put $k^2 = \frac{1}{2}r^2$. In this case for $x = r$ and $\theta = \pi/2$, v' becomes $2v/3$, that is the velocity of the centre of the disk is reduced to two thirds of its former value.

182. **Further Examples of Impulse. Effect of Freeing and Fixing Axes of Rotation.**—The following example is instructive. A body of any form is given rotating about an axis A. That axis is suddenly freed and a parallel axis B is fixed: it is required to find the angular velocity about the new axis. Let a, b be the distances of a parallel axis through the centroid G from the first axis and the second respectively, θ the angle between perpendiculars GA, GB drawn from the centroid to the axes as shown in Fig. 106. By § 155 above the moment of momentum, about the point B in space, of the body revolving about A is $mk^2\omega + m\omega ab\cos\theta$, where k is the radius of gyration of the body about the axis through G. This cannot be altered by the impulse applied at B to fix that axis. Thus if ω' be the new angular velocity about B, we have

$$m(k^2 + b^2)\omega' = m(k^2 + ab\cos\theta)\omega,$$

so that

$$\omega' = \frac{k^2 + ab\cos\theta}{k^2 + b^2}\omega. \tag{77}$$

As a particular case let the body be a uniform disk of radius r with its plane at right angles to the axes, then

$$\omega' = (r^2 + 2ab\cos\theta)\omega/(r^2 + 2b^2).$$

If, further, A be at the centre and B on the edge of the disk $a = 0$, $\theta = 0$, $b = r$, so that $\omega' = \frac{1}{3}\omega$.

The slightly more difficult problem in which the axes are not parallel may be dealt with in a similar manner.

183. **Body supported under Gravity. Effect of Removal of a Support.**—Again, let a uniform beam be suspended by two vertical cords (or supported by two props) at equal distances a from the centroid. The force applied to the beam by each support is $W/2$, if W be the gravity of the beam. One of the supports is removed, and it is required to determine the initial acceleration of the beam, and the force applied by the remaining support.

The removal of the support is clearly equivalent to the instantaneous application of a downward force $W/2$ to the beam at the point of attachment of the support. Thus for the instantaneous motion of the centroid we obtain, putting m for the inertia of the beam, F for the force applied by the remaining support, and $\ddot{x}$ for the downward acceleration of the centroid,

$$m\ddot{x} = W - F.$$

For the angular acceleration about a horizontal axis at the centroid we get

$$mk^2\ddot{\theta} = Fa.$$

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But clearly $\ddot{\theta} = \ddot{x}/a$, and therefore by the first equation we find

$$F = \frac{k^2}{a^2 + k^2} W.$$

This gives

$$\ddot{x} = \frac{a^2}{a^2 + k^2} \frac{W}{m}, \quad = \frac{a^2}{a^2 + k^2} g,$$

and the motion is determined.

If the length of the beam be great in comparison with either transverse dimension $k^2 = \frac{1}{3}a^2$ and $F = \frac{1}{4}W$.

184. **Force due to a Jet of Particles.**—When a body is bombarded by a stream of particles, it experiences a force which is measured by the rate of change of the momentum of the particles. Thus if the average mass of a particle be m, and its velocity before impact be v, and after impact v', in the same direction, the change of momentum experienced by the particle is $m(v - v')$. If N such particles strike the body in the same direction in each second of time the average force experienced by the body is $Nm(v - v')$.

For example, if each particle have its velocity reversed from v to $-v$, that is if the particles rebound with the velocity of approach to the body, the force is $2Nmv$.

If the particles striking the body be distributed over an area S of the surface of the body perpendicular to the direction of motion, the average force per unit of area exerted on the surface will be $Nm(v - v')/S$. This is called the average *pressure* exerted on the surface. The pressure exerted by a gas on the walls of the containing vessel is explained in this way.

Again, let a body, a truck for example on a railway, carry a tank of water, and let the water issue in a horizontal jet from one end of the carriage. If a be the effective area of the orifice, and v the velocity of efflux of the water, that is the velocity relatively to the orifice, the volume of water that issues in a second is av, and its mass is ρav where ρ is the density of the water. Whatever may be the velocity of the water, relatively to the railway, the momentum lost from the tank and given to the issuing water in unit of time is ρav^2. This is the horizontal force exerted by the contents of the tank. and therefore by the truck, on the issuing water, and measures the force in the opposite direction exerted on the truck by the jet.

Thus if the truck be at rest, the force is still ρav^2.

Again, a wheel may be driven by a jet impinging upon it. Conceive for example a wheel free to turn on a horizontal axis, and played upon by a horizontal jet of particles (say the bullets from a rapidly worked machine-gun) in the plane of the wheel and at a distance h from the axis. If m be the mass of a particle, v its velocity, and N the number received by the wheel per second, the moment of momentum absorbed by the wheel per second is, if the velocities of the particles are just annulled, $Nmvh$, and this is the

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moment round the axis of the force accelerating the motion of the wheel. The force on the bearings is Nmv.

Forces of the nature here indicated will be discussed more fully in connection with their applications.

185. **Dimensions of Momentum, Force, and Impulse.**—Since momentum is the product of mass into velocity the unit of momentum is that of the unit of mass moving with the unit of velocity; thus for a given momentum we may write the equation

$$momentum = mv[LMT^{-1}],$$

where m is the number of units of mass in the body, v the number of units of velocity, L the unit of length, M the unit of mass, T the unit of time. If we change to units L', M', T' such that $L = \lambda L'$, $M = \mu M'$, $T = \tau T'$, this equation may be written

$$momentum = mv\lambda\mu\tau^{-1}[L'M'T'^{-1}],$$

where $\lambda\mu\tau^{-1}$ is the change-ratio.

Since force is rate of change of momentum the unit of force is that which gives the unit of mass the unit of acceleration, so that the unit of force may be denoted by $[LMT^{-2}]$; thus we may write the equation for a given force of numerical value F,

$$force = F[LMT^{-2}].$$

Changing the units as before to L', M', T', we have

$$force = F\lambda\mu\tau^{-2}[L'M'T'^{-2}],$$

where $\lambda\mu\tau^{-2}$ is the change-ratio.

An impulse is the time-integral of a force, and must therefore be capable of being represented by the product of a force into a time; its dimensional formula and change-ratio are therefore the same as those for momentum.

If we change the units from the foot, the pound, the second to the centimetre, the gramme, and the second, we have $\lambda = 30{\cdot}479$, $\mu = 453{\cdot}6$, $\tau = 1$; and therefore the change-ratio for momentum and impulse, namely $\lambda\mu\tau^{-1}$, has the value in this case

$$30{\cdot}48 \times 453{\cdot}6 = 13825.$$

Since $\tau = 1$ the change-ratio for force evidently has the same value for the change of units here specified.

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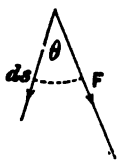
CHAPTER IV.

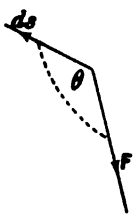
WORK AND ENERGY.

186. **Definition and Measurement of Work.**—We now consider what is termed the work done by any force of a system in any actual displacement of the system. If ds be any displacement of a particle to which the force F is applied, and the direction of this make an angle θ with the direction of the force, as in Fig. 107, the product $F\cos\theta.ds$ is called the *work* done by the force F in the displacement ds.

FIG. 107.



If θ be less than $\pi/2$ the component of the displacement is in the direction of F, and the work done *by* the force is positive. If θ be greater than $\pi/2$ the component is really in the direction opposite to that of F. We have then $-F\cos\theta.ds = F\cos(\pi-\theta)ds$, and each of these quantities is positive. Work is here done *against* the force F by other forces of the system. Work done *by* a force is taken positive, that done *against* a force is negative work done by the force.



For example, let a particle move downwards along a path inclined at an angle θ to the horizontal. Then if W be the gravity of the particle acting vertically down, the work done by the force W in a displacement s along the path is $Ws\sin\theta$.

This is the work done by the gravity of a particle in a displacement of it along an inclined plane, but is not necessarily the whole work done on the particle. If other forces act on the particle their works are to be estimated by the same rule.

187. **Work done in Finite Displacement by a Varying Force.**—If a varying force act on a body so that the place of application of the force is moved along any path, the whole work done by the force must be obtained by supposing the path divided up into a number of indefinitely short elements, $ds_1, ds_2, \ldots, ds_n$, then finding the proper average force, $F_1, F_2, \ldots, F_n$, for each element, and calculating the sum $F_1\cos\theta_1 ds_1 + F_2\cos\theta_2 ds_2 + \ldots + F_n\cos\theta_n ds_n$, where $\theta_1, \theta_2, \ldots, \theta_n$ are the angles between the successive elements and the respective forces $F_1, F_2, \ldots, F_n$. This sum is called the *space-integral* of the force, and

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is usually denoted by the formula $\int F\cos\theta\, ds$. Thus we have, putting W for the work done,

$$W = \int F\cos\theta\, ds,$$

in which the space-integral is taken along the path of displacement.

The space-integral of a force is thus the quantity called work; the time-integral of a force is momentum. We have considered the time-integral of an impulsive force above, and seen that while we may know nothing of the mode of variation of the impulsive force, we are able to reckon the amount of the impulse by the momentum generated. When an impulsive force acts it also does work which, by what has just been said, is expressible as a space-integral. We shall see how to reckon it when we have considered kinetic energy.

188. **Rate of Working or Activity.**—If v be the velocity of the point of application of the force F at any element ds of the path, and dt be the interval of time taken to traverse the element, we have $ds = vdt$, and the equation of work becomes

$$W = \int Fv\cos\theta\, dt, \tag{1}$$

in which the integral is taken over the whole time during which the motion lasts. The work is thus from a space-integral of F converted into a time-integral of the quantity $Fv\cos\theta$. This may be regarded either as the product of F into the component of velocity in the direction of F, or as the product of v into the component of F in the direction of v. $Fv\cos\theta$ is called the rate of working of the force F, or sometimes the *activity* of F. Sometimes also the word *power* is used in this sense.

189. **Unit of Work.**—For the unit of work we have the unit of force, and the unit of displacement in the direction of the force, or such a force F or $F\cos\theta$, and such a displacement $ds\cos\theta$ or ds, that the product $F\cos\theta\, ds$ is unity. When the unit of force is the dyne, and the unit of displacement is the centimetre, the unit of work is the work done by a force of one dyne in a displacement in the same direction of one centimetre. This is called an *erg*. When the unit of force is the gravity of one gramme, and the displacement is one centimetre in the same direction, the work done is g ergs (where g is the number of dynes in the gravity of a gramme, a number, rather greater than 981, depending on the place at which the force is measured), and this work is called a *centimetre-gramme*. Again, when the unit of force is what is called a poundal, and the displacement is one foot, the work done is called a *foot-poundal*; and last, when the unit of force is the gravity of a pound, and the unit of displacement is one foot, the work done is g foot-poundals (where g is the number of poundals in the gravity of a pound, a number, rather greater than 32, depending on the place where the force is measured), and this is called a *foot-pound*. [For values of g see *Gravitation* below.]

190. **Unit of Activity.**—When the activity is such that one unit of work is done per second, it is taken as unity. Thus as

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different units of activity we have 1 *erg per second*, 1 *centimetre-gramme per second*, 1 *foot-poundal per second*, 1 *foot-pound per second*. For many practical purposes these units are too small, and larger units are employed. Thus for the rate of working of steam-engines, gas-engines, and other prime movers, the unit of activity is 1 horse-power, which is defined as an activity of 33,000 *foot-pounds per minute*, or, which is of course the same thing, 550 *foot-pounds per second*. Again, for electrical purposes the unit activity employed is frequently the *watt*, which is an *activity* of 10^7 *ergs per second*. Sometimes also an activity of 1000 watts is used as unit, and is called a *kilowatt*.

191. **Distinction between Work and Activity.**—The reader will note the essential distinction between work and activity or rate of working. Confusion between the two quantities is not uncommon even in dynamical treatises, to the extent at least of referring to a horse-power as 33,000 foot-pounds, or a watt as a unit of energy, which of course it is not. Thus we have heard the question asked, "How many fully charged storage cells of a certain pattern will contain a horse-power?" The cells when charged contain so much energy, which can be measured in ergs, or foot-pounds, or in terms of any other unit of energy or work; the rate at which the cells can work depends on their arrangement and that of the circuit in which their energy is to be developed, and a single cell may be made to give many horse-power or as small a fraction as may be desired of a horse-power. Even a Hertzian vibrator charged by an induction coil of moderate size may, while discharging by oscillations and producing electrical waves, work for a very short time at an enormous rate. The mean activity during the time of oscillatory discharge has been reckoned for such a vibrator at nearly 200 horse-power

192. **Dimensional Formulæ of Work and Activity.** — Let a certain amount of work be expressed numerically by W when the units of length, mass, and time are certain chosen units which we denote by L, M, T. Then we may in the same manner as at §§ 7, 124, 185, write

$$\begin{aligned} Work &= W(unit\ of\ force \times unit\ of\ length) \\ &= W[LMT^{-2} \times L] = W[L^2MT^{-2}]. \end{aligned}$$

Now let the units L, M, T be altered to new units L', M', T' such that $L = \lambda L'$, $M = \mu M'$, $T = \tau T'$. Then we obtain for the same work

$$Work = W\lambda^2\mu\tau^{-2}[L'^2M'T'^{-2}]. \qquad (2)$$

The numerical value of the work is now $W\lambda^2\mu\tau^{-2}$, which is the former numerical value multiplied by $\lambda^2\mu\tau^{-2}$, the factor obtained by substituting for L, M, T in the dimensional formula $[L^2MT^{-2}]$ the ratios λ, μ, τ of the old units to the new. This factor is the change-ratio for energy.

An activity is measured numerically by the ratio W/t, of the

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numerical value of an amount of work done to that of the time occupied in doing it.

Thus for the dimensional formula of activity we have

$$Activity = A[L^2MT^{-3}]. \tag{3}$$

If a change is made to units L', M', T' from units L, M, T, fulfilling the relation $(L, M, T) = (\lambda L', \mu M', \tau T')$, the change-ratio is $\lambda^2\mu\tau^{-3}$.

As a numerical example we may find the number of ergs in a foot-poundal. Here $\lambda = 30{\cdot}48$, $\mu = 453{\cdot}6$, $\tau = 1$. Thus $\lambda^2\mu\tau^{-3} = 30{\cdot}48^2 \times 453{\cdot}6 \times 1 = 421408$, which is the number of ergs in a foot-poundal nearly.

One foot-poundal per minute is $421408/60^2 (= 1{\cdot}95)$ ergs per second.

193. **Work done by Forces which depend on the Co-ordinates.**—It is desirable to give here some examples of work done in different circumstances. First consider a particle under the action of forces such that the resultant force F on the particle depends only on the position in which it is placed, that is on its co-ordinates x, y, z with reference to a given set of axes. In any element ds of the path at which the force is F the work done is $F\cos\theta.ds$, where θ is the angle between F and ds. Now if X, Y, Z be the components of F, the direction cosines of F are X/F, Y/F, Z/F; and if dx, dy, dz be the projections of the element ds on the axes, the direction cosines of ds are dx/ds, dy/ds, dz/ds. Hence, § 24,

$$\cos\theta = \frac{X}{F}\frac{dx}{ds} + \frac{Y}{F}\frac{dy}{ds} + \frac{Z}{F}\frac{dz}{ds},$$

and

$$F\cos\theta.ds = Xdx + Ydy + Zdz. \tag{4}$$

But since F is a function of the co-ordinates, X, Y, Z are also. Let them, further, be functions such that the integrals $\int Xdx, \ldots$ taken round any closed path in the field of force (that is the portion of space in which the force exists and in which the work is done) are zero, that is such that the work done in any portion of the closed path is exactly equal and opposite to the work done in the remaining portion. If the work from A to B, Fig. 107, along any chosen path be W_{AB}, and integration along the path from A to B be denoted by $\int_A^B$, we have

$$W_{AB} = \int_A^B (Xdx + Ydy + Zdz), \tag{5}$$

and each of the integrals on the right must depend only on the terminals A and B of the path, that is must be independent of the course of the path between the initial point A and the final B. For

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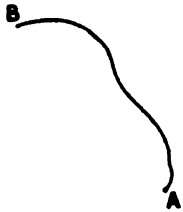
each must be exactly equal and opposite to the corresponding integral along any path from B to A making with the path AB a complete circuit in the field. X is $\partial W/\partial x$ the rate, for the point B, of variation of W with x, when y and z are kept unchanged. Similarly for Y and Z. The quantity in brackets in (5) is what is called a *perfect differential of W*. Such forces are called *conservative* forces, and will be discussed more fully in the following sections and in other parts of this work (see § 218).

FIG. 108.



194. **Impossibility of a "Perpetual Motion."**—The case here described is of great practical importance, as it represents the action of natural forces, and therefore asserts the impossibility of a "perpetual motion," that is a self-acting machine which without having work done upon it from without will continue indefinitely long to do useful work. For such a machine can only continue to work by making successive "strokes," that is by passing through changes of configuration which run in cycles, so that the machine comes back again and again to the same configuration relatively to the bodies exerting forces on its parts, and therefore doing work on these parts as they move. Just as much work must therefore be done by such a machine against external forces as is done upon it by external forces in a *complete stroke*, or closed cycle of changes of configuration, and so there is no gain of work on the whole. On the contrary, there must in all cases, as we shall see presently, be a balance of work spent in overcoming frictional resistances; and this, as well as any useful work done by a machine, must be done by putting some other system through a partial or unclosed cycle of changes.

195. **Work done by Gravitational Forces. Theory of Potential.**—A very good example of forces fulfilling the conditions stated in § 193 is furnished by gravitation. According to the law of gravitation discovered by Newton we may, with proper choice of a reference system of axes, say that every particle of matter acts on every other particle with a force which, for every pair, is proportional to the product of the masses of the particles, and inversely proportional to the square of the distance between them, and acts along the line between them. Thus if m, m' are the masses of two particles, r their distance apart, there exists along the line joining them a force of mutual attraction of amount kmm'/r^2, where k is a constant multiplier required for the expression of the force in ordinary dynamical units. Thus if m, m' be taken in grammes, r in centimetres, the force will be expressed in dynes if k be the attractive force in dynes between two particles, each a gramme in mass, concentrated at two points a centimetre apart. Thus reckoned, k is approximately $6{\cdot}66 \times 10^{-8}$. (See chap. xii.)

The reference system here chosen must be one entirely independent of the action of the particles which are being considered (see § 220). It may be such a system, *e.g.*, one fixed relatively to the

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fixed stars, as the consistency of the results of observation shows to be capable of being regarded for practical purposes as "fixed in space." Since the centroid of a system of mutually influencing particles is not affected by their mutual action it is sometimes very conveniently taken as the origin of co-ordinates.

Now consider the work done by attraction in diminishing from r_2 to r_1 the distance of a particle of mass m from another of mass m'. Let a positive value of dr denote an *increase* of this distance, then the direction of the attraction is that of $-dr$, and (if force tending to increase r be taken positive) the attraction must be reckoned as a force $-kmm'/r^2$. The work done by this force in the displacement $-dr$ is therefore $kmm'dr/r^2$. Hence the whole work done by the force of attraction is

$$W = kmm'\int_{r_1}^{r_2} \frac{dr}{r^2} = kmm'\left(\frac{1}{r_1} - \frac{1}{r_2}\right). \tag{6}$$

The quantity km'/r is called the *potential* at the position of the mass m produced by the mass m', or the potential due to m' at any point distant r from it; and similarly km/r is the potential due to the mass m at the distance r from it. The difference of potential $km(1/r_1 - 1/r_2)$ is the work done by the attraction of m in bringing a particle of unit mass from a distance r_2 to a distance r_1, and $kmm'(1/r_1 - 1/r_2)$ is the work done by the attraction of m in changing the distance of m' from r_2 to r_1. The work done by the attraction of m' in bringing m through the same change of distance has the same value, as we should expect from the fact that the attraction is mutual.

When $r_2 = \infty$, kmm'/r_1 is the value of W, in other words the potential of any mass m, at a point distant r_1 from it, is the work done by the attraction of m in bringing a particle of unit mass from an infinite distance to a distance r_1, or by applied force in carrying it from distance r, against attraction, to distance infinite.

196. **Potential produced at any Point by a System of Particles. Work-Values of Particles in different Positions.**—The potential at any point P due to an assemblage of particles is the sum of the potentials which each mass produces separately at P, for the attraction of any particle on any other is, as observation and experiment show, unaffected by the presence of other particles. Thus if we have n particles of masses $m_1, m_2, \ldots, m_n$, at points $P_1, P_2, \ldots, P_n$, distant $r_1, r_2, \ldots, r_n$ from P, the potential at P (which we shall denote by V) is given by

$$V = k\left(\frac{m_1}{r_1} + \frac{m_2}{r_2} + \ldots\right) = k\Sigma\frac{m}{r}, \tag{7}$$

where the summation includes the value of m/r for each particle.

Thus the work done by attractive forces in bringing, from an infinite distance from every particle of the attracting system, a particle of mass m to a point P at which the potential is V is mV. This we shall call for shortness at present the *work-value* of the particle in the given position P relatively to the other particles. If P coincide

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with the position P_1 of one of the particles $m_1, m_2, \ldots, m_n$, and the particle of mass m be the particle of mass m_1 there situated, if V_1 denote the potential at P_1 due to $m_2, m_3, \ldots, m_n$, and $r_{12}, r_{13}, \ldots, r_{1n}$ denote the distances of P_1 from $P_2, P_3, \ldots, P_n$, the work-value of m_1 in the position P_1 is

$$m_1 V_1 = km_1\left(\frac{m_2}{r_{12}} + \frac{m_3}{r_{13}} + \ldots + \frac{m_n}{r_{1n}}\right) = km_1 \Sigma \frac{m_k}{r_1}, \tag{8}$$

where Σ denotes summation of m_k/r_{1k} taken for all values of k from 2 to n.

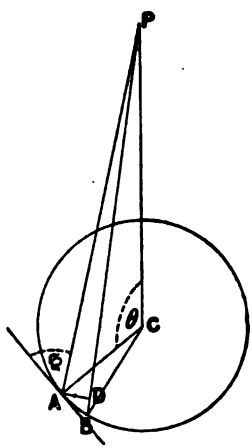
Similarly the work-value of m_2 in its position P_2 is $km_2 \Sigma m_k/r_{2k}$, where k has all values from 1 to n excluding 2. In the same way the work-values of the other particles may be expressed.

197. **Potential Energy of a System of Gravitating Particles.**—If r_{jk} be put for the distance of the particle at any point P_j from the particle at any other point P_k, the total work-value of the particles, $-E$, of the system, taken as the sum of the work-values of the particles, is expressed by the equation

$$-E = \frac{k}{2} \Sigma\Sigma \frac{m_j m_k}{r_{jk}}, \tag{9}$$

where one summation is for all values of j from 1 to n, and the other for the same range of values of k. The factor $\frac{1}{2}$ is necessary as each term of the work-value is brought by the summations twice into the account.

FIG. 109.



This work-value depends only on the relative configuration of the particles, and is therefore independent of the paths or processes by which the particles are brought together by their attractions.

We shall see later that what have been called above the *work-values* of the particles are component parts of the *exhaustion of configurational energy*, or *exhaustion of the potential energy*, of the system of particles. The potential energy is the sum with its sign changed of the work-values of the individual particles as here defined or this negative sum with any convenient constant added, that is $E + C$.

198. **Potential of Uniform Spherical Shell.**—We shall now show that if we have an assemblage of particles composing a thin spherical shell of uniform mass per unit of area, the potential of the shell at any external point is precisely equal to the potential at the same point of a particle of mass equal to that of the shell and situated at its centre. Let P, Fig. 109, be the point, C the centre of the shell, and A, B two adjacent points of the shell on a plane section through C and P. Let ϕ be the angle the tangent to the section at A makes with the radius, θ the angle

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between CP and CA, and draw AD at right angles to PA. Further, let a denote the length of the radius, r the length of PA, and l that of PC. Then $PB = r + dr$, $DB = dr$, and $d\theta = ACB$. Also $AB = ad\theta$. But $\sin PAC = \cos\phi = BD/AB = dr/ad\theta$, and therefore, since $\sin PAC/\sin\theta = l/r$, $rdr = al\sin\theta d\theta$.

Now if μ be the mass of the shell per unit of area, the mass of the zone swept out by AB, when the diagram is made to turn completely round PC, is $\mu ad\theta \times 2\pi a\sin\theta = 2\pi\mu a^2\sin\theta d\theta = 2\pi\mu ardr/l$. This is all at the distance r from P, and hence its potential at P is $2\pi k\mu adr/l$. We have then for the potential at P due to the whole shell

$$V = \frac{2\pi k\mu a}{l}\int dr,$$

where $\int dr$ means the difference between the extreme values of l. But this is $2a$, and therefore we obtain

$$V = \frac{4\pi k\mu a^2}{l}. \tag{10}$$

The mass of the shell is $4\pi\mu a^2$, and $4\pi k\mu a^2/l$ is the potential which that mass concentrated at C would produce at P.

199. **Potential of Solid Sphere.**—Since this result holds for each shell of a solid sphere made up of thin concentric shells each of uniform mass per unit area, it holds for the whole sphere. Thus a solid sphere of density uniform throughout, or of density varying in any manner with distance from the centre, but fulfilling the condition that the density is the same at all points at the same distance from the centre, produces the same potential at an external point P as would be produced by a particle of mass equal to that of the sphere and concentrated at the centre C.

200. **Work done by Attractive Forces in Condensing a Nebula.**—Consider a sphere of uniform density ρ and radius r, the potential at an external point at a distance $c + r$ from the centre is

$$\tfrac{4}{3}\pi\rho kr^3/(c + r).$$

The potential at the surface is therefore $\frac{4}{3}\pi\rho kr^2$. The work done by attractive forces in adding on a layer of thickness dr to the sphere by matter brought from infinity is therefore $\frac{4}{3}\pi\rho kr^2 \times 4\pi\rho r^2dr$, that is $\frac{16}{3}k\pi^2\rho^2r^4dr$. Hence for the work W done by attractive forces in building up a sphere of radius a and density ρ by means of matter brought from infinity we have

$$W = \tfrac{16}{3}k\pi^2\rho^2\int_0^a r^4dr = \tfrac{16}{15}k\pi^2\rho^2a^5,$$

or

$$W = \tfrac{3}{5}k\frac{M^2}{a}, \tag{11}$$

where M is the mass of the sphere.

It is important to notice that the work done by the attractive

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forces in carrying the surface of this sphere in towards the centre through a distance da is

$$dW = \tfrac{3}{5}k\frac{M^2}{a^2}da, \tag{12}$$

and the work done by these forces in bringing the surface in from a to a' is

$$W' - W = \tfrac{3}{5}kM^2\left(\frac{1}{a'} - \frac{1}{a}\right). \tag{13}$$

In the estimation of the work-value in (11) all the matter is supposed to come from infinity to the surface of the growing sphere; but it is not difficult to see that this is the work done by the mutual attractions of the particles in coming together from uniform diffusion through infinite space to form the sphere of radius a. For suppose that matter of amount M is brought in the manner described from infinity to build up a sphere of radius a, and again suppose that it is brought together to form a sphere of greater radius D. The work-values of the aggregations of particles are $\tfrac{3}{5}kM^2/a$, $\tfrac{3}{5}kM^2/D$, and, by § 197, depend entirely on the configurations of the particles in the two cases, and not in the manner in which the particles were brought together. Hence the difference $\tfrac{3}{5}kM^2(1/a - 1/D)$ is the work done by attractive forces in compressing the material sphere of radius D and mass m to the sphere of radius which becomes $\tfrac{3}{5}kM^2/a$ when D is infinite. Thus the work done by the mutual attractive forces of the particles in bringing together universally diffused matter to a sphere of radius a is exactly $\tfrac{3}{5}kM^2/a$.

201. **Work done in condensing the Earth from Nebular Matter. Amount of Heat generated.**—Let us apply this result to a case which may be taken to represent the earth regarded as built up of infinitesimal particles brought by their mutual attraction from universal diffusion. Take the radius a as $6{\cdot}367 \times 10^8$ centimetres, ρ as $5{\cdot}5$; then to reckon W in ergs we must take $k = 6{\cdot}66 \times 10^{-8}$, and therefore have

$$W = \tfrac{3}{5}6{\cdot}66.10^{-8}.\tfrac{16}{9}\pi^2.5{\cdot}5^2.6{\cdot}367^5.10^{40} = 2{\cdot}216 \times 10^{39}. \tag{14}$$

This is an underestimate for the earth, as the density increases from the surface inwards, and therefore as the matter is on the whole nearer the centre of the sphere than in the case here considered, the real value of W must be much greater.

As we shall see later, the quantity of heat required to warm a gramme of water from a certain temperature of the Centigrade scale to a temperature one degree higher is taken as a unit of heat, and called a *calorie*, and 42×10^6 ergs of work spent in producing heat would generate one calorie. Hence the above value of W would generate $5{\cdot}29 \times 10^{31}$ calories. The combustion of a gramme of anthracite coal yields about 8000 calories of heat. Thus the heat produced by W is equal to that which would be produced by burning $6{\cdot}6 \times 10^{27}$ grammes, or $6{\cdot}6 \times 10^{21}$ tons, of such coal.

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202. **Work done in condensing the Sun from Nebular Matter. Source of the Sun's Radiant Heat.**—The sun's mass is about 332,000 times that of the earth, and its radius is 110 times that of the earth. Hence W, for the sun, regarded as formed by the collection of universally diffused matter, is $332{,}000^2/110$ or 10^9 times the value of W just found for the earth, that is $2{\cdot}216 \times 10^{46}$ ergs. Thus the heat produced in bringing together a body of mean density and radius equal to those of the sun at present, would be about $5{\cdot}29 \times 10^{40}$ calories, or an amount equal to that produced by the combustion of $6{\cdot}5 \times 10^{30}$ tons of coal.

The amount of radiant heat which would be received in one minute by a square centimetre of surface held in the direct rays of the sun at the distance of the earth, at a place where there was no atmosphere, is estimated at about 3 calories. The amount radiated by the sun in one second is that received by a sphere of radius equal to the sun's distance from the earth. Now the sun's distance is about $110^2 \times$ *earth's diameter*, and therefore the area of a sphere of radius equal to the sun's distance is, roughly, $4\pi \times 4 \times 110^4 \times 6{\cdot}367^2 \times 10^{16}$ in sq. cms. The total number N of calories of heat radiated by the sun in one second is $\cdot 05 \times 4\pi \times 4 \times 110^4 \times 6{\cdot}367^2 \times 10^{16}$, and the number n emitted per second by each sq. cm. of the sun's surface is this number divided by $4\pi \times 110^2 \times 6{\cdot}367^2 \times 10^{16}$, that is $\cdot 05 \times 4 \times 110^2$. Thus approximately $N = 1{\cdot}5 \times 10^{26}$ and $n = 2420$. This heat would be produced by the expenditure of work at the rate of over 135,000 horse-power per square metre of the sun's surface, or by the combustion on each sq. cm. of the sun's surface of over 1000 grammes of the best coal per hour, or about a ton per square foot per hour.

The received theory, which is mainly due to v. Helmholtz, of the production of this heat is that of contraction of the sun. To produce the energy radiated the sun's radius would have to shorten at the rate of rather more than 1/5000 of a centimetre per second (or by about 17 centimetres per day, that is about 200 feet per year) on the supposition that no alteration of temperature is involved in the contraction. This is, however, much too great an estimate, as the density of the sun is no doubt much greater in the central portions than near the surface. Von Helmholtz has estimated about 125 feet per year as the radial contraction.

The sun is in part certainly gaseous, and it is known that a mass of gas cooling, and at the same time contracting, will increase in temperature, a result we shall prove later when dealing with heat. The sun, however, no doubt very nearly fulfils the condition of constancy of temperature.

203. **Work done by Impact of a Jet of Particles. Reaction Water-Wheel.**—Take the jet issuing from a tank on a truck as described in § 184 above. Let the velocity of the liquid in space be v, and that of the carriage be u in the opposite direction. The mass of liquid issuing per second is therefore $a(v+u)\rho$, where ρ is the density of the fluid and a the cross-section of the jet. The momentum is thus $a\rho(v+u)^2$, and this is the force exerted on the

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carriage. The jet does work on the carriage at a rate $A = Fu$ where F is the force, and thus

$$A = a\rho(u+v)^2u. \qquad (15)$$

One extreme example of this is got by putting $u = 0$, and is zero.

Again, consider a simple reaction water-wheel like that shown in Fig. 110, fed by water which may be taken as initially at rest at height h above the orifices. Let v be the velocity of the water, ω the angular velocity of the wheel, r its radius, a the cross-section of each jet, and let the water issue tangentially to the circle in which the orifices move; we have for the mass of water issuing from one of them per second $a\rho(\omega r + v)$. The momentum communicated to the water per second is $a\rho(\omega r + v)v$, and the moment of this force round the axis is $a\rho(\omega r + v)vr$. The total moment for three jets is $3a\rho(\omega r + v)vr$ and this is the couple tending to increase the angular velocity of the wheel. We thus have for the rate of working

FIG. 110.

$$A = 3a\rho(\omega r + v)v\omega r. \qquad (16)$$

204. **Work done in stretching a contractile Film.**—As still another example, consider the work done in increasing the area of a contractile film such as that which may be regarded as existing on the surface of a liquid. The film tends to contract, and to keep any narrow band of it stretched there must be applied to the part of it on each side of any section at right angles to the length of the band a pulling force of amount Tb, where b is the breadth of the band. The value of T is for a water surface at 10° about 75 dynes per lineal centimetre. T is called the *surface tension* of the film. In the case of a bubble the tensions of the two surfaces give an effective tension of $2T$ for the film. It is found that this is independent of the thickness of the film, unless the film be made exceedingly thin.

Now consider a plane film of breadth b and length l, and let this be changed to l' by pulling it out in the direction of its length; we have for the force $2Tb$, and for the work done $2Tb(l' - l)$, that is the work done in stretching the film is $2T \times$ *increase of area.* [As we shall see under *Thermodynamics* the film when drawn out tends to sink in temperature. If, however, the extension goes on slowly the temperature is kept constant by heat drawn from the surroundings.]

205. **Normal Force exerted by a Curved Film.**—A contractile film, whatever its form, exerts a force along the principal normal at

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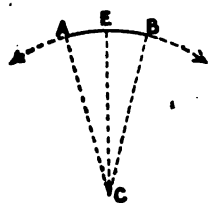
each point of the surface, that is along the perpendicular to the plane *touching* the surface at the point. For consider first a cylindrical film, and take a ribbon at right angles to the generating lines of the cylinder. Let AB, Fig. 111, be a small element of length ds, E its middle point, c the centre of curvature, r the radius of curvature EC, $d\theta$ the angle ACE or BCE. If b be the breadth of the strip (perpendicular to the paper) the stretching force at A and B, if T be the tension, is Tb. Then we get for the force P along EC, $2Tb\sin d\theta = 2Tbd\theta$. But $2d\theta = ds/r$, therefore $P = Tbds/r$. If p be the force exerted by the strip per unit of area along EC, $p = P/bds = T/r$. This is called the *pressure* along the normal EC, and is exerted on the fluid on the concave side of the surface.

FIG. 111.



206. **Film of Double Curvature.**—Now consider any surface, and take any section of it containing the normal; each such section has a radius of curvature of its own along the principal normal. The section by any plane nearly coincident with this, intersecting the former plane in a line through the centre of curvature at right angles to the radius of curvature, has approximately the same curvature near the point of contact, if the form of the surface changes continuously.

Let the principal normal be drawn to any point P on the surface. Two planes at right angles to one another intersecting along the principal normal will give curves of intersection with the surface having radii r, r' along that line, and in the same direction, if the curves of section are both concave or both convex outwards from the surface, and in opposite directions when one curve is concave, the other convex. These have the corresponding curvatures $1/r$, $1/r'$. Let these planes be denoted by A, B, and the centres of curvature of their curves of intersection with the surface be C, C', Figs. 112 and 113. Take now two planes A_1, A_2 nearly coincident with A, which contain a line through C perpendicular to the principal normal, and cut the surface in two curves at equal distances from P. Near P these curves will be approximately parallel lines. They include between them there a short narrow strip of the contractile film with parallel edges, which are clearly similar curves of curvature $1/r$, practically identical with the curve of section between them through P made by A. Take now other two planes B_1, B_2 in the same way, one on each side of the plane B intersecting in a line through C' also perpendicular to the principal normal, and cutting the surface in two curves at equal distances on the two sides of P, and having, at points near P, curvature $1/r'$. These will give the other two edges of the portion of surface, which thus will be very approximately a small rectangle.

A line drawn across this surface parallel to either pair of edges does not deviate more than infinitesimally from a straight line perpendicular to the normal section A or B through P to which the

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edges are perpendicular. Fig. 112 shows the element when the curvatures in the two sections are in the same direction; Fig. 113 when they are in opposite directions, as in a saddle-shaped surface.

Let the curvature of one pair of edges be $1/r$, of the other pair of edges $1/r'$; the stretching force in the film along the first pair of edges gives by § 204 a force T/r per unit of area, and directed along

FIG. 112.

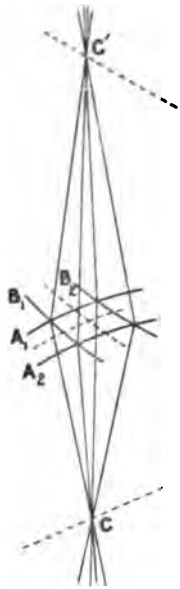
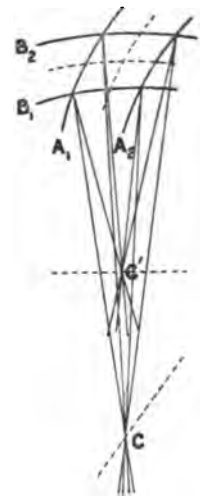


FIG. 113.



the principal normal, and similarly, the stretching force in the film along the other pair of edges gives a force T/r', also along the principal normal. If the numerical magnitudes of these forces be taken positive the forces are to be regarded as acting towards the centres of curvature respectively referred to. Thus in Fig. 112 the forces are oppositely directed. But $1/r$, $1/r'$ the curvatures may be regarded as directed quantities, so that $1/r'$ is taken negative if r' is oppositely directed from r; and so for the total pressure p we have

$$p = T\left(\frac{1}{r} + \frac{1}{r'}\right). \tag{17}$$

207. **Euler's Theorem of Sum of Curvatures at any Point of a Surface.**—It is a theorem of the geometry of surfaces that if any pair of rectangular sections of a surface through the principal normal be taken the sum of the curvatures $1/r + 1/r'$ is a constant. For one such pair the radius of curvature in one of the sections must be the

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least, and therefore the radius of curvature of the other section of the pair must be the greatest radius for the point considered. These are called the principal radii of curvature. We shall denote them by R and R'. Thus

$$p = T\left(\frac{1}{R} + \frac{1}{R'}\right). \tag{18}$$

The constancy of this sum of curvatures will be proved in chap. xvi., but it may be inferred from the physical fact that p must be independent of the direction of the two pairs of planes adopted above to define the rectangular element.

If $1/R + 1/R' = 0$, $p = 0$. This is the case of a film with equal pressure on its two sides. The radii of curvature must in this case be oppositely directed, unless the film is plane, in which case they are both infinite.

208. **Problem of Minimum Surface with Given Boundary.**—In the general case of a film with a fixed curvilinear boundary, as a soap film attached to a wire bent into a closed curve of any form, the film is curved, and the radii are finite and fulfil the condition $1/R + 1/R' = 0$. But the film contracts so as to have the smallest area possible under the conditions; and thus solves the isoperimetrical problem, to find a surface of minimum area with a given boundary.

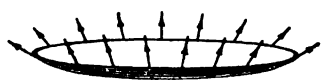
The analytical condition fulfilled by such a surface is well known to be precisely

$$\frac{1}{R} + \frac{1}{R'} = 0, \tag{19}$$

so that the solution is given at once by the dynamical process.

209. **Work done in Blowing a Spherical Soap-Bubble.**—Next consider a spherical soap-bubble filled with a gas which expands,

FIG. 114.



blowing out the soap-bubble, and let this expansion go on so slowly that the temperature of the film, and therefore its tension, remains constant.

The pressure exerted on the fluid inside by a contractile film of spherical form may be evaluated in the following manner. Consider a portion of the film bounded by a small circle on the sphere; that is pulled tangentially at all points of its boundary by the film beyond with a force $2T$ per unit of length of the boundary (see Fig. 114). Draw a plane through the vertex of this portion and the centre of the sphere, and let $2d\theta$ be the angle subtended at the centre by the intersection of the plane with the surface of the segment. The tangential forces round the circumference give a resultant force through the vertex and toward the centre of $4\pi Trd\theta^2$, if $d\theta$ be small. This divided by $\pi r^2 d\theta^2$, the area of the portion of the surface, is $2T/r$.

A gas exerts a force normally outwards on every part of the

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surface, and the amount of this in all ordinary cases is sensibly the same at every part. Let p be this force per unit area (or pressure) exerted on the inner surface of the soap-bubble.

If P be the external pressure there is a resultant outward force of amount $p - P$ per unit area on each element of the film. This is balanced in the case of a soap-bubble in equilibrium by an inward force $4T/r$ where $2T$ is the tension for the film (taking account of both sides). Thus we have

$$p - P = \frac{4T}{r}. \tag{20}$$

If the radius be increased by an amount dr the force on each element of surface of area dS acts through a distance dr, and does work of amount $4TdSdr/r$. The whole work done in an extension of radius is therefore $4\pi r^2 4Tdr/r = 16\pi Trdr$. If the radius be increased from a to a' the work done is

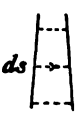
$$W = 8\pi T(a'^2 - a^2). \tag{21}$$

The area has been increased by an amount $4\pi(a'^2 - a^2)$, and thus we have for the work

$$W = 2T \times \textit{increase of area}. \tag{22}$$

210. **Work done by an Expanding Gas.**—As a last example, consider a quantity of gas compressed in the interior of a closed space. As will be seen under *Hydrostatics* on each element of the surface of the containing vessel there is exerted a force acting normally outwards the amount of which per unit of area of the element may vary from point to point of the surface. This force per unit area is called the *pressure* on the element. If p be its amount at any point P, the outward force perpendicular to an element of small area dS surrounding the point is pdS. Now let the volume of the space occupied by the gas be increased by a small outward motion of the whole or part of the boundary. The element dS takes up a new position, as in Fig. 115, having been displaced bodily through a mean distance ds in a direction perpendicular to its former position. The work done is $pdSds$. The whole work done by the expanding gas on the containing walls is the sum of the works done in this way on the elements making up the surface. Thus we have

FIG. 115.



$$W = \Sigma pdsdS, \tag{23}$$

where the summation is taken for every element dS of surface.

If p is the same for every element, as may be taken to be the case for a compressed gas, we have

$$W = p\Sigma dsdS \tag{24}$$

Here $\Sigma dsdS$ is evidently the increase of volume of the vessel, and if this be denoted by dv the work equation becomes

$$W = pdv. \tag{25}$$

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Thus if a gas is confined in a cylinder of area A by a piston, and the pressure applied at each point of the lower surface of the piston be p, the upward force on the piston is pA. If the piston rise a distance h while the pressure remains p, the work done is pAh. But Ah is dv, the increase of volume, and so the work is pdv.

211. **Finite Expansion**: (1) **Isothermal**; (2) **Adiabatic**.—The pressure of an expanding gas generally does vary with the volume, and may vary in an infinite number of ways. Two modes of expansion are, however, of special importance: (1) expansion subject to the condition that the temperature is constant; (2) expansion under the condition that no heat is received or parted with by the gas. In each case there is, as will be shown under *Heat*, a definite relation between the pressure and the volume of the gas. To a high degree of approximation for several gases the relation is, in (1) that the product pv, of the pressure and volume of the space occupied by the gas, is constant; in (2) that the product of the pressure p by v^γ, where γ is the ratio of the specific heat at constant pressure to the specific heat at constant volume, is a constant. For oxygen, hydrogen, nitrogen, dry air, the value of γ is very nearly 1·41. The conditions in the two cases are expressed by the equations

$$pv = C,\ pv^\gamma = C', \tag{26}$$

where C, C' are constants. These are called the *characteristic equations* of a gas under the stated conditions. Generally when they are used in thermodynamics v is understood to be the volume of unit mass of the gas.

In these cases it is important to obtain expressions for the work done by the gas in a *finite* expansion, say from volume v_0 to volume v_1. First let us calculate in each case the variation of the pressure produced by a small increase in the volume dv.

(1) The relation being $pv = C$, where C is a constant, let v be changed to $v + dv$, and let p become $p + dp$ in consequence. We have then

$$(p + dp)(v + dv) = C,$$

and therefore

$$pdv + vdp + dpdv = 0.$$

If dv be small, dp will be correspondingly small, and $dpdv$ will be very small in comparison with pdv or vdp. Hence we have

$$pdv + vdp = 0,$$

and for the work dW done in an expansion dv

$$dW = pdv = -vdp. \tag{27}$$

In (2) a similar change gives

$$(p + dp)(v + dv)^\gamma = C',$$

or

$$(p + dp)v^\gamma\left(1 + \frac{dv}{v}\right)^\gamma = C';$$

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or expanding $(1+dv/v)^\gamma$ by the binomial theorem, and reducing using the relation $pv^\gamma = C$, we obtain

$$\left(1+\gamma\frac{dv}{v}+\ldots\right)dp+p\left(\gamma\frac{dv}{v}+\ldots\right)=0.$$

If dv be small we may neglect all squares and higher powers of dv, and products of dp and dv.

Thus we obtain

$$vdp+\gamma pdv=0,$$

and we have

$$dW=pdv=-\frac{1}{\gamma}vdp. \tag{28}$$

212. **Work done in Isothermal and Adiabatic Expansions.**—Now proceeding to the calculations proposed above, we have by (25) for the whole work W done by the gas in changing its volume from v_0 to v_1 under constant temperature the sum of the works pdv for the successive steps dv of volume, that is

$$W=\int_{v_0}^{v_1} pdv = C\int_{v_0}^{v_1}\frac{dv}{v}. \tag{29}$$

since $p=C/v$.

Consider now for a moment the equation $u=e^x$, where u is real and positive, and x is real: e^x is the function defined in § 54 above. Here x is called the logarithm of u to the base e, or the Napierian logarithm of u. This is usually denoted by the symbol $\log u$, so that we may write instead of $u=e^x$ the equation $x=\log u$.

Clearly if w be another real positive quantity, and y be real, and $w=e^y$, we have

$$\log(uw)=x+y=\log u+\log w$$

$$\log\frac{u}{w}=x-y=\log u-\log w.$$

Now we have seen that

$$du=e^x dx,$$

so that

$$dx=\frac{du}{e^x}=\frac{du}{u}.$$

This we may write in the form $d(\log u)=du/u$.

We may proceed from this, which is an equation between the differential of $\log u$, that is $d(\log u)$, and the differential du of u, to the integral equation, by taking the sum of the increments of $\log u$ as u increases say form u_0 to u_1, where u_0 and u_1 are real positive quantities. This sum is $\log u_1-\log u_0$ or $\log(u_1/u_0)$, and therefore we have

$$\log u_1-\log u_0=\log\frac{u_1}{u_0}=\int_{u_0}^{u_1}\frac{du}{u}.$$

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Now returning to the equation (29)

$$W = C\int_{v_0}^{v_1} \frac{dv}{v},$$

we have

$$W = C(\log v_1 - \log v_0). \tag{30}$$

This is the work done by the gas in expanding at constant temperature from volume v_0 to volume v_1, and is in general spent in overcoming external forces through the increase of volume. It is also the work which must be done by external forces in compressing the gas at constant temperature from volume v_1 to v_0.

In case (2), since $pv^\gamma = C'$, (28) becomes

$$dW = C'\frac{dv}{v^\gamma},$$

where γ is greater than 1, equal in most cases to 1·41 nearly. We have then

$$W = C'\int \frac{dv}{v^\gamma}, \tag{31}$$

Consider the function $1/v^{\gamma-1}$ and let v be changed to $v+dv$. Then we have, as the reader may verify, if dv be small compared with v, $dv/v^\gamma = -d(1/v^{\gamma-1})/(\gamma-1)$; and by taking all the steps of increment of $1/v^{\gamma-1}$ from $v=v_0$ to $v=v_1$, we obtain

$$\int_{v_1}^{v_1} \frac{dv}{v^\gamma} = \frac{1}{\gamma-1}\left(\frac{1}{v_0^{\gamma-1}} - \frac{1}{v_1^{\gamma-1}}\right).$$

Equation (31) therefore becomes

$$W = \frac{C'}{\gamma-1}\left(\frac{1}{v_0^{\gamma-1}} - \frac{1}{v_1^{\gamma-1}}\right). \tag{32}$$

This is the work done by pressure in an expansion of volume v_0 to volume v_1, subject to the condition that there is no passage of heat. It is the work also which must be spent by external forces in compressing the gas from volume v_1 to volume v_0, subject to the same condition. This is called *adiabatic* expansion or compression.

213. **Work done by a System of Forces.**—Let F_1, F_2, F_3, ... denote any forces acting on the particles m_1, m_2, m_3, ... of a material system, and δs_1, δs_2, δs_3, ... be any displacements in the direction of these forces which are possible under the conditions to which the system is subject. The sum $F_1\delta s_1 + F_2\delta s_2 + \ldots$ is called the *work*

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done by these forces in the specified displacements. We may denote it by δW, and write the equation

$$\delta W = F_1 \delta s_1 + F_2 \delta s_2 + \ldots = \Sigma F \delta s. \tag{33}$$

If $X_1, Y_1, Z_1, \ldots, \delta x_1, \delta y_1, \delta z_1, \ldots$ be the components of $F_1, \ldots, \delta s_1, \ldots$ parallel to the axes, we have

$$\delta W = \Sigma(X \delta x + Y \delta y + Z \delta z). \tag{34}$$

If $F_1, F_2, \ldots$ be the resultant forces on the particles, and $\delta s_1, \delta s_2, \ldots$ be any possible displacements in the directions of the forces, so that the system changes in consequence of these displacements from one configuration to another, δW is called the work done by the forces in the change of configuration. Here X, Y, Z denote the components of the resultant forces on a particle of mass m; but we may, as in (24) of § 156 above, put instead of these $X + X', Y + Y', Z + Z'$, where X, Y, Z now denote the external applied forces on the particle, and X', Y', Z' the internal component forces on the particle due to the mutual connections of the particles of the system. With this new meaning of X, Y, Z

$$\delta W = \Sigma(X \delta x + Y \delta y + Z \delta z) + \Sigma(X' \delta x + Y' \delta y + Z' \delta z). \tag{35}$$

We thus get by equations (24) of § 156

$$\begin{aligned} \delta W &= \Sigma m(\ddot{x} \delta x + \ddot{y} \delta y + \ddot{z} \delta z) \\ &= \Sigma m(X \delta x + Y \delta y + Z \delta z) + \Sigma m(X' \delta x + Y' \delta y + Z' \delta z). \end{aligned} \tag{36}$$

In the general case neither of the sums on the right of this equation vanishes. The second sum is the work done by internal forces, and this does not vanish unless the internal forces are only those which maintain invariable relations between the particles of the system. For consider a system composed of two mutually attracting particles A, B. Their mutual action does not change the position of their centroid. But in any small interval of time A is brought nearer the centroid by a distance δr_1, and B by a distance δr_2. If F be the numerical value of the mutual attracting force the work done is $F(\delta r_1 + \delta r_2)$, which is not zero. It will be noticed that while the directions of the displacements are opposite, the forces producing them are also opposed, so that the works done are both positive.

214. **Work of Applied Forces and Forces of Constraint. Kinetic Energy.**—Let us now denote components of force on any particle by $X_a + X_c, Y_a + Y_c, Z_a + Z_c$, where X_a, Y_a, Z_a are the forces on the particle other than those which do no work on the whole in any displacement to which the system may be subjected. The forces $X_a, \ldots$ we shall call the *work-forces* on the particle. They are sometimes also called the *applied forces*, but it is to be remembered that they are not necessarily forces arising from matter outside the system. The forces X_c, Y_c, Z_c are the components of force which arise from those connections of the system which are invariable, that is from the unvarying kinematical conditions to which the system is subject. We shall call

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them *forces of constraint.* Thus the forces on two particles exerted in consequence of an elastic link joining them are forces due to the connections of the system, but they must be included in the group X_a, ... On the other hand, forces applied to two particles by a rigid rod, which keeps them always at the same distance apart, are forces of constraint in the sense here adopted, and must be included with X_c, ... These forces do no work on the whole.

If the kinematical conditions are invariable, that is if they are expressed by equations involving the co-ordinates, but not depending on the time explicitly, the group of forces of the type X_c, Y_c, Z_c, as will be proved, will do no work in any possible displacement of the system, and so no sum of work-terms due to them appears in the equation of work. Thus we obtain instead of (36)

$$\delta W = \Sigma m(\ddot{x}\delta x + \ddot{y}\delta y + \ddot{z}\delta z) = \Sigma(X_a\delta x + Y_a\delta y + Z_a\delta z). \qquad (37)$$

The variables δx, δy, δz are any arbitrary possible displacements of the particles, but it is to be most carefully observed that this variational equation, as it is called, cannot be used by itself to obtain the equations of § 156, since there exists along with it the relation

$$\Sigma(X_c\delta x + Y_c\delta y + Z_c\delta z) = 0. \qquad (38)$$

There are to be included in the forces of constraint X_c, Y_c, Z_c, ... forces, if any, applied to keep particles moving on specified surfaces or along guide-pieces, for the conditions are capable of being expressed in the kinematical equations. It is to be understood that these are forces everywhere perpendicular to the direction of motion, and therefore no work is done by or against them. They do not, therefore, necessarily appear in the equation of work. That the kinematical conditions are invariable and that the corresponding forces do no work on the whole will be assumed unless the contrary is stated. If the system is rigid X_c, Y_c, Z_c are simply the internal forces X', Y', Z' referred to above in that connection.

Now for δx, δy, δz put $dx = \dot{x}dt$, $dy = \dot{y}dt$, $dz = \dot{z}dt$, so that dx, dy, dz are the *actual* displacements which take place in the motion of the system in the short interval of time dt. We get then for (37)

$$\Sigma m(\dot{x}d\dot{x} + \dot{y}d\dot{y} + \dot{z}d\dot{z}) = \Sigma(X_a dx + Y_a dy + Z_a dz). \qquad (37')$$

If we write for the system of particles

$$T = \tfrac{1}{2}\Sigma m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2), \qquad (39)$$

(37′) becomes

$$\frac{dT}{dt} = \Sigma(X_a\dot{x} + Y_a\dot{y} + Z_a\dot{z}). \qquad (40)$$

The quantity T is called the kinetic energy, or energy of motion of the system, while $X_a\dot{x}$, $Y_a\dot{y}$, $Z_a\dot{z}$, which are the rates at which work is being done by the forces X_a, Y_a, Z_a respectively, are called the *activities* of the impressed forces.

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By integration we obtain

$$T = \Sigma \int (X_a dx + Y_a dy + Z_a dz) + C, \tag{40'}$$

where C is a constant, and the integral is taken over the time of motion from the initial to the final configuration.

*215. **Forces due to Constraints. Activity due to Varying Kinematical Conditions.**—To find the forces on the particles when the system is constrained to fulfil certain kinematical conditions we may proceed as follows. Let the masses be $m_1, m_2, \ldots$, and their positions at any instant of time be the points $x_1, y_1, z_1, x_2, y_2, z_2, \ldots$, as indicated above. The number of co-ordinates required to fix their positions is thus $3n$. Let these co-ordinates, however, be connected by m equations,

$$f_1(x_1, y_1, z_1, x_2, y_2, z_2, \ldots) = 0, \quad f_2(x_1, y_1, z_1, x_2, y_2, z_2, \ldots) = 0, \ldots, \tag{41}$$

so that only $3n - m$ of them are independent. These give the differential relations

$$\frac{\partial f_1}{\partial x_1}\delta x_1 + \frac{\partial f_1}{\partial x_2}\delta x_2 + \ldots = 0, \quad \frac{\partial f_2}{\partial x_1}\delta x_1 + \frac{\partial f_2}{\partial x_2}\delta x_2 + \ldots = 0, \ldots, \tag{41'}$$

which are equivalent to (41). Such conditions are said to be *invariable.*

If the first of (41′) be multiplied by λ_1, the second by λ_2, and so on, and the products be added, we obtain

$$\left(\lambda_1 \frac{\partial f_1}{\partial x_1} + \lambda_2 \frac{\partial f_2}{\partial x_1} + \ldots\right)\delta x_1 + \left(\lambda_1 \frac{\partial f_1}{\partial x_2} + \lambda_2 \frac{\partial f_2}{\partial x_2} + \ldots\right)\delta x_2 + \ldots = 0. \tag{42}$$

With this the equation of work for any set of arbitrary displacements which fulfil the kinematical conditions may be written

$$\Sigma m(\ddot{x}\delta x + \ddot{y}\delta y + \ddot{z}\delta z) = \Sigma\{(X_1 + \lambda_1 \frac{\partial f_1}{\partial x_1} + \lambda_2 \frac{\partial f_2}{\partial x_2} + \ldots)\delta x_1 + \ldots\}, \tag{43}$$

where $X_1, Y_1, Z_1, X_2, \ldots$ are the *work*-forces (§ 214) on the particles $m_1, m_2, \ldots$. This equation holds along with (40).

So far the multipliers $\lambda_1, \lambda_2, \ldots$ are undetermined. We now choose them so that the co-efficients of $\delta x_1, \delta x_2, \ldots$ on the two sides of (43) may be the same, so that

$$\left.\begin{aligned} m\ddot{x}_1 &= X_1 + \lambda_1 \frac{\partial f_1}{\partial x_1} + \lambda_2 \frac{\partial f_2}{\partial x_1} + \ldots \\ m\ddot{y}_1 &= Y_1 + \lambda_1 \frac{\partial f_1}{\partial y_1} + \lambda_2 \frac{\partial f_2}{\partial y_1} + \ldots \\ &\ldots\ldots\ldots\ldots \end{aligned}\right\} \tag{44}$$

These are called the equations of motion, and are $3n$ in number. With the m kinematical equations (41) they give $3n + m$ equations in all for the co-ordinates $x_1, y_1, z_1, x_2, y_2, z_2, \ldots$, and the m multipliers

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$\lambda_1, \lambda_2, \ldots$; so that the system of values proposed for $\lambda_1, \lambda_2, \ldots$ is a possible one.

The quantities $\lambda_1 \partial f_1/\partial x_1 + \lambda_2 \partial f_2 \partial x_1 + \ldots, \ldots$, are the forces on the particles arising from the constraints. For if a constraint, say that given by $f_1 = 0$, were annulled, the motion of m_1 could only be maintained unaltered by adding to X_1 a force $\lambda_1 \partial f_1/\partial x_1$, to Y_1 a force $\lambda_1 \partial f_1/\partial y_1$, to Z_1 a force $\lambda_1 \partial f_1/\partial z_1$, and so on for the other particles.

It is clear from (42) and (43) that the total work due to the invariable constraints is zero. This is not the case if the kinematical equations involve the time explicitly as well as the position co-ordinates, that is, are of the form

$$f(t, x_1, y_1, z_1, x_2, y_2, z_2, \ldots) = 0, \ldots, \tag{45}$$

so that we have

$$\frac{\partial f_1}{\partial t} + \frac{\partial f_1}{\partial x_1}\dot{x}_1 + \frac{\partial f_1}{\partial x_2}\dot{x}_2 + \ldots = 0, \quad \frac{\partial f_2}{\partial t} + \frac{\partial f_2}{\partial x_1}\dot{x}_1 + \frac{\partial f_2}{\partial x_2}\dot{x}_2 + \ldots = 0, \ldots \tag{46}$$

Equations (43) will still hold, provided $\delta x_1, \delta x_2, \ldots$ be variations of $x_1, x_2, \ldots$ consistent with the kinematical conditions *which exist at time t*, and the equations of motion are still valid. If, however, (44) be multiplied, the first by λ_1, the second by λ_2, and so on, and the products be added, we obtain

$$\frac{dT}{dt} = \Sigma(X\dot{x} + Y\dot{y} + Z\dot{z}) - \left(\lambda_1 \frac{\partial f_1}{\partial t} + \lambda_2 \frac{\partial f_2}{\partial t} + \lambda_3 \frac{\partial f_3}{\partial t} + \ldots\right), \tag{47}$$

that is the rate of increase of the kinetic energy is less than the rate of working of the non-constraining forces by the quantity $\lambda_1 \partial f_1/\partial t + \lambda_2 \partial f_2/\partial t + \ldots$, which represents the rate at which work is done by the forces called into play by the relations of the system. The equation of work (37), however, still holds, as $\delta x, \delta y, \delta z, \ldots$ are variations of the co-ordinates consistent with the invariability of the conditions.

If $X, Y, Z, \ldots$ be conservative, we have § 218

$$\frac{d(T+V)}{dt} = -\left(\lambda_1 \frac{\partial f_1}{\partial t} + \lambda_2 \frac{\partial f_2}{\partial t} + \ldots\right), \tag{47'}$$

so that the varying kinematical conditions are inconsistent with constancy of energy.

Now consider a little more in particular the forces on the particle at x_1, y_1, z_1, which exist in consequence of the condition $f_1 = 0$. These forces, as we have seen, are $\lambda_1 \partial f_1/\partial x_1$, $\lambda_1 \partial f_1/\partial y_1$, $\lambda_1 \partial f_1/\partial z_1$, and are in the directions of x, y, z. $\partial f_1/\partial x_1$, $\partial f_1/\partial y_1$, $\partial f_1/\partial z_1$ are the rates of variation of f_1 taken with respect to x_1, y_1, z_1 respectively, when the other co-ordinates are taken as constants. Hence we may regard $f_1 = 0$ as a relation connecting different values of x_1, y_1, z_1 with those of certain constants, which are any proper chosen values of the other co-ordinates. It is then the equation of a surface on which the point (x_1, y_1, z_1) lies, when the other variables remain unchanged, and $\partial f_1/\partial x_1$,

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$\partial f_1/\partial y_1$, $\partial f_1/\partial z_1$ are proportional to the direction cosines of the normal to that surface at the point. The resultant force due to the constraint is thus in the direction of the normal to the surface $f_1 = 0$.

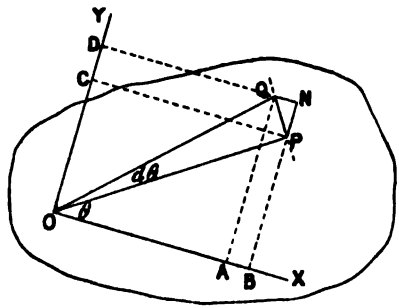
In the same way the force on the particle at (x_2, y_2, z_2) due to the constraint $f_1 = 0$ is obtained by taking the surface obtained by supposing x_2, y_2, z_2 the only variables in f_1; and so on. (Lagrange, *Theorie des Fonctions*, 3ième Partie, chap. v.)

The term in the variational equation of work due to the constraining force on the particle m_1 arising from $f_1 = 0$ is

$$\lambda_1\left(\frac{\partial f_1}{\partial x_1}\delta x_1 + \frac{\partial f_1}{\partial y_1}\delta y_1 + \frac{\partial f_1}{\partial z_1}\delta z_1\right).$$

If δx_1, δy_1, δz_1 are, as they may be, any variations of x_1, y_1, z_1 giving a displacement δn, at right angles to the surface defined by $f_1 = 0$, with

FIG. 116.



a particular set of values of the other variables, and R_1 denote the resultant force due to the condition, that is

$$\lambda_1\sqrt{\left\{\left(\frac{\partial f_1}{\partial x_1}\right)^2 + \left(\frac{\partial f_1}{\partial y_1}\right)^2 + \left(\frac{\partial f_1}{\partial z_1}\right)^2\right\}},$$

the work term just written becomes $R_1\delta n_1$. Thus the force on m_1 necessary for the fulfilment of the condition $f_1 = 0$ is a force which tends to vary the surface $f_1 = 0$.

216. **Work done in Angular Displacement of Rigid Body about a Fixed Axis.**—As a particular case consider a rigid body free to rotate about a fixed axis. Without any loss of generality we may take this axis as coincident with the axis of z. Consider a particle at a point P, Fig. 116, distant $r = \sqrt{x^2 + y^2}$ from the axis, which we suppose to be through the origin O, and let θ be the angle a perpendicular OP drawn from the particle to the axis makes with a fixed perpendicular to the axis, for convenience say a line OX parallel to the axis of x. Take OX, OY as axes. The motion possible to the particle is one in which $dx = NQ = -r\sin\theta\, d\theta$, $dy = PN = r\cos\theta\, d\theta$, $dz = 0$, or $\dot{x} = -r\sin\theta.\dot{\theta}$, $\dot{y} = r\cos\theta.\dot{\theta}$. Also $x = r\cos\theta$, $y = r\sin\theta$, and therefore

$$\ddot{x} = -r\cos\theta.\dot{\theta}^2 - r\sin\theta.\ddot{\theta}, \quad \ddot{y} = -r\sin\theta.\dot{\theta}^2 + r\cos\theta.\ddot{\theta},$$

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values which can be obtained either analytically, or by resolving along x, y the accelerations $r\ddot{\theta}$, $r\dot{\theta}^2$ which P has perpendicular to OP, and along PO respectively. Thus $\ddot{x}dx + \ddot{y}dy = r^2\ddot{\theta}d\theta$. Since the body is rigid $\ddot{\theta}$ and $d\theta$ must be the same for every particle, though of course the values of θ are different for the different particles.

Again,

$$\Sigma(X_a dx + Y_a dy + Z_a dz) = \Sigma(-X_a r\sin\theta + Y_a r\cos\theta)d\theta = \Sigma(Rr)d\theta,$$

where R is the component of the force X_a, Y_a, Z_a at right angles (in the direction of increasing θ) to the perpendicular OP let fall upon the axis from the particle to which the force is applied.

Hence for the body free to rotate as described the last equation becomes

$$\Sigma(mr^2)\ddot{\theta}d\theta = \Sigma(Rr)d\theta,$$

which we have already obtained in § 159 above. Also, since $d\theta = \dot{\theta}dt$, and $\ddot{\theta}dt = d\dot{\theta}$, we obtain instead of (37′)

$$\Sigma m(\dot{x}d\dot{x} + \dot{y}d\dot{y} + \dot{z}d\dot{z}) = \Sigma(mr^2).\dot{\theta}d\dot{\theta} = \Sigma(Rr).\dot{\theta}, \tag{48}$$

that is the sum of the moments of the impressed forces about the axis multiplied into the angular velocity of the body is equal to the rate of increase of the kinetic energy. Also the rate of working of the applied forces is equal to the sum of their moments about the axis, multiplied into the angular velocity of the body, or, what is the same, the work done in any angular displacement of the body about the fixed axis, is equal to the total moment about the axis of the impressed forces multiplied by the angle which measures the displacement.

217. **Kinetic Energy of Rotation.**—By integration of (48) we obtain for the rotating body, putting $\Sigma mr^2 = Mk^2$, where k is the radius of gyration about the fixed axis

$$T = \tfrac{1}{2}Mk^2\dot{\theta}^2 = \Sigma(Rr)\theta + C. \tag{49}$$

Thus to a constant C the kinetic energy is equal to the product of the total moment of the forces into the angular displacement to which the body has been subjected.

It has been shown in §§ 156–159 that the action of the forces on a rigid body may be considered as producing (1) acceleration of the centroid; (2) angular acceleration of the body about an axis through the centroid, and that these accelerations are independent. It will be convenient to prove in this connection the important theorem that the kinetic energy of the body is $\frac{1}{2}M\dot{s}^2 + \frac{1}{2}Mk^2\dot{\theta}^2$, where M is the mass of the body, k its radius of gyration and $\dot{\theta}$ its angular velocity about the axis through the centroid, and $\dot{s}$ the velocity of the centroid. Let P be a point of the body, G the centroid, GH the direction in which the centroid is moving, p the distance of P from the axis. The square of the velocity of a particle at P is

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$\dot{s}^2+\dot{\theta}^2p^2-2\dot{s}\dot{\theta}p\cos\phi$, where ϕ is the angle between the velocity $\dot{\theta}p$ and GH. The kinetic energy is

$$\tfrac{1}{2}\Sigma m(\dot{s}^2+\dot{\theta}^2p^2)=\tfrac{1}{2}M\dot{s}^2+\tfrac{1}{2}Mk^2\dot{\theta}^2, \tag{50}$$

since $\Sigma mp\cos\phi=0$ and $\Sigma mp^2=Mk^2$. Thus the proposition is proved.

218. **Potential Energy. Conservative Forces. Conservation of Energy.**—If now the impressed forces can be derived from a function V of the co-ordinates such that

$$X_a=-\frac{\partial E}{\partial x},\quad Y_a=-\frac{\partial E}{\partial y},\quad Z_a=-\frac{\partial E}{\partial z}, \tag{51}$$

where $-\partial E/\partial x$ means the rate of variation of $-E$ with x, when y, z are supposed constant, $-\partial E/\partial y$ the rate of variation of $-E$ with y, when z, x are constant, and similarly for $-\partial E/\partial z$. Then the variation of V with t, say, will be

$$\frac{dE}{dt}=\Sigma\left(\frac{\partial E}{\partial x}\dot{x}+\frac{\partial E}{\partial y}\dot{y}+\frac{\partial E}{\partial z}\dot{z}\right), \tag{52}$$

when the sum is taken for all the applied forces. Thus (40) becomes

$$\frac{dT}{dt}+\frac{dE}{dt}=0,\ \text{or}\ T+E=C, \tag{53}$$

where C is a constant. The quantity E is called the *potential energy* of the system. Since it is a function of the co-ordinates of all the particles, it depends upon the configuration of the particles, and is sometimes called energy of configuration.

Equation (53) expresses the fact that, however the particles may be displaced and however their motions may be varied, the sum of the kinetic and the potential energies remains constant, provided the forces are derivable in the manner specified from a function E of the co-ordinates.

A system in which the forces depend thus upon the configuration of the particles at each instant is called a *conservative system*, and the forces are called *conservative forces*.

The value of the sum C depends upon the configuration from which the potential energy is reckoned, that is the configuration for which E is taken as zero, and that can be chosen arbitrarily. If T_0 be the kinetic energy in that configuration, T_0 is the value of C.

219. **Relativity of Kinetic Energy. Reckoning of Energy for a Self-contained System. Condition necessary for Definiteness.**—It will be observed that the value of the kinetic energy T, since it depends upon $\dot{x}$, $\dot{y}$, $\dot{z}$ for each particle, and these depend upon the system of reference to which the motion of the particles is referred, is essentially relative. The equation (40), however, always holds if X_a, Y_a, Z_a are measured by the mass-accelerations of the particles relatively to the reference system employed to define $\dot{x}$, $\dot{y}$, $\dot{z}$, and no ambiguity can arise.

The doctrine of energy belongs really to what we have called a

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self-contained material system. The potential energy is that of the relative configuration of different parts of the system which act upon one another with forces depending on the distances between these parts; the kinetic energy is the energy of the motions of the parts of the system relatively to a reference frame entirely independent of the material system the energy of which is under consideration. It is essential for the definiteness of changes of kinetic energy (the absolute amount of the kinetic energy is of no consequence) that no part of the material system under consideration should form the reference frame. The equality of action and reaction would not then hold. For consider two particles of masses m, m', at distance r apart, attracting one another according to the law of gravitation. Let their motions be referred to the centroid of the two particles, the position of which cannot be altered by their mutual action, and to axes through this point as origin which may, in the sense already explained, be regarded as being "at rest in space" (see § 195). The forces on m and m' act toward the centroid, and are of the same numerical amount, kmm/r^2. The acceleration of m is km'/r^2, of m', km/r^2, and these are along the same line in opposite directions. This is all in accordance with the Third Law of Motion.

But now let m be supposed to be at rest, the acceleration of m', which must now be taken relatively to m, is no longer km/r^2, but $k(m+m')/r^2$, since an acceleration km'/r^2 equal and opposite to that of m must be imposed on both particles to reduce the acceleration of m to zero, and since that does not affect the relative motions of the two particles, the acceleration of m' relatively to m must be as stated. The corresponding force on m' is $km'(m+m')/r^2$, the force on m is zero; and the Third Law of Motion is not fulfilled.

The consideration in this way of the motions of the particles of a system relatively to some chosen one taken as at rest is frequently convenient, especially in the consideration of the motions of the satellites of a planet; but it is necessary always to remember that it violates the Third Law of Motion, and renders the estimation of energy-changes liable to ambiguity.

220. **Non-Conservative or Dissipative Forces. Work spent in overcoming Friction.**—So far we have considered only conservative forces, and have supposed the work done by these forces to have its equivalent in the kinetic energy produced in the system. There are, however, forces such as those of friction, with which we may class all resistances to motion not arising from inertia, against which work is done, apparently, at first sight, without the production of any equivalent. Thus a body is moved along the surface of another body—a book pushed along a table will serve as an illustration—and its motion is *resisted* by a force acting tangentially in the surface of contact of the bodies. This force is reversed when the relative motion is reversed, and work is still done against it in the motion. It is therefore non-conservative. Again, a body—for example, a pendulum bob—moves through a fluid, and its motion is *resisted* by a

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force depending on the speed of the bob relatively to the fluid. This force also is non-conservative.

221. **Conversion of Heat into Ordinary Dynamical Energy. Heat Engines.**—It appeared, until the discovery of modern thermodynamics set the matter in its proper light, that work done in overcoming friction was without equivalent. It is now known, chiefly by the experiments of Joule, that when work is so spent heat is produced of amount in direct proportion to work spent, and that heat is a form of energy which can be expended in doing mechanical work. We shall deal under *Thermodynamics* with the subject of the transformation of heat energy into mechanical work; at present we shall give merely the chief modifications of the equations of motion of a material system, produced by the introduction of non-conservative or, as they are sometimes called, *dissipative* forces. It may be stated here, however, that for the conversion of heat energy into energy of configuration of the mutually acting parts of a material system, or into relative motions of visible bodies, two bodies at different temperatures are necessary, and that, even in the most favourable circumstances of transformation, unless one of these bodies is at what is called the absolute zero of temperature, a balance of the heat operated on (which is always received from the hotter body) will be transferred to the colder body, so that the conversion is incomplete. The hotter body, from which heat is taken by the working substance, is called the *source;* the colder body, which receives heat from the working substance, is called the *refrigerator*.

The process consists in putting a material system, called the working substance, through a closed cycle of changes, in the course of which more work is done by the working substance against external forces than is done by external forces upon the substance, so that for each cycle or stroke of the arrangement (or *heat engine* as it is called) there is a balance of useful work obtained. This is at the expense of heat absorbed by the substance from the source, and, if there is no passage of heat to external bodies in the course of the stroke, the difference, $H-h$, between the heat H received from the source and the heat h given to the refrigerator in the stroke has its equivalent in the work done on external bodies. The cycle is said to be *reversible* if when the operations of the stroke are performed in the reverse order all the agencies are exactly reversed throughout the series of changes, so that heat h is taken from the refrigerator and heat H is given to the source while work $H-h$ is done by external forces on the working substance; and the engine performing the cycle is said to be a *reversible engine*. Such an engine will be proved later to have the greatest possible *efficiency*, that is one in which the ratio W/H, or $(H-h)/H$, of the useful work done in a stroke to the heat taken in from the source, has the highest possible value, so that all such engines have the same efficiency, which is therefore independent of the nature of the working substance. An engine reversible in this sense is also called a *perfect engine*.

If during the stroke there is a passage of heat to outside bodies,

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or if, in any way, the ratio $(H-h)/H$ is less than that for a perfect engine, the engine is not perfect, and the cycle is non-reversible.

222. **Non-Reversible or Dissipative Forces.**—A most important peculiarity of the so-called dissipative forces here considered is the non-reversibility of cycles of changes in which work is done against such forces. When one particle attracts another work must be done *against* the attraction by externally applied forces in separating the particles; and work is done *by* the attraction against external forces when the particles are allowed to approach one another. Not so with resistances of the kind here considered: whatever the direction of the displacement may be, work must be done against these resistances, and thus, when a system is put through any closed cycle of displacements, the work done against such forces is not zero, but a finite positive quantity. Such cycles may therefore be called non-reversible cycles. We shall in future refer to forces of this character as *dissipative* forces.

Any system in which such forces have play, like every system for which the conservation of energy does not hold, is really only part of a self-contained system. Closed cycles of changes of the latter are reversible, and the law of conservation holds. The heat generated by dissipative forces is possibly, in the first instance, only kinetic energy of particles of the partial system under consideration, or of some other partial system in relation with it, the co-ordinates of which we have no means of specifying. There is no means of retaining this energy within any system with which we are concerned, or even of completely utilising it if it were retained. See § 224.

223. **Dissipation of Energy.**—The name *dissipative forces* is appropriate inasmuch as the work done against them has its equivalent in the form of heat, and this by what has been stated above it is impossible even in the most favourable circumstances to completely retransform into energy of relative displacement or motion of visible bodies. The balance in general of heat given to the refrigerator or to external bodies by a heat engine is diffused by conduction and radiation of heat throughout all bodies accessible by such processes, and goes to raise their general temperature. Also bodies heated by friction invariably lose by these processes a portion, often the whole, of the heat generated in them. In the uniformly diffused state, heat energy will be unavailable, owing to the necessity of bodies at different temperatures for its utilisation; and the continual leakage of energy to this form, arising from conduction and radiation, and the rejection of heat to colder bodies in thermodynamic processes, steadily increases the amount of energy locked up in the unavailable form. This constant tendency of energy to become unavailable has been called by Lord Kelvin, to whom the doctrine is due, the *dissipation of energy*.

224. **Introduction of Dissipative Forces into Equations of Motion.**—Going back to the equations of motion of a system of particles, let the components of the force really acting on any particle be X, Y, Z, and let the forces actually producing acceleration

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be $X - X_d$, $Y - Y_d$, $Z - Z_d$, where X_d, Y_d, Z_d are the components of the dissipative forces which must be applied to the particle. The forces X, Y, Z in general are natural forces which are conservative, that is are derivable from a potential V. The equations of motion of the particle if of mass m are

$$m\ddot{x} = X - X_d, \quad m\ddot{y} = Y - Y_d, \quad m\ddot{z} = Z - Z_d. \tag{54}$$

The variational equation becomes in this case

$$\Sigma m(\ddot{x}\delta x + \ddot{y}\delta y + \ddot{z}\delta z) = \Sigma(X_a\delta x + Y_a\delta y + Z_a\delta z) - \Sigma(X_d\delta x + Y_d\delta y + Z_d\delta z), \tag{55}$$

where X_a, Y_a, Z_a have the same meanings as in § 214. Instead of (40) we obtain

$$\frac{dT}{dt} = \Sigma(X_a\dot{x} + Y_a\dot{y} + Z_a\dot{z}) - \Sigma(X_d\dot{x} + Y_d\dot{y} + Z_d\dot{z}), \tag{56}$$

or, putting $X_a = -\partial E/\partial x$, $Y_a = -\partial E/\partial y$,

$$\frac{d(T+E)}{dt} = -\Sigma(X_d\dot{x} + Y_d\dot{y} + Z_d\dot{z}). \tag{57}$$

225. **Controllable and Uncontrollable Co-ordinates. Rate of Dissipation of Energy.**—$T + E$ is the sum of the kinetic and potential energies of the *visible* bodies of the system, that is of the bodies as distinguished from the molecules or ultimate particles composing them. Heat energy consists no doubt in the kinetic energy of molecular motions, which are changed when a body is raised in temperature, but the particles concerned cannot have their motions traced individually, nor can such particles be laid hold of so as to have their motions subjected to conditions arbitrarily imposed. Sometimes $T + E$, the ordinary energy, is called the energy depending on the *controllable* co-ordinates of the system, while heat energy depends on what are called *uncontrollable* co-ordinates, that is the co-ordinates by which molecular motions are to be defined and measured. It is not impossible that potential or configurational energy may be really kinetic energy depending on a system of uncontrollable co-ordinates of matter (or it may be of ether) in the space surrounding the bodies.

The expression on the right of (57) shows that the sum of the potential and kinetic energies of the system diminishes at a rate which is numerically $\Sigma(X_d\dot{x} + Y_d\dot{y} + Z_d\dot{z})$. This is the rate of dissipation.

226. **Body on a Rough Inclined Plane.**—The following simple case will illustrate this result. A body slides a distance s down an inclined plane under the action of gravity, and is resisted by friction between it and the plane. Let α be the inclination of the plane to the horizontal, m the mass of the body, and F the frictional resistance on the body, acting up the plane, and supposed constant during the displacement. At the beginning of the displacement let the velocity

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of the body be v_0, and its potential energy E_0, then $T+E=\frac{1}{2}mv_0^2+E_0$. The force due to gravity acting down the plane during the displacement is $mg\sin\alpha$, and therefore the resultant downward force is $mg\sin\alpha-F$. The work done by this force in the displacement is $(mg\sin\alpha-F)s$, and the loss of potential energy is $mgs\sin\alpha$. The acceleration down the plane is $g\sin\alpha-F/m$ throughout the displacement, and is uniform, and if v be the final velocity,

$$(v-v_0)/(g\sin\alpha-F/m)$$

is the time, t, of describing s. Thus, by § 90,

$$s=\tfrac{1}{2}(v+v_0)t=\tfrac{1}{2}(v^2-v_0^2)/(g\sin\alpha-F/m).$$

The kinetic energy $\frac{1}{2}mv^2$ after s has been described is therefore

$$mgs\sin\alpha-Fs+\tfrac{1}{2}mv_0^2.$$

Hence then

$$T+E=mgs\sin\alpha-Fs+\tfrac{1}{2}mv_0^2+E_0-mgs\sin\alpha=\tfrac{1}{2}mv_0^2+E_0-Fs.$$

Thus the sum of the kinetic and potential energies has been diminished by Fs, the work done in overcoming friction.

227. **Laws of Friction. Static and Kinetic Friction.**—We may here conveniently discuss frictional resistance between bodies in relative motion. It is usual to distinguish between the friction which exists when solids are in contact with only air between them, and the resistance to motion when the solids are separated by a layer of lubricant such as oil. Thus the terms "solid friction," or "dry friction," and "fluid friction" are in common use to denote the two kinds of friction, and two sets of "laws" are given for them. There are, however, indications that these are only particular cases of fluid friction, that in fact when there is no lubricant in the ordinary sense between rubbing surfaces the presence of air plays an important part.

FIG. 117.

Frictional force exists between every pair of elements of the surfaces of the two bodies if these elements are in contact and are in relative motion. It acts tangentially on each of the two elements in the direction opposing the relative motion which there exists. Only further research can reveal its actual cause, but it is of the nature of the shearing stress which would resist the sliding of one surface on another to which it adheres. It is not to be confounded with the resistance to motion exerted on a body which abrades or cuts into the surface along which it moves.

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Experiments in friction have been made in various ways, but the apparatus shown in Fig. 117 (due to Professor Perry) enables all the ordinary information to be obtained very easily. The instrument consists of a vertical pulley with a broad smooth circumference, on the highest part of which a transverse horizontal piece, P, is made to bear by a weight W. The pulley is driven (by hand or by a steam engine) at a constant speed which is measured by a speed-counter. Owing to friction between the pulley surface and the bearing piece, the latter would be carried in the direction of motion of the pulley until arrested by a stop were it not for the horizontal force applied by means of the weights in the scale-pan to prevent the motion. Thus by varying the surfaces of the pulley and the bearing piece, and the speed, all the so-called laws of friction can be verified. These laws are:

1. The frictional resistance is nearly independent of the speed of the relative motion, provided the speed be not excessively small, in which case the friction is found to increase towards a maximum value, called *static* friction, as the speed falls off to zero.

2. The friction depends on the nature of the rubbing surfaces, but is independent of the extent of them in contact.

3. The friction is proportional to the force with which one surface is pressed against the other, and is equal to this force multiplied by a number which is less than unity. This number is called the *co-efficient of friction*, and, in all cases in which the possibility of abrasion of the surfaces is excluded, is considerably less than unity.

For wood on wood (dry) the co-efficient of friction varies from ·25 to ·5, and for metal on metal from ·15 to ·2.

Static friction, it has been stated, is greater than the friction between bodies in relative motion or *kinetic friction*, as it is sometimes termed. The amount of static friction developed in different cases varies from zero up to its maximum value, and is never more than that required to prevent motion. Thus let a body rest on a plane inclined at an angle α to the horizontal; the downward force on the body due to gravity is $mg \sin \alpha$. If this be less than the greatest possible value of the static friction, the frictional force developed will just be $mg \sin \alpha$ in amount, and will act up the plane. This is clear from the fact that if a greater force were developed the body would receive an upward acceleration. The ratio of the maximum of static friction to the normal force is called the co-efficient of static friction.

228. **Experiments on Friction. Determination of Co-efficient of Friction.**—In experiments made with the apparatus shown in Fig. 117, in which, however, the bearing piece or slide extended over a considerable arc of the pulley, Professor Perry found the friction to diminish as the speed was increased until it reached a minimum value, after which it increased as the speed was raised. This was found to be due to the pumping of air into the curved space between the surfaces so that the pressure was made to vary from point to point, as was proved by the insertion of small pressure gauges.

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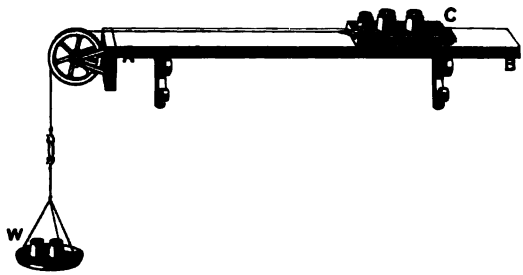
Accordingly Professor Perry has replaced the apparatus by an improved instrument in which the slide is pressed by a weight against the upper surface of a disk rotating in a horizontal plane, so that the bearing surfaces are both quite flat. (See Professor Perry's *Applied Mechanics.*)

But very instructive experiments can be made with the simple laboratory apparatus shown in Fig. 118, which explains itself. The surfaces in contact can be altered by placing on the shelf, AB, a plate of one of the substances, and using for the sliding plate, C, a piece of the other substance. The friction is measured by the weight W in the scale-pan, while the normal force exerted by one surface on the other is the combined weight of the plate C and its load. This

FIG. 118.



method was used by Morin in his experiments on the co-efficients of friction between different substances.

The student observes with this apparatus that the resistance before motion is set up is always greater than that which exists after motion, so that the weight W in the scale-pan which just suffices to start the motion will produce acceleration after the motion has been set up. A slightly less weight must be used, and the motion just started by a slight impulse given to C. If then the motion continues uniformly just the right force is applied. The observer will soon learn to judge pretty accurately of the uniformity of the motion by simple observation. He will then find by varying the load that F is proportional to N, the normal force between the surfaces, that is that $F = \mu N$, and that practically the same result is obtained whatever the relative speed between the two surfaces may be.

In recording the results of such experiments the observer should not merely enter the numbers obtained in his note-book, but examine their significance by careful graphical comparison. This he should do by plotting them in curves on what is called "squared paper," that is paper ruled by not too obtrusive lines into small squares. In fact the note-books of students generally, and all note-books used in a physical laboratory, should be made of paper thus ruled. The author has found that such books can be obtained, at

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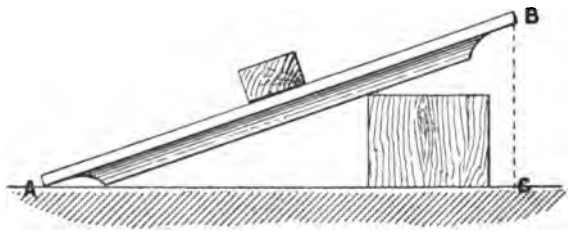
only a very slight increase of cost, though the paper is ruled into squares of one inch in side, which are subdivided into squares of one tenth of an inch by rather fainter lines.

The test of the law $F = \mu N$ would thus be obtained by laying down values of N as distances on a horizontal line along the page, from a vertical line at the left-hand side, which can easily be done on any scale by means of the divisions along the common boundary of a horizontal row of squares, and then laying off the values of F as ordinates from the points given by the values of N. If these points lie on a straight line the law is verified, and μ is given by any value of F divided by the corresponding value of N, or better, since the values of F and N will not always give the same ratio exactly, a certain mean line will be indicated, the tangent of the inclination of which to the horizontal line is the value of μ.

229. **Co-efficient of Friction measured on Inclined Plane.**—When a body moves uniformly down an inclined plane, under the action of

FIG. 119.



gravity and the friction between the body and the plane, the tangent of the angle of inclination of the plane to the horizontal is the value of μ for the surfaces in contact. If α be the inclination of the plane to the horizontal, and W be the gravity of the body, the component force down the plane is $W \sin \alpha$, and since there is no acceleration this must be just balanced by the friction, F, acting up the plane. But the force normal to the plane is $W \cos \alpha$. Hence we have

$$F = W \sin \alpha = \mu W \cos \alpha,$$

that is

$$\mu = \tan \alpha \tag{58}$$

The co-efficient of friction may be found by this relation with great ease. Have an inclined plane made stiffened, as shown in Fig. 119, by two parallel longitudinal pieces on the under side to prevent perceptible bending by any load used. Prop it up on a horizontal table until the body slides uniformly down. Then place a large set square with one edge along the table and the other intersecting the lower edge of the side of the plane at B, and mark the point of intersection on the vertical edge of the set square, and the point C on the horizontal plane. Measure AC, then BC/AC is μ.

Let the body be moving down the plane. The action of the

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plane on the body is then a force $W\cos\alpha$ normal to the plane, and from the plane to the body, and a force $\mu W\cos\alpha$ up the plane. The resultant action of the plane on the body is therefore $\sqrt{1+\mu^2}W\cos\alpha$ along a line in the vertical plane of motion, and inclined at an angle $\tan^{-1}1/\mu$ to a line drawn up the plane, or an angle $\tan^{-1}\mu$ to the normal to the plane. If the body moves up the plane the friction acts down, and the resultant action of the plane on the body has the

FIG. 120.

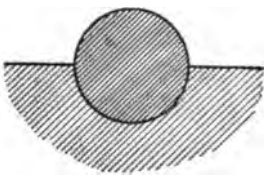
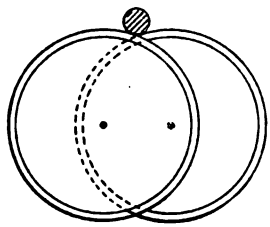


FIG. 121.



same value as before, and is inclined to the normal at the same angle, but on the opposite side. This angle is called the *limiting angle of friction.*

Consider a body at rest on a plane surface between which and the plane the co-efficient of friction is μ, and let a force F be applied through the centroid of the body in a line making an angle θ, less than $\tan^{-1}\mu$, with the normal drawn to the surface. The normal force is $F\cos\theta$, and the tangential force is $F\sin\theta$. The greatest value the friction resisting motion can have is $\mu F\cos\theta$, and since $\cos\theta > 1/\sqrt{1+\mu^2}$, $\mu F\cos\theta > \mu F/\sqrt{1+\mu^2}$. On the other hand $\sin\theta < \mu/\sqrt{1+\mu^2}$, and therefore $F\sin\theta < \mu F/\sqrt{1+\mu^2}$. Hence the tangential force due to F tending to move the body along the surface is less than the greatest frictional force which can be called into play by the normal component $F\cos\theta$, and hence the body is not moved by the force. The line of the force, therefore, to produce motion along the surface must lie outside the cone of which the normal through the centroid is the axis, and the semi-vertical angle is $\tan^{-1}\mu$.

The general case of the equilibrium of a particle on a rough surface will be considered under *Statics*. Also the use of friction-ergometers for the measurement of the activities of machines will be described under *Measures and Instruments*.

230. **Anti-Friction Rollers and Ball-Bearings.**—We shall consider friction further in chap. vi. and in vol. ii. under *Viscosity*, but before dismissing the subject here it is desirable perhaps to indicate the general principle of the mode adopted in many machines for the diminution of its effects. In such cases it is the friction of axles on bearings that exerts the retarding action. Thus consider the cylindrical bearing shown in Fig. 120. To the relative motion, which is everywhere tangential to the cylindrical surfaces, resistance is

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opposed, and though this may be different at different points of a bearing the friction at each point acts at the same distance from the axial line of the axle, and so exerts on the axle a couple of moment $r\Sigma F$, where r is the radius of the axle, and ΣF is the sum of the tangential forces. The work done in overcoming friction in any displacement is $s\Sigma F$, where s is the distance each point of the axle has moved against the retarding force applied at a point of the bearing. But for unit of time s is $2\pi rn$, where n is the number of revolutions made by the axle in unit of time; hence the work done at the bearing per unit of time is $2\pi nr\Sigma F$, that is the product of the couple $r\Sigma F$ into the angle $2\pi n$ turned through by the axle in unit of time. In other words, the activity spent in overcoming friction is the retarding couple multiplied into the angular velocity of the axle. If then there be in a machine a number of bearings at which the couples are L_1, L_2, ..., and the angular velocities are ω_1, ω_2, ..., the activity spent against friction is $\omega_1 L_1 + \omega_2 L_2 + \ldots$, or $\Sigma\omega L$. This work is less the smaller the value of ω at each bearing, and the smaller L, that is for the same frictional resistance, the smaller the radius of the bearing.

The effect of friction can thus be reduced very much by diminishing the diameter of the bearings, and further by adopting an arrangement like that shown in Fig. 121. There each end of the axle of a wheel is laid in the angle between a pair of overlapping wheels supported, we shall suppose, on ordinary bearings. The ends of the axle roll on the supporting wheels, and, if there is no slipping, no activity (or only a very little on account of what is called rolling friction) is spent in consequence of this relative motion. If ω be the angular velocity of the axle, r its radius, and R the radius of each of the wheels on which it rests, the angular velocity of each supporting side wheel is $\omega r/R$. If the average frictional couple applied to a side wheel be L, the activity spent in overcoming friction is for the four $4\omega Lr/R$. If the radius of the axle of the supported wheel be the same as that of each side wheel, and there be only an imperceptible increase of weight introduced by the side wheels, the rate of working against friction, if the supported wheel were on ordinary bearings, would be $4\omega L$. Hence the activity spent against friction is diminished by this device in the ratio r/R.

If the radius of the axles of the side wheels be made r', and the weight be the same, the couple applied to each side wheel becomes Lr'/r, where L has the same value as before. The angular velocity of of each side wheel is $\omega r/R$. Hence the whole activity spent against friction is $4\omega Lr'/R$. But the couple applied to each end of the axle of the supported wheel when on ordinary bearings would now be $2L$, if the diameter of the axle at the bearings is r, and the activity spent against friction would be $4\omega L$. By the side wheels, then, the frictional activity is reduced in the ratio r'/R, which is independent of the radius of the bearing part of the axle of the supported wheel. But the frictional activity of the supported wheel on ordinary bear-

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ings, which is thus reduced by the side wheels, is in proportion to r; so that it is well to turn down the bearing parts of the axle of the supported wheel as small as is consistent with strength.

The explanation of the diminution of the retarding effect of friction on a bicycle wheel produced by ball-bearings depends in the same way on the substitution of rolling for slipping in the bearings of the wheel. Ball-bearings have the advantage that slipping is by their means more completely prevented, since the balls roll in contact with the turning axle, and in contact with the bearing fixed relatively to the machine.

231. **Rolling Friction.**—It only remains here to say a word or two as to rolling friction. A wheel or cylinder never touches a rail or plane in a point or line, but always in a surface produced by the elastic deformation of the bodies. There is thus a slight sliding motion of parts of the surfaces in contact introduced which retards the motion. Thus as a locomotive wheel turns a deformation of the rim where the weight is borne is produced, and there is a corresponding indentation of the rail. This produces a retarding couple of amount proportional to the weight W on the wheel, say Wc. A wheel given rotating on a rough horizontal surface would gradually be brought to rest, in consequence of this couple, and the introduction of sliding friction necessary to ensure the fulfilment, which still exists in such a case, of the kinematical equation $a\theta = s$, where a is the radius of the wheel, θ the angle turned through by the wheel in any time, s the corresponding displacement of the centroid. For let this sliding friction be F, and mk^2 be the moment of inertia of the wheel about its axis, the moment retarding the wheel is $Fa + Wc$. Thus we get $mk^2\ddot{\theta} = -(Fa + Wc)$.

Along with this equation $a\theta = s$, so that $mk^2\ddot{s} = -(Fa + Wc)a$.

But we have, transferring F to the centroid, $m\ddot{s} = F$, and therefore $k^2F = -(Fa^2 + Wca)$ or

$$F = -\frac{Wca}{a^2 + k^2}. \tag{59}$$

The couple of rolling friction thus retards rotation and tends to bring about a forward slipping of the lowest part of the wheel. This is resisted by the force F which, it will be seen from its sign, acts in the direction opposed to the forward motion, and tends to bring the wheel to rest. Energy is dissipated partly in working against sliding friction, partly because of the viscosity of the substances in contact, which are continually undergoing and recovering from deformation.

The couple of rolling friction was experimentally determined by Coulomb by a method which practically consisted in placing a cylinder on a rough table, and hanging over it a cord to the ends of which weights W and $W + w$ were attached. The value of w which just produced rotation gave the couple due to rolling friction by the formula wgr, where r denotes the radius of the cylinder and g the

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force of gravity on unit of weight. From this the sliding friction can be deduced as just shown.

The magnitude of the couple of rolling friction developed, it was found, can never be greater than a certain limiting value, and the amount developed in any case is that which is sufficient to prevent rolling. It depends on the materials in contact, and is proportional to the normal force between the surfaces in contact. There seems also to be some dependence on the curvatures of the surfaces in contact; but the subject requires further investigation.

232. **Rolling of a Body on an Inclined Plane.**—We may consider here as a simple example of work done by gravity in presence of frictional forces, which, however, do not bring about any dissipation, the rolling of a sphere or cylinder down an inclined plane. We shall neglect the couple of rolling friction. Let α be the inclination of the plane to the horizontal, m the mass of the body, a its radius, and mk^2 its moment of inertia about a horizontal axis through the centroid. The centroid will be supposed to move along the line of greatest slope of the plane, so that if the body is a cylinder the axis is always parallel to the intersection of the inclined plane with a horizontal plane. Thus k is the radius of gyration of the cylinder about its axis, and of the sphere about a diameter. The forces on the body are F applied to the body at its place of contact with the plane, and $mg\sin\alpha$ acting downward parallel to the plane through the centroid of the body.

Thus for the motion of the centroid we have, if $\ddot{s}$ be the downward acceleration,

$$m\ddot{s} = mg\sin\alpha - F, \tag{60}$$

and for the motion of the body about the centroid

$$mk^2\ddot{\theta} = Fa, \tag{61}$$

where θ is the angle a radius of the body in the plane of rolling makes with a line fixed in space. These are the equations of motion.

But we have the kinematical equation

$$a\theta = s,$$

and therefore the second equation of motion becomes

$$mk^2\ddot{s} = Fa^2.$$

Substituting in the first equation of motion we get

$$F = \frac{k^2}{a^2 + k^2} mg\sin\alpha. \tag{62}$$

This is the amount of friction necessary to produce pure rolling, and no more is developed. If the surfaces, however, are so imperfectly rough or the plane is so much inclined that pure rolling is not produced, the kinematic equation given above is not applicable. The further consideration of the question is left to the reader, who may

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show that if the friction developed be smaller than that stated in (62), that is $\mu < \tan\alpha . k^2/(a^2+k^2)$, the space described from rest in time t is $\frac{1}{2}g(\sin\alpha - \mu\cos\alpha)t^2$, and the angle turned through $\frac{1}{2}\mu a/k^2 . g\cos\alpha . t^2$.

If the value of F given in (62) be substituted in (60) we obtain

$$\ddot{s} = \frac{a^2}{a^2+k^2} g\sin\alpha, \tag{63}$$

and the acceleration of the centroid is uniform. The velocity acquired and distance travelled from rest in time t are therefore

$$\dot{s} = \frac{a^2}{a^2+k^2} gt\sin\alpha, \quad s = \tfrac{1}{2}\frac{a^2}{a^2+k^2} gt^2\sin\alpha, \tag{64}$$

If the body is a sphere $k^2 = \frac{2}{5}a^2$, and

$$\dot{s} = \tfrac{5}{7}gt\sin\alpha, \quad s = \tfrac{5}{14}gt^2\sin\alpha\,;$$

and if the body is a uniform cylinder $k^2 = \frac{1}{2}a^2$, and therefore

$$\dot{s} = \tfrac{2}{3}gt\sin\alpha, \quad s = \tfrac{1}{3}gt^2\sin\alpha.$$

If the plane were frictionless and the body were to slide down, the velocity acquired and the distance traversed in time t would be in both cases

$$\dot{s} = gt\sin\alpha, \quad s = \tfrac{1}{2}gt^2\sin\alpha. \tag{65}$$

Thus the velocity of the centroid and the space described in any time are diminished in the ratio $a^2/(a^2+k^2)$ by the rolling.

But it has been proved in § 217 above that the kinetic energy of any body which rotates with angular velocity ω, and whose centroid has a velocity $\dot{s}$, is $\frac{1}{2}m(\dot{s}^2+k^2\omega^2)$.

Thus the total kinetic energy of the body comes out

$$\tfrac{1}{2}m\frac{a^2}{a^2+k^2}g^2t^2\sin^2\alpha. \tag{66}$$

But the loss of potential energy is mgh, where h is the vertical height through which the body has descended, that is $s\sin\alpha$. Thus by the value of $\dot{s}$

$$mgh = \tfrac{1}{2}m\frac{a^2}{a^2+k^2}g^2t^2\sin^2\alpha, \tag{67}$$

that is the gain of kinetic is equal to the loss of potential energy.

This principle, with the kinematic equation $s = a\theta$, might have been applied to find the velocity acquired by the body in rolling a distance s down the plane.

233. **Solution of an Old Problem.**—The greater k^2 the smaller is $\dot{s}$, and therefore the smaller is s for a given time. This gives a solution of the old problem, to distinguish between a solid ball of brass or copper gilded and a hollow sphere of gold when both are of the same size and weight. Clearly the value of k^2 will be greater for the hollow sphere than for the solid one. In fact, as the reader will

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easily prove, the moment of inertia of the hollow sphere, supposing r, r' to be the external and internal radii, and M its mass, is $\frac{2}{5}M(r^5 - r'^5)/(r^3 - r'^3)$. The moment of inertia of the solid sphere is $\frac{2}{5}Mr^2$. Hence $\frac{5}{2}k^2$ for the former is $r^2 + r'^2(r + r')/(r^2 + rr' + r'^2)$, and for the latter r^2. Thus the solid sphere rolls down an inclined plane the more quickly of the two, so that they can be distinguished at once by placing them on any inclined plane and allowing them to start together from rest.

234. **Work done by an Impulse.**—The work done by an impulse can now be reckoned. No matter how the impulse is applied to a body, it is measured by the time-integral of the impulsive force, that is by the momentum it produces. Let it be applied to a single free particle of mass m at rest, then, if v be the velocity acquired by the particle, the impulse is mv. The kinetic energy acquired by the particle is $\frac{1}{2}mv^2$, and this is the work done. Again, let the particle have initial velocity u, and the velocity acquired be v, the increase of momentum is $m(v - u)$, and this measures the impulse. The energy gained is $\frac{1}{2}m(v^2 - u^2)$, and measures the work done. The work done in this case is therefore equal to *impulse* $\times \frac{1}{2}(v + u)$, and is quite independent of the manner in which the impulsive force varies during the time τ of action of the impulse.

Now if by an impulse applied at any point of a material system, the velocity of that point be changed from u to v in an infinitesimal interval τ, the work done cannot differ from that which would be done if the point were a free particle, since in the time τ the point has not been displaced relatively to the rest of the system, and resistance to the motion of the point cannot arise until there has been such displacement. Thus if I be an impulse, u, v the initial and final velocities of its point of application,

$$\textit{work done by impulse } I = \tfrac{1}{2}I(u + v). \tag{68}$$

235. **Rod started by an Impulse applied at one End. Case of Minimum Energy.**—We may consider here the work done by the impulse applied in the case described at § 180 above. Here the body is set into motion with given velocity v of the extremity to which the impulse is applied. We have seen that the rod begins to turn round a point at a distance of two thirds of its length from that end. Its angular velocity, if the length of the rod is $2l$, is $\frac{3}{4}v/l$.

A motion of the rod, consistent with the velocity condition at the extremity, is one of rotation about a point at distance $2x$ from the end that is struck. We shall calculate the kinetic energy T in this case, and then show that T is a minimum when the body turns about the point specified above. By considering the turning rod made up of the two parts of lengths $2x$ and $2(l - x)$ rotating with angular velocity $v/2x$, about their common extremity, the point that remains at rest, we easily obtain, putting μ for the mass of the rod per unit length,

$$T = \frac{\mu}{3}v^2 l \frac{l^2 - 3lx + 3x^2}{x^2}. \tag{69}$$

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To see when this is a minimum let $(l^2 - 3lx + 3x^2)/x^2 = u$, so that we have

$$x^2(3 - u) - 3lx + l^2 = 0.$$

This is a quadratic to determine x, which it is clear must have real values. In order that the roots may be real we must have $9 > 4(3 - u)$, that is $u > \frac{3}{4}$. The least value of u possible is therefore $\frac{3}{4}$, and this will convert the quadratic expression into a perfect square, namely $(\frac{3}{2}x - l)^2$. The roots of the quadratic are then each $x = \frac{2}{3}l$.

Accordingly the kinetic energy is a minimum for the motion actually taken by the rod. This result, as we shall see, is a particular case of a very general theorem of minimum energy discovered by Lord Kelvin.

236. **Rod started by Impulse applied at one End. Case of Maximum Energy.**—As a companion to this theorem we have one of maximum energy, which was previously discovered by Bertrand, and of this we shall have an example if, in the case of motion just considered, we suppose given, not the velocity, but the impulse at the end struck.

If the impulse is I, and v be the velocity taken by the point struck, the kinetic energy is, as we have seen above, $\frac{1}{2}Iv$, since this is the work done by the impulse. By the preceding example, if the rod turn about a point at distance $2x$ from the end struck, the kinetic energy is given in (69). Hence $\frac{1}{2}Iv = \frac{1}{3}\mu v^2 l(l^2 - 3lx + 3x^2)/x^2$, that is

$$v\frac{l^2 - 3lx + 3x^2}{x^2} = \frac{3}{2}\frac{I}{l}.$$

The factor $(l^2 - 3lx + 3x^2)/x^2$ on the left is a minimum when $x = \frac{2}{3}l$. Hence, since its product by v is a constant $\frac{3}{2}I/l$, v must then have its maximum value. Therefore, when I is given, $\frac{1}{2}Iv$ must be a maximum, that is the rod moves so that its kinetic energy is a maximum.

Before proceeding to the theorems of Bertrand and Lord Kelvin, it will be convenient to discuss shortly the very general dynamical method given by Lagrange.

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CHAPTER V.

GENERAL DYNAMICAL METHODS.*

*237. **Motion of a System of Particles. Kinematical Conditions and Independent Co-ordinates.**—The kinematical conditions to which the system is subject have been several times referred to; we shall deal with them here a little more particularly. Considering first a system consisting of n free particles, we see that to fix their position $3n$ equations are required, three for each particle. The system may be said to have $3n$ *degrees of freedom.*

Again, a rigid body has three directions of displacement in which its centroid may be moved, and three directions round which it may be rotated. All other motions or rotations can be compounded of the motions in these directions, or the rotations about these axes, as the case may be. The rigid body is said to have 6 degrees of freedom.

It consists of a large number, n say, of particles, but the degrees of freedom are reduced, by the connections involved in rigidity, from $3n$ to 6.

To fix the positions of a system of n particles $3n$ equations are required. If the system be subject to kinematic conditions, these can be expressed by equations connecting the co-ordinates of the particles. We shall suppose these equations to be m in number, where of course $m < 3n$. These m kinematical equations determine any m of the co-ordinates in terms of the remaining $3n - m$ co-ordinates. The degrees of freedom are thus reduced to $3n - m$, and for each of these we must have an equation of motion. The undetermined co-ordinates have to be obtained from the equations of motion. And if they are determined for any instant the configuration of the system is completely known for that instant.

They may be called the *independent* co-ordinates. It is, however, to be remembered that the independence of co-ordinates does not justify the idea that superposition in any order of a number of variations of these co-ordinates will lead to the same resultant displacement. Thus the turning of a rigid body through successive angles α, β, γ about the axes of x, y, z will lead to different results according to the order in which the rotations are taken. The position taken by a body in consequence of a rotation about a fixed axis is a function of the initial position of the body relatively to the axis.

* §§ 237–252 are intended for advanced students and may be omitted. They are not required for the problems discussed in the remainder of this chapter.

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The co-ordinates which enter into the equations of motion may either be $3n-m$ ordinary position co-ordinates left undetermined, or they may be $3n-m$ parameters, as they are called, which are connected with the position co-ordinates by $3n-m$ equations. Thus when the parameters have been found the position co-ordinates can all be obtained by means of the $3n-m$ relations just referred to and the m kinematical equations.

There are, in this view, in all $6n-m$ quantities, $x_1, y_1, z_1, x_2, y_2, z_2, \ldots, x_n, y_n, z_n, p, q, r, \ldots$ to be determined, and for this purpose there are available $3n-m$ equations of motion, $3n-m$ equations connecting the parameters with the position co-ordinates, and the m kinematical equations.

*238. **Expression of Kinetic Energy in Terms of Generalised Co-ordinates.**—The kinematical equations as at § 215 may be written in this form—

$$\left.\begin{array}{c} f_1(x_1, y_1, z_1, x_2, y_2, z_2, \ldots)=0 \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ f_m(x_1, y_1, z_1, x_2, y_2, z_2, \ldots)=0 \end{array}\right\} \quad (1)$$

and since each of the position co-ordinates is expressible in terms of the parameters $p, q, r, \ldots$, we may write

$$\left.\begin{array}{l} x_1=\phi_1(p, q, r, \ldots),\quad y_1=\chi_1(p, q, r, \ldots),\quad z_1=\psi_1(p, q, r, \ldots) \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ x_n=\phi_n(p, q, r, \ldots),\quad y_n=\chi_n(p, q, r, \ldots),\quad z_n=\psi_n(p, q, r, \ldots) \end{array}\right\} (2)$$

Thus we obtain

$$\left.\begin{array}{c} \dot{x}_1=\dfrac{\partial x}{\partial p}\dot{p}+\dfrac{\partial x}{\partial q}\dot{q}+\dfrac{\partial x}{\partial r}\dot{r}+\ldots \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \end{array}\right\} \quad (3)$$

to $3n$ equations. The partial differential co-efficients $\partial x/\partial p, \ldots$ are supposed to be expressed in terms of the parameters $p, q, r, \ldots$ The substitution of these values of $\dot{x}_1, \dot{y}_1, \dot{z}_1, \ldots$ in the expression

$$\tfrac{1}{2}\Sigma m(\dot{x}^2+\dot{y}^2+\dot{z}^2)$$

converts the kinetic energy, T, into a homogeneous quadratic function of the velocities $\dot{p}, \dot{q}, \dot{r}, \ldots$ with co-efficients which are functions of the co-ordinates. Thus we may write

$$T=\tfrac{1}{2}\{(p, p)\dot{p}^2+2(p, q)\dot{p}\dot{q}+\ldots+(q, q)\dot{q}^2+2(q, r)\dot{q}\dot{r}+\ldots\}, \quad (4)$$

where $(p, p), (p, q), \ldots, (q, q), \ldots$ denote co-efficients of the powers and products of the velocities as indicated.

The parameters $p, q, r, \ldots$ are commonly called generalised co-ordinates, from the fact that they need not represent merely position co-ordinates, but may be any quantity characteristic of the motion of the system. Thus dp may be an elementary rotation, or a linear displacement, or a small quantity of fluid which passes a cross section in a pipe, and so on. To every generalised displacement

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there is a generalised "force" of corresponding type. Thus in the case just mentioned the corresponding forces are respectively a couple, a force in the ordinary sense, a fluid pressure, and so on. The product of the force into the displacement is always an amount of work, and quantities of any physical nature whatever may be taken as force and corresponding displacement, provided their product have the dimensional formula of work, namely $[ML^2T^{-2}]$, § 191.

*239. **Generalised Forces and Equation of Work.**—We shall denote the forces corresponding to the co-ordinates $p, q, r, \ldots$ by $P, Q, R, \ldots$ They are to be found in the following manner from the equation of work. We have seen that

$$\delta W = \Sigma(X_a \delta x + Y_a \delta y + Z_a \delta z), \tag{5}$$

and by (3)

$$\left.\begin{aligned} \delta x &= \frac{\partial x}{\partial p}\delta p + \frac{\partial x}{\partial q}\delta q + \ldots, \\ \delta y &= \frac{\partial y}{\partial p}\delta p + \frac{\partial y}{\partial q}\delta q + \ldots, \\ &\ldots\ldots\ldots \end{aligned}\right\} \tag{6}$$

to $3n$ equations. These values substituted in (40) of chap. iii. give

$$\left.\begin{aligned} \delta W = \Sigma\left(X_a\frac{\partial x}{\partial p} + Y_a\frac{\partial y}{\partial p} + \ldots\right)\delta p + \Sigma\left(X_a\frac{\partial x}{\partial q} + Y_a\frac{\partial y}{\partial q} + \ldots\right)\delta q \\ + \ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}\right\} \tag{7}$$

We *define* $P, Q, R, \ldots$ by the equations

$$\left.\begin{aligned} P &= \Sigma\left(X_a\frac{\partial x}{\partial p} + Y_a\frac{\partial y}{\partial p} + \ldots\right) \\ Q &= \Sigma\left(X_a\frac{\partial x}{\partial q} + Y_a\frac{\partial y}{\partial q} + \ldots\right) \\ &\ldots\ldots\ldots \end{aligned}\right\} \tag{8}$$

to $3n - m$ equations. Thus we obtain

$$\delta W = P\delta p + Q\delta q + R\delta r + \ldots, \tag{9}$$

which is the equation of work expressed in terms of generalised forces and displacements.

*240. **Lagrange's Equations of Motion.**—We can now obtain Lagrange's equations of motion. There are several ways of obtaining them, but the simplest is by transformation of the equations of motion of a set of free particles. Thus consider the equation

$$m\ddot{x} = X.$$

X is the actual force producing the acceleration x. Now in the

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equation of work, and therefore in (7), we can replace X_a, Y_a, ... by the actual forces on the particles, since the connection forces which involve no work, when introduced, each bring in a set of terms the sum of which is identically zero. Hence we may write instead of (8)

$$\left.\begin{aligned} &\Sigma\left(X\frac{\partial x}{\partial p}+Y\frac{\partial y}{\partial p}+\dots\right)=P\\ &\Sigma\left(X\frac{\partial x}{\partial q}+Y\frac{\partial y}{\partial q}+\dots\right)=Q\\ &\quad\dots\dots\dots \end{aligned}\right\}\quad(10)$$

Thus we have

$$\left.\begin{aligned} &\Sigma m\left(\ddot{x}\frac{\partial x}{\partial p}+\ddot{y}\frac{\partial y}{\partial p}+\dots\right)=P\\ &\Sigma m\left(\ddot{x}\frac{\partial x}{\partial q}+\ddot{y}\frac{\partial y}{\partial q}+\dots\right)=Q\\ &\quad\dots\dots\dots \end{aligned}\right\}\quad(11)$$

Now

$$\ddot{x}\frac{\partial x}{\partial p}=\frac{d}{dt}\left(\dot{x}\frac{\partial x}{\partial p}\right)-\dot{x}\frac{d}{dt}\frac{\partial x}{\partial p},$$

and it can easily be proved (see § 241) that

$$\frac{\partial x}{\partial p}=\frac{\partial \dot{x}}{\partial \dot{p}},\quad \frac{d}{dt}\frac{\partial x}{\partial p}=\frac{\partial \dot{x}}{\partial p},$$

so that

$$\ddot{x}\frac{\partial x}{\partial p}=\frac{d}{dt}\left(\dot{x}\frac{\partial \dot{x}}{\partial \dot{p}}\right)-\dot{x}\frac{\partial \dot{x}}{\partial p}.$$

Therefore we have

$$\begin{aligned}\Sigma m\left(\ddot{x}\frac{\partial x}{\partial p}+\ddot{y}\frac{\partial y}{\partial p}+\dots\right)&=\Sigma m\frac{d}{dt}\left(\dot{x}\frac{\partial \dot{x}}{\partial \dot{p}}+\dot{y}\frac{\partial \dot{y}}{\partial \dot{p}}+\dots\right)\\ &\qquad-\Sigma m\left(\dot{x}\frac{\partial \dot{x}}{\partial p}+\dot{y}\frac{\partial \dot{y}}{\partial p}+\dots\right)\\ &=\frac{d}{dt}\frac{\partial T}{\partial \dot{p}}-\frac{\partial T}{\partial p},\end{aligned}$$

and similar equations are obtainable in the same way for the other co-ordinates. Thus we have finally

$$\left.\begin{aligned} &\frac{d}{dt}\frac{\partial T}{\partial \dot{p}}-\frac{\partial T}{\partial p}=P\\ &\frac{d}{dt}\frac{\partial T}{\partial \dot{q}}-\frac{\partial T}{\partial q}=Q\\ &\quad\dots\dots \end{aligned}\right\}\quad(12)$$

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These are Lagrange's celebrated equations of motion.

*241. **Remarks on Lagrange's Equations.**—The reader to whom this view of the subject is new must be careful to observe the meaning of the differential co-efficients. It is presupposed that x, y, z for every particle are expressed in terms of $p, q, r, \ldots$ by equations (2), so that $\dot{x}, \dot{y}, \dot{z}$ can be expressed in terms of $\dot{p}, \dot{q}, \dot{r}, \ldots$ to enable $\partial\dot{x}/\partial\dot{p}, \partial\dot{x}/\partial\dot{q}, \ldots$ to be calculated. The differential co-efficients are all partial, that is, for example, $\dot{x}$ is supposed to be expressed in terms of $\dot{p}, \dot{q}, \dot{r}, \ldots$, and $\partial\dot{x}/\partial\dot{p}$ is taken on the supposition that $\dot{p}$ varies while $\dot{q}, \dot{r}, \ldots$ are constant, and so for the other differentiations. That $\partial x/\partial p = \partial\dot{x}/\partial\dot{p}$ follows at once from (3).

Again, the substitution, $\partial\dot{x}/\partial p$ for $d(\partial x/\partial p)/dt$ may be justified as follows. Put w for $\partial x/\partial p$, then w can be expressed in terms of $p, q, r, \ldots$. Then

$$\frac{dw}{dt} = \frac{\partial w}{\partial p}\dot{p} + \frac{\partial w}{\partial q}\dot{q} + \ldots.$$

But by (3)

$$\frac{\partial\dot{x}}{\partial p} = \frac{\partial w}{\partial p}\dot{p} + \frac{\partial w}{\partial q}\dot{q} + \ldots,$$

so that dw/dt and $\partial\dot{x}/\partial p$ are identical.

The partial differential co-efficients $\partial T/\partial\dot{p}, \partial T/\partial\dot{p}, \ldots$ are to be obtained from the expression of the kinetic energy in terms of $p, q, r, \ldots, \dot{p}, \dot{q}, \dot{r}, \ldots$ given above.

An important caution in the use of Lagrange's equations may be stated here. The co-ordinates $p, q, r, \ldots$ must not only be independent, but they must be capable with the kinematical equations of expressing the configuration of the system at any instant. This is ensured if $\dot{x}, \dot{y}, \dot{z}, \ldots$ are expressible as in (3). Neglect of this point has led some writers into very serious error; for example, the expression of the kinetic energy of a rigid body in terms of its angular velocities at a particular instant about its principal axes has been used as T in Lagrange's equations. This is not a proper choice of velocities of co-ordinates for this purpose, inasmuch as these axes move with the body, and the angular velocities tell nothing as to the position of the body in space.

*242. **Lagrange's Equations with Dissipative Forces.**—If the system be acted on by dissipative forces so that the actual force producing in a particle m the rate of change of momentum $m\ddot{x}$ is the force $X - X_d$, where X is the force applied and X_d is the dissipative force, we have equations (10) as above, but instead of (11)

$$\left.\begin{aligned} \Sigma m\left(\ddot{x}\frac{\partial x}{\partial p} + \ddot{y}\frac{\partial y}{\partial p} + \ldots\right) &= P - \Sigma\left(X_d\frac{\partial x}{\partial p} + Y_d\frac{\partial y}{\partial p} + \ldots\right) \\ \Sigma m\left(\ddot{x}\frac{\partial x}{\partial q} + \ddot{y}\frac{\partial y}{\partial q} + \ldots\right) &= Q - \Sigma\left(X_d\frac{\partial x}{\partial q} + Y_d\frac{\partial y}{\partial q} + \ldots\right) \\ \ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}\right\} \quad (13)$$

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The sums on the right-hand sides of these equations are the generalised dissipative forces. Denoting them by P_d, Q_d, ... we get for Lagrange's equations with dissipative terms included

$$\left.\begin{aligned} \frac{d}{dt}\frac{\partial T}{\partial \dot{p}} - \frac{\partial T}{\partial p} &= P - P_d \\ \frac{d}{dt}\frac{\partial T}{\partial \dot{q}} - \frac{\partial T}{\partial q} &= Q - Q_d \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \end{aligned}\right\} \quad (14)$$

If the forces P, Q, ... are conservative we have $P = -\partial E/\partial p$, $Q = -\partial E/\partial q$, ..., where E is the potential energy, which must be a function of the co-ordinates only. If, for example, it contained the time explicitly as well as the co-ordinates, the forces would be derivable from the function, but the conservation of energy would not hold. Also besides the conservative forces and the dissipative forces any other generalised components of external force P_e, Q_e, ... may act on the system. Finally then we have

$$\left.\begin{aligned} \frac{d}{dt}\frac{\partial T}{\partial \dot{p}} - \frac{\partial T}{\partial p} + \frac{\partial E}{\partial p} &= P_e - P_d \\ \frac{d}{dt}\frac{\partial T}{\partial \dot{q}} - \frac{\partial T}{\partial q} + \frac{\partial E}{\partial q} &= Q_e - Q_d \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \end{aligned}\right\} \quad (15)$$

In general E is a function of the co-ordinates p, q, r, ... only. It therefore does not involve $\dot{p}$, $\dot{q}$, ..., and $\partial E/\partial \dot{p}$, $\partial E/\partial \dot{q}$, ... are all zero identically. If, then, we write L for $T - E$ the equations may be written

$$\left.\begin{aligned} \frac{d}{dt}\frac{\partial L}{\partial \dot{p}} - \frac{\partial L}{\partial p} &= P_e - P_d \\ \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} &= Q_e - Q_d \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \end{aligned}\right\} \quad (16)$$

*243. **The Dissipation Function. General Form of Lagrange's Equations.**—The dissipative forces P_d, Q_d, ... are capable, in an important class of cases, of being derived from a homogeneous quadratic function of the velocities $\dot{p}$, $\dot{q}$, ..., namely

$$F = \tfrac{1}{2}(a_{11}\dot{p}^2 + 2a_{12}\dot{p}\dot{q} + \ldots + a_{22}\dot{q}^2 + 2a_{23}\dot{q}\dot{r} + \ldots).$$

Then

$$P_d = -\frac{\partial F}{\partial \dot{p}} \quad Q_d = -\frac{\partial F}{\partial \dot{q}}, \ldots$$

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The equations of motion can then be written

$$\left.\begin{aligned} &\frac{d}{dt}\frac{\partial T}{\partial \dot{p}} - \frac{\partial T}{\partial p} + \frac{\partial F}{\partial \dot{p}} + \frac{\partial E}{\partial p} = P_e \\ &\frac{d}{dt}\frac{\partial T}{d\dot{q}} - \frac{\partial T}{\partial q} + \frac{\partial F}{\partial \dot{q}} + \frac{\partial E}{\partial q} = Q_e \\ &\ldots\ldots\ldots\ldots \end{aligned}\right\} \quad (17)$$

The function F has been called by Lord Rayleigh the *dissipation function.* It is of great service in the general theory of frictional resistances which depend on the first powers of the velocities of the parts of a system, and by analogy in the dynamical treatment of a system of mutually influencing electric currents, as we shall see later.

*244. **Explicit Appearance of Time in Kinematical Equations.**—It is to be remarked that the process by which Lagrange's equations are established in § 241 is not affected by the explicit appearance of the time t in the kinematic equations (2). The only effect of this is to give kinematic conditions varying with the time, and instead of (3)

$$\left.\begin{aligned} &\dot{x} = \frac{\partial x}{\partial t} + \frac{\partial x}{\partial p}\dot{p} + \frac{\partial x}{\partial q}\dot{q} + \ldots \\ &\ldots\ldots\ldots \end{aligned}\right\} \quad (18)$$

Lagrange's equations therefore hold for this case also.

The kinetic energy, however, is no longer a homogeneous function of the generalised velocities, but in consequence of the quantities $\partial x/\partial t$, $\partial y/\partial t$, ... consists of three parts, a homogeneous quadratic function of the velocities, a linear function, that is a function involving only first powers and no products of the velocities, but depending on the co-ordinates, and a third part, not involving the velocities, but squares and products of $\partial x/\partial t$, ... The co-efficients of the quadratic and linear parts as well as the third part itself are of course functions of the co-ordinates and the time.

It can be shown that if the kinematic conditions vary work must be spent on the system to ensure their fulfilment, but into this question and others we have not space here to enter (see however § 215).

*245. **Generalised Components of Momentum.**—If T expressed as in (4) be partially differentiated with respect to $\dot{p}$, $\dot{q}$, $\dot{r}$, ..., then, as the reader may verify by inspection,

$$\left.\begin{aligned} &\frac{\partial T}{\partial \dot{p}} = (p, p)\dot{p} + (p, q)\dot{q} + (p, r)\dot{r} + \ldots \\ &\frac{\partial T}{\partial \dot{q}} = (p, q)\dot{p} + \ldots + (q, q)\dot{q} + (q, r)\dot{r} + \ldots \\ &\ldots\ldots\ldots\ldots \end{aligned}\right\} \quad (19)$$

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If the first of these be multiplied by $\dot{p}$, the second by $\dot{q}$, and so on, and the results be added we obtain evidently

$$2T = \dot{p}\frac{\partial T_v}{\partial \dot{p}} + \dot{q}\frac{\partial T_v}{\partial \dot{q}} + \dot{r}\frac{\partial T_v}{\partial \dot{r}} + \dots \tag{20}$$

The quantities $\partial T/\partial \dot{p}$, $\partial T/\partial \dot{q}$, ... are, as exhibited above, linear functions of the velocities, and are called the *generalised components of momentum* of the system.

We shall denote them by the letters ξ, η, ζ, ..., so that

$$2T = \dot{p}\xi + \dot{q}\eta + \dot{r}\zeta + \dots \tag{21}$$

*246. **Kinetic Energy expressed in Terms of Generalised Momenta. Reciprocal Equations.**—There are just as many components of momentum as there are independent generalised co-ordinates, and therefore the equations (19) are independent. By means of (19) $\dot{p}$, $\dot{q}$, $\dot{r}$, ... can be expressed in terms of ξ, η, ζ, ...; and so T can be expressed as a homogeneous quadratic function of the generalised momenta. Thus

$$T_v = \tfrac{1}{2}\{[p, p]\xi^2 + 2[p, q]\xi\eta + \dots + [q, q]\eta^2 + \dots\}, \tag{22}$$

where square brackets are used to distinguish the co-efficients $[p, p]$, .. in this expression for T from the quite different co-efficients (p, p), ... in (4).

To distinguish the expression of T as in (4) from its expression as on the right of (22) we shall denote the former by T_v, the latter by T_m. We have by (21)

$$T_m = -T_v + \dot{p}\xi + \dot{q}\eta + \dot{r}\zeta + \dots \tag{23}$$

Now in T_m the velocities $\dot{p}$, $\dot{q}$, ... are supposed expressed in terms of ξ, η, ζ, ... Hence

$$\frac{\partial T_m}{\partial \xi} = -\left(\frac{\partial T_v}{\partial \dot{p}}\frac{\partial \dot{p}}{\partial \xi} + \frac{\partial T_v}{\partial \dot{q}}\frac{\partial \dot{q}}{\partial \xi} + \dots\right) + \dot{p} + \xi\frac{\partial \dot{p}}{\partial \xi} + \eta\frac{\partial \dot{q}}{\partial \xi} + \dots$$

But $\partial T_v/\partial \dot{p} = \xi$, ..., and therefore the equation just written reduces to $\partial T_m/\partial \xi = \dot{p}$. We thus have the reciprocal relations

$$\left.\begin{aligned} &\frac{\partial T_v}{\partial \dot{p}} = \xi, \quad \frac{\partial T_v}{\partial \dot{q}} = \eta, \quad \frac{\partial T_v}{\partial \dot{r}} = \zeta, \dots \\ &\frac{\partial T_m}{\partial \xi} = \dot{p}, \quad \frac{\partial T_m}{\partial \eta} = \dot{q}, \quad \frac{\partial T_m}{\partial \zeta} = \dot{r}, \dots \end{aligned}\right\} \tag{24}$$

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Also putting dT_m/dp for the total variation of T_m with respect to p so far as it appears in T_m through its co-efficients and through $\xi, \eta, \ldots$ we have by (23)

$$\frac{dT_m}{dp} = \frac{\partial T_m}{\partial p} + \frac{\partial T_m}{\partial \xi}\frac{\partial \xi}{\partial p} + \ldots = -\frac{\partial T_v}{\partial p} + \dot{p}\frac{\partial \xi}{\partial p} + \ldots$$

But it has been shown that $\partial T_m/\partial \xi = \dot{p}, \ldots$. Hence this relation gives

$$\frac{\partial T_m}{\partial p} = -\frac{\partial T_v}{\partial p}, \quad \frac{\partial T_m}{\partial q} = -\frac{\partial T_v}{\partial q}, \ \ldots \tag{25}$$

*247. **Hamilton's Dynamical Equations.**—By equations (24) and (25) Lagrange's equations can evidently be written in the form

$$\left.\begin{aligned} \frac{d\xi}{dt} + \frac{\partial T_m}{\partial p} &= P \\ \frac{d\eta}{dt} + \frac{\partial T_m}{\partial q} &= Q \\ \ldots \end{aligned}\right\} \tag{26}$$

This is the form given to the equations by Sir W. R. Hamilton. T, it is to be remembered, is supposed expressed as an explicit function of the momenta $\xi, \eta, \ldots$. In this form the equations state that the time rate of increase of momentum, together with the rate of variation of T (expressed as a function of the momenta with co-efficients depending on the co-ordinates) with the co-ordinate p, when all the other co-ordinates and the momenta remain constant, is equal to the applied force of type P. Dissipative forces may of course be introduced as already indicated.

If $P = -\partial E/\partial p$, $Q = -\partial E/\partial q$, ..., and $\mathbf{E}$ be put for $T_m + E$ we get instead of (26),

$$\frac{d\xi}{dt} = -\frac{\partial \mathbf{E}}{\partial p}, \quad \frac{d\eta}{dt} = -\frac{\partial \mathbf{E}}{\partial q}, \ \ldots, \tag{26'}$$

and since we may write $\partial T_m/\partial \xi = \partial(T_m + E)/\partial \xi$, ..., we have by (24),

$$\frac{dp}{dt} = \frac{\partial \mathbf{E}}{\partial \xi}, \quad \frac{dq}{dt} = \frac{\partial \mathbf{E}}{\partial \eta}, \ \ldots, \tag{26''}$$

Equations (26′), (26″) are the famous canonical equations of motion, the integration of which has engaged the attention of so many mathematicians.

*248. **Lagrange's Equations for Impulsive Forces.**—Lagrange's equations (12), if the forces $P, Q, \ldots$ be impulsive, take the form

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$$\left.\begin{aligned} &\left[\frac{\partial T}{\partial \dot{p}}\right]_0^\tau = P, \quad \left[\frac{\partial T}{\partial \dot{q}}\right]_0^\tau = Q, \ldots \\ \text{or} \quad &\xi - \xi_0 = P, \ \eta - \eta_0 = Q, \ldots \end{aligned}\right\} \tag{27}$$

where ξ_0, ξ, η_0, η, ... are the values of the momenta at the beginning and end of the time τ of the action of the impulses P, Q, ..., which are the time-integrals of the generalised forces over the duration τ of the impulse. The other terms $\partial T/\partial p$, ... in equations (26) do not appear, inasmuch as their values are finite during the interval τ, which is vanishingly small. Thus the generalised momenta generated by the impulses are equal to the time-integrals of the generalised forces.

If impulses P, Q, ... are simultaneously applied during a very short interval τ, and the velocities at the beginning and end of τ are $\dot{p}_0$, $\dot{p}$, $\dot{q}_0$, $\dot{q}$, ..., the work done by the impulses P, Q, ... is

$$\tfrac{1}{2}(\dot{p}+\dot{p}_0)P + \tfrac{1}{2}(\dot{q}+\dot{q}_0)Q + \ldots;$$

for provided τ be very short the work done by the impulses does not depend on the order or manner of their application, though this will not be the case for the work done by any particular impulse.

Hence by (27) we obtain

$$\tfrac{1}{2}\{(\xi-\xi_0)(\dot{p}+\dot{p}_0)+(\eta-\eta_0)(\dot{q}+\dot{q}_0)+\ldots\} = \tfrac{1}{2}\{P(\dot{p}+\dot{p}_0)+Q(\dot{q}+\dot{q}_0)+\ldots\}$$

*249. **Kinetic Energy in Impulsive Generation of Motion.**—But identically, as may be verified by writing down the values of ξ, ξ_0, η, η_0, ... and performing the multiplications,

$$\xi\dot{p}_0 + \eta\dot{q}_0 + \ldots = \xi_0\dot{p} + \eta_0\dot{q} + \ldots, \tag{28}$$

so that we have for the work done by the impulses

$$\tfrac{1}{2}(\xi\dot{p} - \xi_0\dot{p}_0 + \eta q - \eta_0\dot{q}_0 + \ldots) = \tfrac{1}{2}\{P(\dot{p}+\dot{p}_0)+Q(\dot{q}+\dot{q}_0)+\ldots\}. \tag{29}$$

If $\dot{p}_0$, $\dot{q}_0$, ... be all zero, so that the system is started from rest by the impulses

$$T = \tfrac{1}{2}(\xi\dot{p}+\eta\dot{q}+\zeta\dot{r}+\ldots) = \tfrac{1}{2}(P\dot{p}+Q\dot{q}+R\dot{r}+\ldots). \tag{30}$$

Thus the kinetic energy generated is the sum of the impulses each multiplied by half the corresponding velocity generated.

*250. **Reciprocal Relations. Application.**—A reciprocal relation similar to (28) is easily obtained with forces instead of momenta, and displacements δp, δq, ... from a position of equilibrium, provided the *potential energy* is a homogeneous quadratic function of the co-ordinates. Thus if the forces be conservative and δp, δq, ... be

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displacements from a configuration of equilibrium produced by forces $P_1, Q_1, \ldots$, and $\delta p_2, \delta q_2, \ldots$ be another set of displacements from the same configuration produced by the set of forces $P_2, Q_2, \ldots$, we have

$$P_1\delta p_2 + Q_1\delta q_2 + \ldots = P_2\delta p_1 + Q_2\delta q_1 + \ldots. \tag{31}$$

This theorem it is left to the reader to prove. It is due to Lord Rayleigh, and is of the greatest service in a great many practical cases.

For example, let a beam be deflected by a load applied at any point A_1, and let the deflection at any point A_2 be observed, then if the load be applied at A_2 the same deflection will be produced at A_1. Thus the deflection produced at any chosen point by a load applied at any other point of the beam may be obtained by the process of placing the load at the former point, and measuring the deflection at the other points.

The reader should experimentally verify this by means of a lath placed on two supports on the same level near its ends, and a weight attached to a fine wire or string which can be placed round the lath, and slipped along to different points. (See also chap. xv.)

*251. **Lord Kelvin's and Bertrand's Theorems.** (1) **Lord Kelvin's Theorem.**—We shall now state and prove the general dynamical theorems due to Lord Kelvin and M. Bertrand respectively.

Lord Kelvin's theorem asserts that if any material system is suddenly set into motion with any specified velocities imposed on certain parts of the system by the application of suitable impulses to those parts, and to those only, while the other parts are left free to take such velocities as result from the connections of the system, the resulting motion is that for which the kinetic energy has the smallest possible value consistent with the fulfilment of the prescribed velocity conditions.

Bertrand's theorem states that if the system be set into motion by given impulses applied to definite points of the system, the kinetic energy is greater than that for any other motion which the system could have been made to take by the same impulses, together with a merely constraining set of impulses, that is a set of impulses which do no work on the whole.

These theorems may be proved together as follows. Let the generalised co-ordinates be divided into two sets, $p_1, p_2, \ldots, q_1, q_2, \ldots$, and let $\xi_1, \xi_2, \ldots, \eta_1, \eta_2, \ldots$ be corresponding impulses. Thus

$$2T = \Sigma\xi\dot{p} + \Sigma\eta\dot{q}. \tag{32}$$

Let $\dot{p} + \delta\dot{p}$, $\dot{q} + \delta\dot{q}$ be representative velocities for impulses $\xi + \delta\xi$, $\eta + \delta\eta$ of another set applied to the system. Then we have for the kinetic energy

$$2T' = \Sigma(\xi + \delta\xi)(\dot{p} + \delta\dot{p}) + \Sigma(\eta + \delta\eta)(\dot{q} + \delta\dot{q}). \tag{33}$$

Thus

$$2(T' - T) = \Sigma(\dot{p}\,\delta\xi + \xi\,\delta\dot{p} + \delta\xi\,\delta p) + \Sigma(\dot{q}\,\delta\eta + \eta\,\delta\dot{q} + \delta\eta\,\delta\dot{q}) \tag{34}$$

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Now let the condition imposed be that the p-velocities are given, then in the values of T and T' every $\delta\dot{p}$ is zero. When the p-velocities are produced by impulses solely of that type every η is zero. By (32) $2T$ is then $\Sigma\xi\dot{p}$. Again, when the p-velocities are produced by impulses of type $\xi+\delta\xi$, $\delta\eta$, $2T'$ is $\Sigma(\xi+\delta\xi)\dot{p}+\Sigma(\dot{q}\,\delta\eta+\delta\eta\delta\dot{q})$. Thus

$$2(T'-T)=\Sigma(\delta\xi.\dot{p}+\dot{q}\,\delta\eta+\delta\eta\delta\dot{q}).$$

But the velocities $\dot{p}$, $\dot{q}$ with corresponding impulses ξ, 0, and the velocities $\dot{p}$, $\dot{q}+\delta\dot{q}$ with corresponding impulses $\xi+\delta\xi$, $\delta\eta$, are typical of the two possible motions of the system. Thus by the reciprocal theorem (28) $\Sigma\dot{p}(\xi+\delta\xi)+\Sigma\dot{q}\,\delta\eta=\Sigma\xi\dot{p}$, so that $\Sigma\dot{p}\,\delta\xi+\Sigma\dot{q}\,\delta\eta=0$. Hence

$$2(T'-T)=\Sigma\delta\eta\,\delta\dot{q}, \tag{35}$$

or the kinetic energy of the first motion is less than that of any other for which the p-velocities are the same, by the kinetic energy of the motion which compounded with the former would produce the latter.

*252. (2) **Bertrand's Theorem.**—According to the conditions of Bertrand's theorem, as stated above, the ξ impulses are given in the first instance, while the η impulses are all zero. The velocities in this case are those which result from the connections and are represented by the type $\dot{p}$, $\dot{q}$. The energy in the motion is to be compared with that of the motion in which the impulses are represented by ξ and $\delta\eta$. The velocities are $\dot{p}+\delta\dot{p}$, $\dot{q}+\delta\dot{q}$, with the condition as to constraints stated in the theorem, and expressed by $\Sigma\delta\eta(\dot{q}+\delta\dot{q})=0$. This gives $\Sigma\dot{q}\,\delta\eta=-\Sigma\delta\eta\delta\dot{q}$.

Now by (34)

$$2(T'-T)=\Sigma(\xi\,\delta\dot{p}+\dot{q}\,\delta\eta+\delta\eta\delta\dot{q})$$

the reciprocal theorem (28) gives for the two possible motions here considered $\Sigma\xi\delta\dot{p}=\Sigma\dot{q}\,\delta\eta$, so that we obtain finally

$$2(T-T')=\Sigma\delta\eta\delta\dot{q}. \tag{36}$$

Thus the energy of the first motion is greater than that of any other motion produced by the same impulses, together with another system of impulses which do no work on the whole, by the kinetic energy of the motion which must be compounded with the former motion to give the latter.

These theorems both express the fact that any constraint imposed on a system virtually increases its inertia. Thus for a single coordinate the inertia is $\xi/\dot{p}$. If ξ is fixed $\dot{p}$ is diminished and so the energy is diminished by any constraint; if $\dot{p}$ is fixed ξ is increased and so the energy is increased by any constraint.

*253. **Principle of Least Action.**—A very general dynamical principle from which the equations of motion of a material system can be derived, and which may therefore be regarded as containing all abstract dynamics, can be little more than mentioned here. It is called the *principle of least action*, and in its first form was given, though rather vaguely, by Maupertuis. The real theory of action is

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due to Euler, Lagrange, and in a very high degree to Sir William Rowan Hamilton. The modern development of Hamilton's theories is mainly due to Jacobi, to whose writings the reader who wishes to become thoroughly acquainted with the subject is referred.*

Let a system move from any configuration to any other configuration in the interval of time from t_0 to t_1. The *action* of the system is defined as twice the time-integral of the kinetic energy, that is

$$A = 2\int_{t_0}^{t_1} T dt. \tag{37}$$

Let the different particles describe paths of which s is a representative, then $2Tdt = \Sigma m\dot{s}^2 dt = \Sigma m\dot{s}ds$, so that

$$A = \Sigma \int_{s_0}^{s} m\dot{s}ds, \tag{38}$$

where s_0, s_1 are limits of the path s of the particle m, for the epochs t_0, t_1, and the summation is taken for all the particles of the system. The latter form shows that the action is the sum of the space-integrals of the momenta of the particles composing the system.

Now the system may move from the initial to the final configuration in an infinite number of ways, and on the transition we can impose any possible conditions. Let, then, the initial and final configurations be specified, and let the transition take place subject to the condition that the sum $T+E$ of the kinetic and potential energies shall be constant, but not, however, the interval $t_1 - t_0$ of time occupied in the transition. The mode of transition for which the action is (as it is usually put) least, is that for which the motion takes place in accordance with the ordinary equations of a conservative system, that is for which we have the equation of work

$$\delta E + \Sigma m\ddot{s}\delta s = 0. \tag{39}$$

If we call δA the variation of A from one mode of transition to another, the theorem merely asserts that, subject to the condition of no variation of $T+E$, if the motion takes place according to the general variational equation of motion—that is if the transition takes place without any constraint applied to the system from without—the variation of the action from one mode of transition to another very near the first is of the second order of small quantities. This is better described, as it was by Hamilton, by saying that for the unguided motion the action is stationary. It can be shown, however, that if there is only one possible mode of motion consistent with the variational equation, for that the action is truly a minimum.

* Werke, Bd. vii. The reader will find also a very suggestive brief account of action in Tait's *Dynamics*; also a fairly full treatment of the general theory in Routh's *Rigid Dynamics*, Part ii.

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Another theorem which may be described also as one of action is that, subject to the condition that $t_1 - t_0$ is constant, the value of

$$S = \int_{t_0}^{t_1} L dt, \tag{40}$$

in which L denotes $T - E$, is stationary when the motion takes place according to the variational equation of work (39), that is when the motion is unconstrained.

For proofs of these theorems the reader may refer to Routh's *Rigid Dynamics*, part ii., or to the author's *Magnetism and Electricity*, vol i. p. 183. We shall find examples of an analogous principle in geometrical optics.

254. **Solution of Equations of Motion.**—The solution of the equations of motion of a dynamical system has been studied by various writers, among whom may be mentioned Lagrange, Laplace, and Jacobi, and in recent years in connection with questions of physical astronomy by Professor Hill and M. Poincaré. The elaborate work of the latter entitled *Méthodes Nouvelles de la Mécanique Céleste* contains a very valuable discussion of various problems of celestial mechanics of great interest.

The expressions for the momentum of a system and its moment of momentum about any axis, and the expression for the kinetic energy of the system may be obtained as first integrals from the equations of motion and the geometrical conditions which the system fulfils. They are steps towards the solution of the problem, to find equations which determine explicitly the configuration of the system at any time t. When the problem has been thus solved, the velocities and accelerations of its various parts can be at once obtained.

It is not always possible to solve the equations of motion, but it is possible sometimes to write down at once the expressions for the kinetic and potential energy and the moment of momentum of the system, and to obtain valuable information from these by means of the equations of motion.

255. **Motion of a Top.**—As an example of this process we may consider the motion of a body which moves under gravity about a fixed point, the problem, in fact, of a top spinning without friction round a sharp peg on which it rests.

We shall suppose that the moments of inertia about two of its principal axes (see § 166) are equal, and that the fixed point lies on the axis of unequal moment of inertia. Thus let O (Fig. 122) be the fixed point, OC the axis of figure of the body, and let C denote the moment of inertia of the body about the axis OC, and A the moment of inertia about an axis OE, which for convenience we shall take perpendicular to OC in the plane ZOC. It will be convenient also to take a third axis OD perpendicular to the plane ZOC and to OE, as shown in Fig. 122. The directions of motion are as shown in the small circles, so that OD, OE, OC form a set of axes corresponding to OX, OY, OZ in Fig. 9, along which distances may be taken

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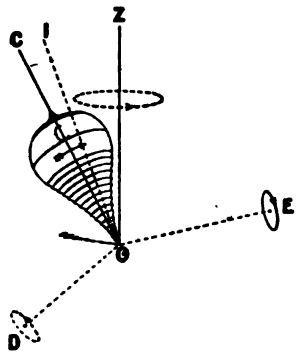
to represent the angular velocities or the moments of momentum about them. The directions of motion are in each case against the clock to an observer looking along the axes towards the point O.

Let the top be rotating about the axis OC *fixed in the body* with angular velocity ω, and the plane ZOC of the axis of the top and the vertical OZ be turning with angular velocity $\dot{\psi}$ about OZ, and ψ be the angle the plane ZOC makes with a plane fixed in space. The motion about OZ will be called frequently in what follows the *precessional motion* of the body, from the fact that it is perfectly analogous to the motion of the earth which gives rise to the precession of the equinoxes. The angular velocity ω is that of any plane through OC, and fixed in the body, relatively to the plane ZOC.

FIG. 122.



BE, OC, OD are a rectangular system of axes fixed relatively to the plane ZOC.

Let θ denote the angle ZOC. The angular velocity $\dot{\psi}$ about OZ may be resolved into two components, $\dot{\psi}\cos\theta$ and $\dot{\psi}\sin\theta$ about fixed axes coincident in position with OC and OE, which is the intersection of the planes COZ and EOD. The former component gives with ω, $\omega+\dot{\psi}\cos\theta$ for the angular velocity about an axis *fixed in space*, and coinciding with the instantaneous position OC, and thus the body has the three angular velocities $\omega+\dot{\psi}\cos\theta$ about OC, $\dot{\psi}\sin\theta$ about OE, and $\dot{\theta}$ about the third axis OD.

The kinetic energy is therefore given by

$$T=\tfrac{1}{2}\{C(\omega+\dot{\psi}\cos\theta)^2+A(\dot{\psi}^2\sin^2\theta+\dot{\theta}^2)\}. \qquad (41)$$

Besides this equation we might write down another expressing the fact that, since there are no forces that have moments about OZ and OC, the moment of momentum about the vertical must remain constant. Two of the angular velocities we have just seen are $\omega+\dot{\psi}\cos\theta$ about OC and $\dot{\psi}\sin\theta$ about an axis through O at right angles to OC, and in the plane ZOC. Hence the moment of momentum of the body is made up of $C(\omega+\dot{\psi}\cos\theta)$ about OC, and $A\dot{\psi}\sin\theta$ about the axis just specified. Since the former is constant we have $\omega+\dot{\psi}\cos\theta$, the angular velocity about OC, equal to the initial value, n say. Again, the moment of momentum about the vertical OZ is obtained by resolving these two components, and is $A\dot{\psi}\sin^2\theta+Cn\cos\theta$. Therefore

$$A\dot{\psi}\sin^2\theta+Cn\cos\theta=H, \qquad (42)$$

where H is a constant.

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If the centroid be G, OG be denoted by h, and the mass of the body by m, and the potential energy be taken as zero when G is in the horizontal plane through O, we have for the potential energy the expression $mgh\cos\theta$. Hence

$$E = mgh\cos\theta, \quad T + E = K, \tag{43}$$

where K is a constant, complete the equations required. Equations (41), (42), (43) express the whole motion.

256. **Rising and Falling of Top.**—We may at once draw some important conclusions. First we shall find the values of θ for which $\dot{\theta} = 0$, that is the limits between which the inclination of the top to the vertical lies. Equation (41) may be written by (43) in the form

$$A\dot{\theta}^2 + A\dot{\psi}^2\sin^2\theta + Cn^2 + 2mgh\cos\theta = 2K. \tag{44}$$

Eliminating $\dot{\psi}$ from this by (42), we have

$$A\dot{\theta}^2 + \frac{(H - Cn\cos\theta)^2}{A\sin^2\theta} + Cn^2 + 2mgh\cos\theta = 2K. \tag{45}$$

Let the top be given initially rotating with angular velocity n about OC, with its axis inclined at β to the vertical, and with $\dot{\psi}$ and $\dot{\theta}$ each zero. The value of H is therefore $Cn\cos\beta$. The initial value of $2T$ is Cn^2, of $2E$ is $2mgh\cos\beta$, so that $2K = Cn^2 + 2mgh\cos\beta$. Putting then $\dot{\theta} = 0$ in (45), and substituting these values of H and $2K$, we obtain

$$(\cos\theta - \cos\beta)\{2mghA(\cos^2\theta - 1) - C^2n^2(\cos\theta - \cos\beta)\} = 0. \tag{46}$$

This is a cubic equation for the determination of $\cos\theta$, and obviously one root is $\cos\theta = \cos\beta$. For the other two roots we have the quadratic

$$2mghA(\cos^2\theta - 1) - C^2n^2(\cos\theta - \cos\beta) = 0. \tag{47}$$

The expression on the left is positive for $\cos\theta = \infty$, negative for $\cos\theta = 1$, and positive for $\cos\theta = -1$. Hence one root lies between ∞ and 1, and another between $+1$ and -1. The former root is, of course, inadmissible.

Now solving (47) we have, putting k for $4mghA/C^2n^2$, the result

$$\cos\theta = \frac{1}{k}\{1 \pm \sqrt{(1 - 2k\cos\beta + k^2)}\}.$$

The quantity under the square root sign is essentially positive since $\cos\beta < 1$. Hence the admissible root is

$$\cos\theta = \frac{1}{k}\{1 - \sqrt{(1 - 2k\cos\beta + k^2)}\}. \tag{48}$$

The inclination of the top to the vertical thus lies between the limits β and $\cos^{-1}[\{1 - \sqrt{(1 - 2k\cos\beta + k^2)}\}/k]$, or α say, and oscillates from one value to the other.

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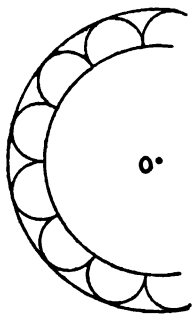
If the top be started rotating very rapidly, that is if n be very great, we get by (48), including terms up to k^2,

$$\cos\theta = \cos\beta - \frac{k}{2}\sin^2\beta. \tag{49}$$

Thus the limits of θ are β and $\cos^{-1}(\cos\beta - \frac{1}{2}k\sin^2\beta_0)$.

257. **Precessional Motion of Top.**—By (44) the value of $\dot{\psi}$ is greatest when $\dot{\theta} = 0$ unless $Cn^2 + 2mgh\cos\theta - 2K = 0$, in which case $\dot{\psi}$ is zero. Since $2K = Cn^2 + 2mgh\cos\beta$, this equation is

FIG. 123.



$$2mgh(\cos\theta - \cos\beta) = 0, \tag{50}$$

so that $\dot{\psi}$ is zero whenever $\cos\theta = \cos\beta$.

When this relation is not fulfilled $\dot{\theta} = 0$ gives

$$\dot{\psi}^2 = \frac{2mgh}{A}\,\frac{\cos\beta - \cos\theta}{\sin^2\theta}, \tag{51}$$

and (42) gives for $\dot{\psi}$ at the other limiting value of θ

$$\dot{\psi} = 2\frac{mgh}{Cn}. \tag{52}$$

The projection of the path of any point of the axis, say the apex or the centroid, on the horizontal plane through O is, as represented in Fig. 123, a periodic curve lying between two circles, about O as centre, corresponding to the two limiting values of θ. At the inner circle $\dot{\psi} = 0$, so that there the curve is cusped; at the outer circle $\dot{\psi}$ has a maximum value (twice the *steady value*, § 260) while $\dot{\theta} = 0$, so that the curve touches the circle.

*258. **Completion of Solution of Problem.**—The motion may be completely worked out and the position of the axis found for any instant in the following manner. Since $2K = Cn^2 + 2mgh\cos\beta$, we have for (45)

$$A\sin^2\theta.\dot{\theta}^2 = 2mgh(\cos\theta - \cos\beta)\{(\cos^2\theta - 1) - \frac{C^2n^2}{2Amgh}(\cos\theta - \cos\beta)\}. \tag{53}$$

If $\cos\alpha$, $\cos\beta$, c be the roots, in ascending order of magnitude, of the cubic equation (46) for $\cos\theta$, α is the larger and β the smaller of the two values of θ between which the inclination of the axis to the vertical oscillates. Hence, if l be written for A/mh, the above equation becomes

$$l\sin^2\theta.\dot{\theta}^2 = 2g(\cos\theta - \cos\alpha)(\cos\theta - \cos\beta)(\cos\theta - c). \tag{54}$$

Now putting

$$\cos\theta = \cos\alpha\cos^2\phi + \cos\beta\sin^2\phi,$$

we obtain

$$\cos\theta - \cos\alpha = (\cos\beta - \cos\alpha)\sin^2\phi$$
$$\cos\theta - \cos\beta = (\cos\alpha - \cos\beta)\cos^2\phi,$$

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which satisfy the inequality $\alpha > \theta > \beta$. These substituted in (54) transform it to

$$\dot{\phi}^2 = p^2(1 - k^2\sin^2\phi),$$

where $k^2 = (\cos\beta - \cos\alpha)/(c - \cos\alpha)$, $p^2 = g(c - \cos\alpha)/2l$. Thus for any value of ϕ corresponding to a value of t reckoned from an instant at which ϕ was zero, or $\theta = \alpha$, we have

$$pt = \int_0^\phi \frac{d\phi}{\sqrt{1 - k^2\sin^2\phi}}. \tag{55}$$

The value of t is thus found as an elliptic integral of the first kind. The values of pt (or K as it is usually written) are given for a gradation of values of ϕ in tables of elliptic functions. Hence both pt and ϕ, that is in effect corresponding values of t and θ, can be found.

To complete the solution it is necessary to show how corresponding values of t and ψ are to be obtained. By (42) and writing $2f$ for Cn/A we get,

$$\dot{\psi} = 2f\frac{\cos\beta - \cos\theta}{1 - \cos^2\theta} = f\left(\frac{1 + \cos\beta}{1 + \cos\theta} - \frac{1 - \cos\beta}{1 - \cos\theta}\right).$$

Using the expression for θ stated above, and writing m_1 for $(\cos\beta - \cos\alpha)/(1 + \cos\alpha)$ and m_2 for $-(\cos\beta - \cos\alpha)/(1 - \cos\alpha)$, we obtain

$$\dot{\psi} = f\left\{(1 + m_1)\frac{1}{1 + m_1\sin^2\phi} - (1 + m_2)\frac{1}{1 + m_2\sin^2\phi}\right\}. \tag{56}$$

But, as has been seen, $p\,dt = d\phi/(1 - k^2\sin^2\phi)^{\frac{1}{2}}$, so that we obtain finally

$$\left.\begin{aligned} p\psi = f\Bigg\{(1 + m_1)&\int_0^\phi \frac{d\phi}{(1 + m_1\sin^2\phi)\sqrt{1 - k^2\sin^2\phi}} \\ &- (1 + m_2)\int_0^\phi \frac{d\phi}{(1 + m_2\sin^2\phi)\sqrt{1 - k^2\sin^2\phi}}\Bigg\}, \end{aligned}\right\} \tag{57}$$

which expresses ψ in terms of elliptic integrals of the third kind. As before, values of ψ and ϕ can be obtained for any actual case from tables. (See for further discussion of this subject Greenhill, *Elliptic Functions.*)

259. **Equations of Motion of Top.**—The discussion just given contains the chief facts regarding the motion of a top. The subject is of great physical interest because of the exact analogy between the motion and the elastic behaviour of a thin round wire of

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flexibility the same in all directions (chap. xv.), and also because of its bearing on the important and difficult subject of the stability of motion. It has always seemed very remarkable that a top or gyrostat (which is only a top running on bearings attached to a case enclosing it) should be stable when rotating in a position in which without spin it would be quite unstable. We have the same phenomenon in the permanence of direction of the earth's axis, and of the axis of a rifle bullet. On account of the importance of this aspect of the subject, we add a short sketch of the method of investigating the stability of the steady motion of a top, that is the motion in which θ and $\dot{\psi}$ are constant.

But first we shall illustrate the Lagrangian method by using it to find the equations of motion for a top, though, as the reader may verify, they may be easily obtained otherwise. Information regarding the steady motion and the small oscillations about that state can be easily obtained by the aid of these equations.

The independent co-ordinates may be taken as θ, ψ, and the angle corresponding to ω. Only the velocity ω, however, not the co-ordinates, enters into the expression for the kinetic energy. Thus we have

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\theta}} = A\ddot{\theta}, \quad \frac{\partial T}{\partial \theta} = A\dot{\psi}^2 \sin\theta\cos\theta - C(\omega + \dot{\psi}\cos\theta)\dot{\psi}\sin\theta,$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\psi}} = A\frac{d}{dt}(\dot{\psi}\sin^2\theta) + C\frac{d}{dt}\{(\omega + \dot{\psi}\cos\theta)\cos\theta\}, \quad \frac{\partial T}{\partial \psi} = 0,$$

$$\frac{d}{dt}\frac{\partial T}{\partial \omega} = \frac{d}{dt}(\omega + \dot{\psi}\cos\theta), \quad \frac{\partial V}{\partial \theta} = -mgh\sin\theta.$$

The equations of motion are therefore

$$\left.\begin{aligned} A\ddot{\theta} - \{A\dot{\psi}\cos\theta - C(\omega + \dot{\psi}\cos\theta)\}\dot{\psi}\sin\theta &= mgh\sin\theta \\ \frac{d}{dt}\{A\dot{\psi}\sin^2\theta + C(\omega + \dot{\psi}\cos\theta)\cos\theta\} &= 0 \\ \frac{d}{dt}(\omega + \dot{\psi}\cos\theta) &= 0. \end{aligned}\right\} \quad (58)$$

The last equation shows that the angular velocity of the body about OC, that is $\omega + \dot{\psi}\cos\theta$, remains equal to the initial value, n say. The same equation converts the second, after integration, into

$$A\dot{\psi}\sin^2\theta + Cn\cos\theta = H.$$

These are the results as to constancy of moment of momentum already obtained in § 254.

260. **Steady Motion of Top.**—By (44), $A\dot{\psi}\sin^2\theta = Cn(\cos\beta - \cos\theta)$, and if the angular velocity n be very great, $\cos\theta$ as we have seen in (49) oscillates between the limits $\cos\beta$ and $\cos\beta - 2mghA\sin^2\beta/C^2n^2$,

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so that the two values of θ are very nearly equal. Hence by (51) the utmost value that $A\dot{\psi}^2\sin^2\theta$ can have is $2mghA\sin^2\beta/Cn$, so that if n is very great $\dot{\psi}$ must be very small. The first of the equations of motion (58) can be written

$$A\ddot{\theta}-\{(A\dot{\psi}\cos\theta-Cn)\dot{\psi}-mgh\}\sin\theta=0.$$

If the motion be steady θ is constant and therefore $\ddot{\theta}=0$. Thus we get the equation of steady motion

$$\{(A\dot{\psi}\cos\theta-Cn)\dot{\psi}+mgh\}\sin\theta=0. \tag{59}$$

This will hold if $\sin\theta=0$, or if $A\dot{\psi}^2\cos\theta-Cn\dot{\psi}+mgh=0$. The first of these gives $\theta=0$ or $\theta=\pi$, that is the centroid is either vertically above or vertically below O. But if $\cos\beta$ is finite this leads to an infinite value of $\dot{\psi}$, and is inconsistent with the fact that θ must lie between the narrow limits stated.

The second relation gives approximately, since $\dot{\psi}$ is small,

$$\dot{\psi}=\frac{mgh}{Cn} \tag{60}$$

for the angular velocity of the steady precessional motion, or half the maximum value given by (52) above. If we solve the equation, putting θ_1 for the constant value of θ, we obtain

$$\dot{\psi}=\frac{Cn}{2A\cos\theta_1}\left\{1\pm\sqrt{\left(1-\frac{4Amgh}{C^2n^2}\cos\theta_1\right)}\right\}. \tag{61}$$

The values of $\dot{\psi}$ must be real, and hence if $h\cos\theta_1$ is positive, that is if the centroid is above the level of the point O, we must have

$$n^2>\frac{4Amgh\cos\theta_1}{C^2}$$

in order that the steady motion may be possible. If the top be started with n very great and $\dot{\psi}$, $\dot{\theta}$ both zero, then since $\dot{\psi}$ remains small the negative sign must be taken before the radical in (61). As a first approximation to $\dot{\psi}$ we get from this

$$\dot{\psi}=\frac{mgh}{Cn}$$

as before.

A second approximation is

$$\dot{\psi}=\frac{mgh}{Cn}\left(1+\frac{Amgh}{C^2n^2}\cos\theta_1\right), \tag{62}$$

The motion corresponding to the other root of (61) is also possible,

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and its stability is proved by the fact that the expression for the period of the small oscillation is real in this case.

261. **Small Oscillations about Steady Motion.**—The small oscillations about steady motion may be investigated by putting $\dot{\psi}=\dot{\psi}_1+\dot{\psi}'$, and $\theta=\theta_1+\theta'$, where $\dot{\psi}_1$, θ_1 denote steady values about which the motion oscillates, and $\dot{\psi}'$, θ' are small variable quantities of which the second and higher powers may be neglected. When these are substituted in the equations of momentum, and the equation (58) for $\ddot{\theta}$, and the steady motion conditions, for example

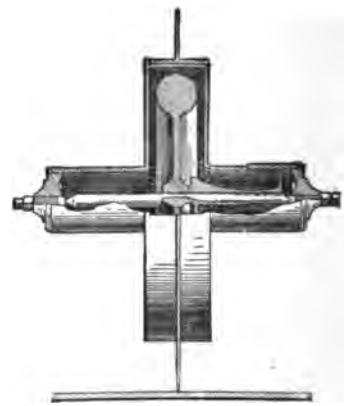
$$Cn=mgh/\dot{\psi}+A\dot{\psi}\cos\theta_1,$$

are given effect to, two simultaneous equations are obtained, which (with assumed values $f\sin(2\pi t/\tau+a)$, $g\sin(2\pi t/\tau+b)$ for $\dot{\psi}_1$, θ_1) enable the period τ of oscillation to be calculated.* It comes out

$$\tau=\frac{2\pi A\dot{\psi}_1}{\sqrt{(A^2\dot{\psi}_1^4-2mghA\dot{\psi}_1^2\cos\theta_1+m^2g^2h^2)}}. \qquad (63)$$

262. **Stability of Motion. Behaviour of Gyrostat.**—The process here illustrated is exceedingly important as it is that which is often adopted to investigate small disturbances from a state of steady or regular motion. It is also one by which the stability of a given motion can be tested, that is by which may be answered the question whether, if the motions be slightly disturbed, the system will perform small oscillations about the state of motion or pass further and further away from it. A recondite example is the question of the stability of the solar system referred to in § 184; a very simple example is the motion of a particle along the lowest generating line inside an inclined straight cylindrical tube, or along the highest generating line outside. In the former case the motion is obviously stable—a small lateral disturbance will only cause the particle to pursue its course making small oscillations about the generating line; in the latter case such a disturbance would cause the particle to permanently leave the generating line and finally the tube.

FIG. 124.



The investigation given above suffices to explain most of the curious phenomena shown by a top formed by a massive fly-wheel

* For further information the reader should refer to Routh's *Advanced Rigid Dynamics*, chap. v.

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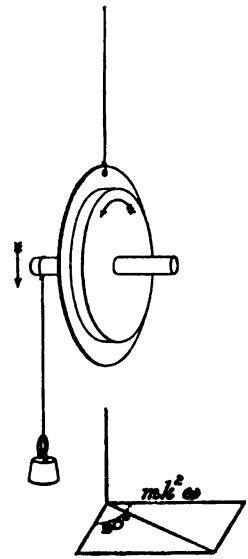
running on bearings attached to a framework or case surrounding it. Such a top is called a gyrostat, and in one form is a well-known toy, in another is an instrument capable of illustrating some of the most recondite physical phenomena; the latter form is shown in Fig. 124, which represents a gyrostat resting on a thin edge on a glass plate. The case is represented as cut open to show the fly-wheel, which is pivoted on a spindle turning on bearings attached to the case. The fly-wheel is partly in section to indicate the construction. As the section shows, the fly-wheel is a thin disk with a massive rim. Fig. 125 illustrates a simple case. The gyrostat is hung by a string from a point of the case in the plane of the fly-wheel so that the centroid is below the point of support. A weight is hung by a string from one end of the axis, and therefore produces momentum about a horizontal axis in the plane of the fly-wheel, which is supposed to rotate rapidly about the (horizontal) axis of the figure. The gyrostat does not incline over sensibly, but turns round in azimuth with angular velocity $\dot{\psi}$ in nearly steady motion, on which, however, are super-imposed oscillations in the period

FIG. 125.



Gyrostat showing precession. The action of the weight causes turning of the axis in a horizontal plane, that is round the cord, in the direction against the hands of a watch to an observer looking from above.

$$\tau = \frac{2\pi A\dot{\psi}}{\sqrt{A^2\dot{\psi}^4 + m^2g^2h^2}}, \qquad (64)$$

where m is the mass hung on the end of the axis, and h is its distance from the line of the string. A is of course the moment of inertia of the gyrostat about a diameter of the fly-wheel. The value of $\dot{\psi} = \frac{mgh}{Cn}$, so that if n is very great

$$\tau = 2\pi\frac{A}{Cn}. \qquad (65)$$

This would also be the formula for the time of oscillation of a rapidly rotating gyrostat or top placed with its axis horizontal and supported at one extremity of the axis, as in Fig. 127. The couple producing the precession would be, as in the discussion above, due to the weight

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of the top. The complete theory of the gyrostat includes terms depending on the motion of the case, which does not spin with the fly-wheel, but these are here neglected. The reader ought not to have much difficulty in modifying the equations given above to take the inertia of the case into account.

If the precession is produced in any other way by an applied couple, we have for the moment of the couple

$$L = Cn\dot{\psi}. \tag{66}$$

263. **The Earth is a Top. Precession.**—The theory of the top is interesting for the reason that the earth is really a gigantic top with a precessional motion due to the attractive forces exerted by the sun and moon, which do not have resultants passing accurately through the earth's centroid, and in a very slight degree to the action of the planets.

The earth spins in a period of one sidereal day about an axis inclined at an angle of 66° 32′ 48″ to the plane of the ecliptic. The plane of the equator is inclined to that of the ecliptic at an angle of 23° 27′ 12″. The attraction of the sun, for example, exerts a couple tending to draw the plane of the equator into coincidence with the ecliptic, just as gravity tends to pull the top down so that its centroid shall be in the horizontal plane through O. Also just as the top does not fall down, but moves with a precessional motion so that when the motion is steady the axis describes a cone round the vertical OC, the earth's axis describes as a result of the moments of all the forces exerted upon it a conical motion in space of period about 26,000 years. The path of the north pole of the earth is thus a circle among the stars on the celestial sphere, about the pole of the ecliptic.

There are three main periodic disturbances of the precessional motion, causing what is called *nutation*, or "nodding," of the earth, of which we shall speak again in the chapter on *Astronomical Dynamics*. They are, however, of the nature of forced oscillation, since they are produced by forces which have their own proper period.

264. **Elementary Calculation of Precession in Steady Motion.**—The following method of investigating the steady motion of a top is simple, and serves to illustrate the idea of an instantaneous axis of rotation explained at § 94 above, and can be applied directly to the precessional motion of the earth. The top is spinning with angular velocity ω about its axis while the centroid turns with uniform angular velocity $\dot{\psi}$. If the rotation about the axis of the top be against the clock when looked down upon from above, the precessional motion will be also against the clock when looked down upon from above Z (Fig. 126). Since the body is rotating with angular velocity $\omega + \dot{\psi}\cos\theta (= n)$ about OC, and angular velocity $\dot{\psi}\sin\theta$ about OE (Fig. 126) in the plane ZOC, it is turning about an instantaneous axis in the same plane. Let this axis be represented by OI, and α be the angle COI. We have at once

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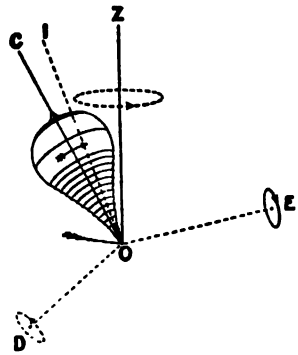
$$\tan\alpha = \frac{\dot{\psi}\sin\theta}{n}, \tag{67}$$

and the angular velocity about OI is $\sqrt{n^2+\dot{\psi}^2\sin^2\theta}$.

In the case of steady motion the instantaneous axis describes a cone in space, and at the same time a cone in the body. The motion may be described by saying that the body-cone rolls on the space-cone. With the directions of rotation in the case considered here the body-cone lies outside the space-cone; in the case of the earth's precessional motion the body-cone rolls round the inside of the space-cone (see § 267).

FIG. 126.



BE, *OC*, *OD* are a rectangular system of axes fixed relatively to the plane *ZOC*.

In a short interval of time dt the gravity mg of the top generates moment of momentum $mgh\sin\theta\,dt$, about the axis OD (Fig. 126) at right angles to OE and OC. This is to be supposed fixed in direction during the small interval of time. This moment of momentum is not represented by that due to generation of angular velocity in dt about OD, for during dt the axes OC and OE do not remain in the same positions. OC is moved by the rotation about OE through an angle $\dot{\psi}\sin\theta\,dt$ towards the direction OD. This gives an increase of moment of momentum about OD of amount $Cn\dot{\psi}\sin\theta\,dt$, as will be shown in § 282 below, or as the reader may easily verify. In the same way OE is turned away from the direction OD in the same time through an angle $\dot{n}dt$, and this causes a decrease of moment of momentum about OD of amount $A\dot{\psi}\sin\theta.\dot{n}dt$. If $d\Omega$ be the angular velocity generated in dt about the temporarily fixed axis OD, the whole moment of momentum about that direction generated in dt is therefore

$$A\,d\Omega + (C-A)n\dot{\psi}\sin\theta\,dt.$$

Hence

$$\left.\begin{aligned} &A\frac{d\Omega}{dt} + (C-A)n\dot{\psi}\sin\theta = mgh\sin\theta \\ \text{or}\quad &d\Omega = \frac{1}{A}\{mgh-(C-A)n\dot{\psi}\}\sin\theta\,dt. \end{aligned}\right\} \tag{68}$$

The generation of the angular velocity $d\Omega$ causes OI to turn round the origin O towards OD through an angle $d\Omega/\sqrt{n^2+\dot{\psi}^2\sin^2\theta}$ Hence a point P on OI at unit distance from O moves a distance of this amount in a direction at right angles to ZOC. But P is at

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a distance from OZ equal to $\sin(\theta - \alpha)$, and therefore turns round OZ through an angle $d\Omega/\{\sqrt{n^2 + \dot{\psi}^2 \sin^2\theta}.\sin(\theta - \alpha)\}$. But from (67) it is easy to show that $\sin(\theta - \alpha) = \omega \sin\alpha/\dot{\psi} = \omega \sin\theta/\sqrt{n^2 + \dot{\psi}^2 \sin^2\theta}$, so that the angle turned through by P is $d\Omega/\omega \sin\theta$. This angle must also be $\dot{\psi}dt$, so that we have $\dot{\psi} = (d\Omega/dt)/\omega \sin\theta$. Hence

$$\dot{\psi} = \frac{mgh - (C - A)n\dot{\psi}}{A\omega}; \tag{69}$$

which, since $\omega = n - \dot{\psi}\cos\theta$, may be written

$$A\dot{\psi}^2 \cos\theta - Cn\dot{\psi} + mgh = 0 \tag{70}$$

the equation already obtained in § 259 for $\dot{\psi}$ in the case of steady motion.

265. **Precession Calculated by compounding Moments of Momentum.**—The same result may be obtained still more simply, perhaps, by direct composition of moments of momentum. The axis, OH say, of resultant moment of momentum lies in the plane ZOC, and makes an angle β with OC given by the equation

$$\tan\beta = \frac{A\dot{\psi}\sin\theta}{Cn}. \tag{71}$$

But by the generation of moment of momentum $mgh \sin\theta\, dt$ about the direction OD the axis OH is displaced towards OD through an angle $mgh \sin\theta\, dt/K$, where K is the resultant moment of momentum. A point Q on OH at unit distance from O therefore turns about OZ through an angle

$$mgh \sin\theta\, dt/K \sin(\theta - \beta).$$

FIG. 127.

But from (71) it is easy to prove that

$$\sin(\theta - \beta) = \sin\theta\left(1 - \frac{A\dot{\psi}}{Cn}\cos\theta\right)\frac{Cn}{K}. \tag{72}$$

Hence the angle turned through by Q about OZ in time dt is $mghdt/(Cn - A\dot{\psi}\cos\theta)$. This must be $\dot{\psi}dt$ since the motion is steady and the axis of resultant moment of momentum always lies in the plane ZOC. We have therefore $\dot{\psi} = mgh/(Cn - A\dot{\psi}\cos\theta)$, or

$$A\dot{\psi}^2 \cos\theta - Cn\dot{\psi} + mgh = 0,$$

the same result again.

If $\theta = \pi/2$, that is if the axis of the top is horizontal, the first term vanishes.

If n be great (70) holds approximately, as we have seen, when the

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first term is omitted, and is applicable to a gyrostat in steady motion, whether supported on the knife-edge surrounding its case (Fig. 124) with its axis in a position inclined to the vertical, or hung by a string attached to the part of the case enclosing the axis (Fig. 127). When the axis is horizontal (70) is strictly true, as the first term, which involves the inertia of the case, disappears from the equation. The quantity mg is, however, the gravity of the whole instrument.

266. **Numerical Example of Precession of Gyrostat.**—As a numerical example we take the following. A gyrostat is hung with its axis horizontal by a cord attached to the case surrounding the axle at a distance of 5 centimetres from the centre of inertia of the whole. The fly-wheel weighs 1500 grammes, has a radius of gyration of 4 centimetres, and rotates at 200 revolutions per second. The mass of wheel and case together is 2200 grammes. Find the angular velocity of the precessional motion, the motion being steady.

The moment of the weight about the point of attachment of the cord, that is mgh, is, in C.G.S. units, nearly 2160×5000. The value of Cn is $1800 \times 16 \times 400\pi$. Hence $\dot{\psi} = \cdot 364$, that is the axis turns round in the horizontal plane with an angular velocity of nearly a third of a radian per second.

The fact that a gyrostat thus supported and moving as described can remain with its axis horizontal, notwithstanding the fact that the centroid is not in the vertical through the point of support, has been considered paradoxical. It is not more so, however, than the behaviour of a common top which spins with its axis inclined to the vertical, provided the proper precessional motion is allowed to take place.

The reader may exercise himself on the preceding discussion by considering the general effect of introducing a couple aiding or opposing the precessional motion. For a top spinning in the ordinary manner on a horizontal plane, or a gyrostat placed in an inclined position to the vertical on the knife-edge opposite the plane of the fly-wheel, or a gyrostat spinning when hung by a cord, as shown in Fig. 127, an opposing couple causes the body to fall in the ordinary way under the action of gravity; an aiding couple causes the centroid to rise. In this way the action of friction on the peg of an ordinary top in raising the centroid may be understood. A complete account of the motion of a top in the case in which it spins on a rough plane on a rounded peg of finite size, so that a frictional couple of finite moment is applied to it, we have no space to give here. The reader may, however, consult Jellett's *Theory of Friction*, in which the subject is very fully discussed.

267. **Precessional Motion of the Earth.**—Precessional motion, and in particular that of the earth, is well illustrated by the model shown in Fig. 128. The globe, the southern end of which is cut away, is weighted so as to rest with the pin projecting from the north pole against the inner side of the horizontal ring, and represents the earth. The cone which has its apex at the centre of the globe, and

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its base at the circle of contact of the pin with the ring, is the space-cone (§ 264); the cone which has its apex at the centre, and its base at the circle in which the pin touches the ring, is the body-cone. The plane of the circle of contact in the ring is to be taken as parallel to the ecliptic; the vertical angle of the cone is twice the angle between

FIG. 128.

the plane of the equator and the ecliptic, that is twice the so-called obliquity of the ecliptic. The rotation of the earth being against the hands of a watch, the precessional motion is with the hands. The body-cone shown in the model is on a scale enormously too large.

The diameter of the rolling circle can easily be calculated for the earth supposed spinning with its centre fixed in space and describing uniformly its precessional motion in the period of 25,868 years. The diameter of the circle of contact of the space-cone, that is the diameter of the ring, is in feet about $2 \times 21 \times 10^6 \times \sin(23^\circ\ 27'\ 12'')$. This must be $25{,}868 \times 366\frac{1}{4}$ times the diameter of the circle of contact of the body-cone, so that the latter diameter is 1·764 feet. In other words, the intersection of the instantaneous axis with the surface of the earth describes a circle round the extremity of the axis of figure of 10·6 inches radius. The diagram of the model is taken from Thomson and Tait's *Natural Philosophy*, vol. i. Part i. The model itself is part of the illustrative apparatus in the University of Glasgow.

268. **Reaction of Top on Support.**—As an example of § 169 we may find the reaction exerted on the top by the supporting horizontal

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plane. If the plane be smooth this force is vertically upward. Let its amount be Y. The motion of the centroid is the same as if all the forces were applied there. The vertically upward component at the centroid is $Y - mg$. But the centroid has an acceleration $h\dot{\theta}^2$ towards O, and another $h\ddot{\theta}$ at right angles to OC in the direction to increase θ. Thus clearly $mh\ddot{\theta} = (mg - Y)\sin\theta$, $mh\dot{\theta}^2 = (mg - Y)\cos\theta$, so that

$$Y = mg - mh(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta). \tag{73}$$

The action on the ground is therefore less than the gravity of the top by $mh(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta)$.

The value of Y can be found by the following method, which is instructive. The kinetic energy of the centroidal motion is $\frac{1}{2}mh^2\dot{\theta}^2$, and the part of this corresponding to the vertical component of the velocity $h\dot{\theta}$ is $\frac{1}{2}mh^2(\dot{\theta}\sin\theta)^2$. Recalling that force in any direction on an unresisted body is space-rate, in that direction, of increase of kinetic energy, we see that the upward force applied by the ground to the top over and above the gravity mg is the upward space-rate of increase of $\frac{1}{2}mh^2(\dot{\theta}\sin\theta)^2$, that is the rate of increase of this quantity per unit of increase of the height $h\cos\theta$ of the centroid.

Thus we obtain $Y = mg + \frac{1}{2}\{d/d(h\cos\theta)\}\{mh^2(\dot{\theta}\sin\theta)^2\}$, or

$$Y = mg + \tfrac{1}{2}mh\frac{d}{d\cos\theta}(\dot{\theta}\sin\theta)^2. \tag{74}$$

The reader may show by differentiating [putting $d/d\cos\theta = -(d/dt)/\dot{\theta}\sin\theta$] that this is exactly the value of Y obtained by the other process.

269. **Gyrostatic Action of Fly-Wheel or of Armature of Dynamo.**—When a fly-wheel runs on bearings which are not fixed, but move about with the body to which they are secured, gyrostatic forces are brought into play which are sometimes overlooked, but which are of practical importance. For example, consider a dynamo-armature spinning rapidly about an axis fixed in bearings on board ship. If the axis is in the thwart-ship direction, the rolling of the ship brings into play forces which at any instant consist of a force towards the bow on one bearing, and an equal force towards the stern on the other. The pitching motion of the ship has no such effect. On the other hand, if the axis of the armature be fore and aft, the pitching produces forces on the bearings, one towards the starboard, the other towards the port side of the ship, while the rolling produces no such effect. These forces are reversed with the reversal of the motion, and thus a constantly reversing stress, alternately on the front and on the back of each bearing, is brought into play.

The same thing happens with the rotating parts of the engines, and with the screw-shaft in its bearings when the ship pitches, but in these cases the angular velocities are not so great as to make the effect so important.

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The amount of the force may be obtained in the following manner. Let the angular velocity with which the ship is rolling (or pitching) be $\dot{\psi}$, and the angular velocity of the rotating part of the machine be n, its moment of inertia about the axis be C, then $\dot{\psi}$ corresponds precisely to the angular velocity (§ 262) of the precessional motion of the gyrostat in the horizontal plane under the couple L, which acts about an axis perpendicular to the axis of rotation and to the plane of the precessional motion. The moment of the couple L is $Cn\dot{\psi}$ (§ 262). For the precessional motion of angular velocity $\dot{\psi}$ of the rotating part of the machine imposed by the rolling of the ship, a couple must be applied by the bearings to the axis of amount $Cn\dot{\psi}$. If l be the distance between the centres of the bearings the force on each is $Cn\dot{\psi}/l$. It is therefore proportional to the moment of inertia of the rotating part and to each of the angular velocities n and $\dot{\psi}$. It is greatest at the middle of a roll (or pitch) of a ship, and zero at the beginning and end, when the ship is reversing its angular motion.

These forces, it must be remembered, are such as to produce a couple about an axis at right angles at once to the axis of rotation and to the axis about which the rolling or pitching is taking place They are sometimes wrongly stated to be forces on the top and bottom of the bearings.

270. **Numerical Example of Gyrostatic Action of Dynamo on board Ship.**—As a numerical example we take the case of a large alternator, the armature of which weighs 10 cwt., has a radius of gyration of 2 feet, and runs at a speed of 600 revolutions. We shall suppose the axis placed in the thwart-ship direction, and the ship to roll through a total range of 30° in a period of 10 seconds. With the pound and foot as units of mass and length, the moment of inertia of the armature is 4480. The angular velocity of rotation of the armature is 20π radians per second. The maximum angular velocity of the ship is $2\pi \times 15/(10 \times 57{\cdot}3) = 3\pi/57{\cdot}3 = {\cdot}165$, in radians per second. Hence the couple parallel to the plane of the base of the dynamo exerted on the bearings is $4480 \times 20\pi \times {\cdot}165$ in pound-foot-second units. If the distance between the centres of the bearings is 2 feet, the average force in pounds applied to each bearing is this number divided by 64, that is about 726. The force on the front or back of each bearing, reversed every half revolution, is thus about a third of a ton.

271. **Uniplanar Motion with reference to Rotating Axes.**—Before leaving the subject of the dynamics of a system of bodies, it will be useful to consider the motion of a particle or of a rigid body with reference to moving axes.

As a first simple example we take (Fig. 129) the case of a particle B moving in a plane, and having its position determined by means of rectangular axes of co-ordinates x, y drawn from an origin A, which is at rest. We shall suppose that the axes revolve with uniform angular velocity n about A, in the direction indicated by the arrow at B. Relatively to the moving axes the velocity-components

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of B are $\dot{x}$, $\dot{y}$. But besides these the particle has velocity components due to the motion of the axes. To find the latter, imagine B to be rigidly fixed relatively to the axes so that $\dot{x}$, $\dot{y}$ are zero. It is then turning round A with angular velocity n, and its linear velocity, which is at right angles to AB, is nr if $r = AB$. The component of this parallel to x is $-ny$, and similarly the component parallel to y is nx. The

FIG. 129.

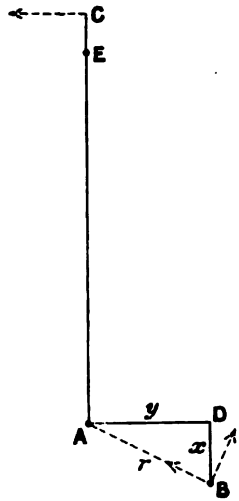
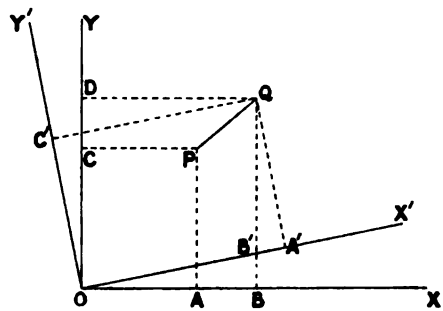


FIG. 130.



total component velocities u, v of the particle relatively to fixed axes coinciding with the positions of the axes of x, y at the instant under consideration are therefore given by the equations

$$u = \dot{x} - ny, \quad v = \dot{y} + nx. \tag{75}$$

The estimation of u, v will be seen more clearly perhaps from Fig. 130. The revolving axes are represented by OX, OY in one position, and by OX', OY' in a subsequent position near the former. In the interval, dt say, let any point P move from P to Q; the displacement along the revolving axis of x is from OA to OA', and along the axis of y is from OC to OC'. $OA' - OA$ is the change in x, $OC - OC$ is the change in $\dot{y}$; the limiting value of $(OA' - OA)/dt$, when dt is taken vanishingly small is $\dot{x}$. Similarly the limiting value of $(OC' - OC)/dt$ is $\dot{y}$.

Now to find the actual velocities of the point P parallel to the axes OX, OY we have for the components of PQ on these axes AB, CD. But AB is less than $OA' - OA$ by $B'A'$, which is $QB' \times \sin B'QA'$. The angle $B'QA'$ is ndt, and if dt be small, $QB' = y$, and $\sin B'QA' = ndt$. Hence $AB = OA' - OA - B'A' = (\dot{x} - ny)dt$. If we denote the *actual* velocity along OX by u, we have $AB = udt$, and so

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$u = \dot{x} - ny$. Similarly if v be the actual velocity along OY, we can show that $v = \dot{y} + nx$.

Now, returning to Fig. 130, consider a particle the co-ordinates of which relatively to the axes of x, y are numerically equal to u, v. The component velocities of this particle in space will be in like manner $\dot{u} - nv$, $\dot{v} + nu$. But these are the time-rates of variation of u, v, and are therefore the accelerations of the former particle relatively to fixed axes coinciding for the instant with the moving axes of x and y. Hence for the acceleration components we have the equations

$$\dot{u} - nv = \ddot{x} - 2n\dot{y} - n^2x, \quad \dot{v} + nu = \ddot{y} + 2n\dot{x} - n^2y. \tag{76}$$

If n vary continuously $\dot{u}$ will include the term $-\dot{n}y$ and $\dot{v}$ the term $\dot{n}x$, which must therefore be added to the quantities on the right of (76).

272. **Components of any Directed Quantity referred to Rotating Axes.**—It may be noticed here that if P, Q be the components parallel to the axes of x, y of any directed quantity whatever, the quantity may be represented by a point the co-ordinates of which are numerically equal, on any scale, to P, Q. Then as P, Q vary the point will change its position, and the time-rates of variation of the components relatively to fixed axes coinciding for the instant with the revolving axes will be respectively

$$U = \dot{P} - nQ, \quad V = \dot{Q} + nP. \tag{77}$$

In precisely the same way we should find for the time-rates of change of U, V relatively to fixed axes coinciding for the instant with the revolving axes the values $\dot{U} - nV$, $\dot{V} + nU$. For n constant

$$\dot{U} - nV = \ddot{P} - 2n\dot{Q} - n^2P, \quad \dot{V} + nU = \ddot{Q} + 2n\dot{P} - n^2Q.$$

This process may be applied, of course, indefinitely to find successive time-rates of change. Thus the time-rates of variation of the last components are

$$\left.\begin{aligned} \ddot{U} - n\dot{V} - n(\dot{V} + nU) &= \ddot{U} - 2n\dot{V} - n^2V = \dddot{P} - 3n\ddot{Q} - 3n^2\dot{P} + n^3Q \\ \ddot{V} + n\dot{U} + n(\dot{U} - nV) &= \ddot{V} + 2n\dot{U} - n^2V = \dddot{Q} + 3n\ddot{P} - 3n^2\dot{Q} - n^3P \end{aligned}\right\} \tag{78}$$

If n be not constant the terms arising from its variation must be supplied in the values of $\dot{U}$, $\dot{V}$, $\ddot{U}$, $\ddot{V}$.

Similar results hold when the directed quantity is not restricted to two dimensions, as we shall see presently (§ 280).

273. **Equations of Motion of Particle referred to Rotating Axes.**—Returning now to the particle at B (Fig. 129), let its mass be m, and the forces acting on it parallel to the positive directions of the axes of x and y be X, Y. The equations of motion are

$$m(\ddot{x} - 2n\dot{y} - n^2x) = X, \quad m(\ddot{y} + 2n\dot{x} - n^2y) = Y. \tag{79}$$

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As an example, let the particle be of unit mass, and be attracted towards the fixed centre A with a force μ/r^2, where $r = AB$. The values of X, Y are $-\mu x/r^3$, $-\mu y/r^3$, and therefore the equations of motion are $\ddot{x} - 2n\dot{y} - n^2x = -\mu x/r^3$, $\ddot{y} + 2n\dot{x} - n^2y = -\mu y/r^3$, or as they may be written

$$\ddot{x} - 2n\dot{y} + \left(\frac{\mu}{r^3} - n^2\right)x = 0, \quad \ddot{y} + 2n\dot{x} + \left(\frac{\mu}{r^3} - n^2\right)y = 0. \tag{80}$$

The reader may, as an exercise, find the expression for the kinetic energy, and from this and the value of the potential energy, μ/r, find these equations of motion by Lagrange's method.

Let us now suppose A to be revolving with angular velocity n about a fixed point E on AC at a distance a from A; we have, as remarked above, simply to add to $\dot{x}$, $\dot{y}$ the corresponding components of the velocity of A. These are na, in the direction of y, and zero in the direction of x. Thus u remains as before, v becomes $\dot{y} + n(x + a)$. The accelerations become therefore $\ddot{x} - 2n\dot{y} - n^2(x + a)$, $\ddot{y} + 2n\dot{x} - n^2y$, and if the forces are the same as before the equations of motion are as written above, with the substitution of $n^2(x + a)$ for n^2x.

274. **Elementary Lunar Theory. Approximate Equations of Motion of Moon. Equation of Energy.**—But we shall now suppose that there are three attracting bodies, one at C of very great mass, one at A of very much smaller mass, and a third at B of mass in its turn small in comparison with that at A, and that the distance AC is very great in comparison with AB. Thus we have three bodies which may be taken to represent the sun-earth-moon system.

The motion of the two first is approximately one of revolution round their common centroid E, which, on account of the great mass of the sun, we shall take as at C, the sun's centre. As a matter of fact it is about 275 miles nearer the earth. We shall regard the moon as an infinitesimal satellite revolving round the earth, that is we take the acceleration of the earth towards the moon as negligible in comparison with that of the moon towards the earth. The mass of the moon is 1/81 of that of the earth, and the common centroid of the earth and moon is nearly 3000 miles from the earth's centre, so that the supposition made is only a rough approximation to the truth.

We thus regard the earth as moving with uniform angular velocity n in a circle of radius a about C as centre, while the moon revolves round the earth in the orbit which results from the motion it has at any instant, and the forces, the attractions of the earth and the sun, which act upon it.

The acceleration of the earth toward the sun at any instant is n^2a. This is the force per unit mass toward the sun at distance a. Hence the force per unit mass toward the sun exerted on unit mass at B is $n^2a.a^2/R^2$ or n^2a^3/R^2, where $R^2 = CB^2 = (x + a)^2 + y^2$. The components of this in the direction of x, y increasing are the accelerations of the particle at B in these directions. These components are $-n^2a^3(x + a)/R^3$, $-n^2a^3y/R^3$. But approximately, since

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x and y are small compared with a, $(x+a)/R^3 = 1/(x+a)^2 = 1/a^2 - 2x/a^3$, and $y/R^3 = y/a^3$, if quantities involving higher powers of $1/a$ than the third are neglected. Thus the component accelerations due to the force towards the sun are $-n^2a + 2n^2x$, $-n^2y$. Besides these there are the accelerations in the same direction due to the earth, namely $-\mu x/r^3$, $-\mu y/r^3$. The total accelerations along x and y are therefore $-n^2a + 2n^2x - \mu x/r^3$, $-n^2y - \mu y/r^3$. Equating these to the accelerations $\ddot{x} - 2n\dot{y} - n^2(x+a)$, $\ddot{y} + 2n\dot{x} - n^2y$ already calculated, we obtain the equations of motion of the moon with reference to the specified axes. They are

$$\ddot{x} - 2n\dot{y} + \left(\frac{\mu}{r^3} - 3n^2\right)x = 0, \quad \ddot{y} + 2n\dot{x} + \frac{\mu}{r^3}y = 0 \tag{81}$$

If the first equation be multiplied by $\dot{x}$, the second by $\dot{y}$, the sum of the products is

$$\tfrac{1}{2}\frac{d}{dt}\left(\dot{x}^2 + \dot{y}^2 - 3n^2x^2 - 2\frac{\mu}{r}\right) = 0.$$

Hence integrating we obtain

$$\tfrac{1}{2}(\dot{x}^2 + \dot{y}^2) - \tfrac{3}{2}n^2x^2 = \frac{\mu}{r} + C, \tag{82}$$

where C is a constant. This is the equation of relative energy.

275. **Graphical Description of Path of Moon relatively to Earth.**—When the curvature of the actual path or the path relatively to the earth is known at every point, the path can be drawn by the method of describing successive short circular arcs from the successive centres of curvature as centres, as explained at § 105 above. The description of the relative path in the present case is of great interest on account of the more recent researches in lunar theory carried out by Mr. G. W. Hill and by M. Poincaré. We shall therefore calculate the curvatures in the present case.

The direction cosines of an element of the actual path at B are $(\dot{x} - ny)/\dot{s}$, $\{\dot{y} + n(x+a)\}/\dot{s}$, where $\dot{s}^2 = (\dot{x} - ny)^2 + \{\dot{y} + n(x+a)\}^2$. $\dot{s}$ is of course the velocity at B. The forces on the unit particle resolved along the normal towards the centre of curvature give the component

$$N = -\left(n^2y + \frac{\mu y}{r^3}\right)\frac{\dot{x} - ny}{\dot{s}} + \left(2n^2x - n^2a - \frac{\mu x}{r^3}\right)\frac{\dot{y} + n(x+a)}{\dot{s}}, \tag{83}$$

and this is $\dot{s}^2/\rho$, where ρ is the radius of curvature. Hence $1/\rho$ is obtained for the actual path by dividing the expression on the right by $\dot{s}^2$.

To find the curvature of the relative orbit, multiply the first equation of motion by $\dot{y}$, the second by $\dot{x}$, and subtract the first product from the second. Writing $\dot{s}_r^2$ for $\dot{x}^2 + \dot{y}^2$, the square of the relative velocity we obtain the equation

$$\ddot{y}\dot{x} - \ddot{x}\dot{y} = \frac{\mu}{r^3}(\dot{y}x - \dot{x}y) - 2n\dot{s}_r^2 - 3n^2x\dot{y}.$$

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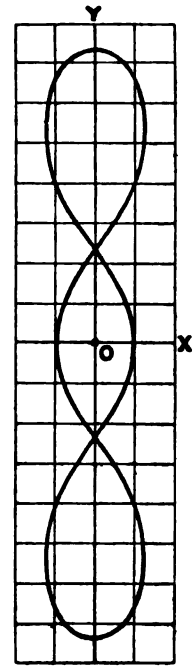
The quantity on the left, when divided by $\dot{s}_r$, becomes the component acceleration in the relative orbit towards the centre of curvature, and therefore equal to $\dot{s}_r^2/\rho$, where ρ_r is the radius of curvature of that orbit at B. Hence the expression on the right divided by $\dot{s}_r^3$ gives

FIG. 131.



$$\frac{1}{\rho_r} = \frac{N}{\dot{s}_r^2} - 2n/\dot{s}_r, \qquad (84)$$

where N now denotes the normal component at B of the force the components of which along x and y are $-(\mu/r^3 - 3n^2)x$, $-\mu y/r^3$. The square of the velocity $\dot{s}_r^2$ in the orbit is given by (82).

This result agrees with one obtained by Lord Kelvin,* and used by Dr. Magnus Maclean to draw by the graphical method already referred to the relative orbit shown in Fig. 131. The data used were $\mu = 1000$, $n = 1$, $2C = -130$, and the initial value of x was 2, that of y zero.

276. **Motion in Space referred to Moving Axes.**—It is of importance to be able to refer the motion of a system to three rectangular axes which are revolving about a system of three axes which are regarded as fixed in space. For example we may refer the motion of a particle on the earth's surface to three axes—one, that of z, say, coinciding with the upward vertical at the place, and the other two axes, those of x and y, drawn from the earth's centre, southward and eastward, say, in a plane at right angles to the vertical. We may consider these axes as moving relatively to axes fixed relatively to the fixed stars. Taking the earth's centre as fixed, we may take as one of these fixed axes the earth's polar axis of rotation, and the other two at right angles to one another in the plane of the equator.

But in many cases it is convenient to consider the moving axes as in motion with reference to a system of three fixed axes coinciding with the moving axes at the instant under consideration. These being known in position with reference to the chosen standard reference system enable the motion, or the position of the body, to be referred to the latter system when required.

Consider then a point referred to three rectangular axes OA, OB, OC which are at any instant revolving with angular velocities θ_1, θ_2, θ_3 about themselves, that is OB, OC are revolving in their own plane with angular velocity θ_1 about OA; OC, OA are revolving in their own plane with angular velocity θ_2 about OB; and similarly

* *Philosophical Magazine*, November 1892.

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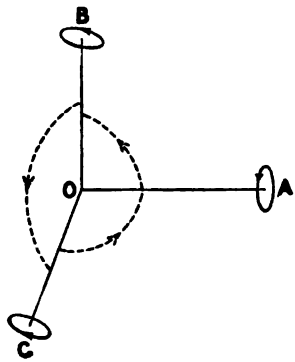
OA, OB are revolving with angular velocity θ_3 about OC. The directions of motion are indicated in Fig. 132.

Then if x, y, z be the co-ordinates relatively to the revolving axes of any particle whatever, its component velocities relatively to the moving axes are $\dot{x}, \dot{y}, \dot{z}$. The exact meaning of the components $\dot{x}, \dot{y}$ is explained in § 271 above for two axes. The same explanation is applicable at once to the present case. To find the other parts of the velocity components it is to be observed that the system of axes is revolving simultaneously with the angular velocities $\theta_1, \theta_2, \theta_3$ about the instantaneous positions of the axes themselves. Consider the effect of the rotation of the pair OA, OB about OC. The result is to add to $\dot{x}$ the term $-\theta_3 y$ just as $-ny$ was introduced in the case already treated in § 262, since θ_3 here plays the part of the angular velocity n. Similarly the motion of the pair OC, OA with angular velocity θ_2 about OB adds to $\dot{x}$ the term $\theta_2 z$, just as nx was added to $\dot{y}$ in the former case.

FIG. 132.



For the actual velocity u parallel to the axis of x we have therefore

$$u = \dot{x} - y\theta_3 + z\theta_2.$$

The change of direction of OA which is going on cannot influence this result. For a vanishingly short interval of time dt the positions of the planes AOB, COA, from which the terms $-y\theta_3$, $z\theta_2$ arise, are altered only infinitesimally in direction, and so in the limit these terms are unaffected.

By the same process similar results are obtained for the actual velocities parallel to the other two axes, OB, OC. Hence we have the three equations

$$\left.\begin{aligned} u &= \dot{x} - y\theta_3 + z\theta_2 \\ v &= \dot{y} - z\theta_1 + x\theta_3 \\ w &= \dot{z} - x\theta_2 + y\theta_1 \end{aligned}\right\} \qquad (85)$$

277. **Any Directed Quantity in Space referred to Rotating Axes. Examples.**—As before, this process is applicable to the calculation of the rates of change of the components of any directed quantity whatever parallel to revolving axes. Thus let F, G, H be the components (relatively to the revolving axes in their instantaneous position) of moment of momentum of a rigid body rotating with one point (the origin) fixed, and let L, M, N be similarly the components round the revolving axes of the moments of the forces in the body. Then we may take co-ordinates of a

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point equal numerically to F, G, H, and we should find for their rates of change

$$\left.\begin{aligned} U &= \dot{F} - G\theta_3 + H\theta_2 \\ V &= \dot{G} - H\theta_1 + F\theta_3 \\ W &= \dot{H} - F\theta_2 + G\theta_1 \end{aligned}\right\} \quad (86)$$

The quantities on the right are clearly the actual rates of change, at the instant, of F, G, H taken parallel to fixed axes coinciding with OA, OB, OC. Therefore, putting L, M, N for U, V, W in (86), we obtain the equations of motion of the rigid body.

As another example take the angular velocities θ_1, θ_2, θ_3 of the axes themselves. Calling the angular velocities about fixed axes coinciding at the instant with the revolving axes θ_x, θ_y, θ_z, we have

$$\left.\begin{aligned} \theta_x &= \dot{\theta}_1 - \theta_2\theta_3 + \theta_3\theta_2 = \dot{\theta}_1 \\ \dot{\theta}_y &- \dot{\theta}_2 - \theta_3\theta_1 + \theta_1\theta_3 = \dot{\theta}_2 \\ \dot{\theta}_z &= \dot{\theta}_3 - \theta_1\theta_2 + \theta_2\theta_1 = \dot{\theta}_3 \end{aligned}\right\} \quad (87)$$

Thus the rates of change of the angular velocities about the moving axes are precisely those about fixed axes coinciding with them, a proposition which does not follow from the fact that at the instant $\theta_x = \theta_1$, $\theta_y = \theta_2$, $\theta_z = \theta_3$.

278. **Motion relative to the Earth.**—As an illustration we take here the problem of the motion of a free particle near a point P on the surface of the earth, the axes being the moving system described in § 276 —viz., axes of x, y drawn southward and eastward through the earth's centre parallel to the horizontal plane at P, and an axis of z drawn upwards along the vertical through P. We shall denote the earth's angular velocity about the polar axis by n, and suppose that the resultant force of gravitation at any place P is directed downwards along a line at that point, making a small angle θ with the vertical. If G be this force for unit mass the component along the vertical upward is $-G\cos\theta$. The x-component at P is $G\cos(\theta + \frac{1}{2}\pi)$ or $-G\sin\theta$, and the y-component is zero, since the axes of x and z lie in the meridian. If the particle be not at P, but have co-ordinates, relative to parallel axes with P as origin, x, y, $z - a$, additional components X, Y, Z will act on the particle, which can easily be computed when x, y, z are known from the value of G at the position of the particle. Besides these we suppose any other applied forces X', Y', Z' to act on the particle.

If λ denote the latitude at P, that is the angle which the vertical there makes with the plane of the earth's equator, the angular velocity of the earth, and therefore of the system of axes, about the vertical is $n\sin\lambda$, the angular velocity about the axis of z is $-n\cos\lambda$, and that about the axis of y is zero. Hence by equations (86),

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putting $(F, G, H) = (u, v, w)$ as given in (85), and noting that $\theta_1 = -n\cos\lambda$, $\theta_2 = 0$, $\theta_3 = n\sin\lambda$, we obtain the equations of motion,

$$\left.\begin{aligned} \ddot{x} - 2\dot{y}n\sin\lambda - n^2\sin\lambda(z\cos\lambda + x\sin\lambda) &= X + X' - G\sin\theta \\ \ddot{y} + 2\dot{z}n\cos\lambda + 2\dot{x}n\sin\lambda - yn^2 &= Y + Y' \\ \ddot{z} - 2\dot{y}n\cos\lambda - n^2\cos\lambda(z\cos\lambda + x\sin\lambda) &= Z + Z' - G\cos\theta \end{aligned}\right\} \quad (88)$$

which are exact.

If the origin be taken on the vertical (*e.g.*, at P) at a distance a from the centre of the earth, the only change required is a substitution of $z + a$ for z. The equations become then:

$$\left.\begin{aligned} \ddot{x} - 2\dot{y}n\sin\lambda - n^2\sin\lambda(z\cos\lambda + x\sin\lambda) &= X + X' - G\sin\theta + n^2a\sin\lambda\cos\lambda \\ \ddot{y} + 2\dot{z}n\cos\lambda + 2\dot{x}n\sin\lambda - yn^2 &= Y + Y' \\ \ddot{z} - 2\dot{y}n\cos\lambda - n^2\cos\lambda(z\cos\lambda + x\sin\lambda) &= Z + Z' - G\cos\theta + n^2a\cos^3\lambda. \end{aligned}\right\} \quad (88')$$

279. **Foucault's Pendulum.**—As an example we consider the motion of a simple pendulum of length l suspended from P, and performing small oscillations about the vertical under gravity. A sufficiently exact solution for practical purposes is obtained by taking the motion along the axis of z as zero, and neglecting the terms in n^2 on the left-hand side of the x- and y-equations (see § 278). The force $-G\sin\theta$ is negligible, since θ is very small; the horizontal applied forces influencing the motion are those due to the stretching force, T, in the thread, and are respectively $-Tx/l$, $-Ty/l$. But by the z-equation of motion, since the left-hand side is zero, and the applied force is $-Tz/l = T$ (since $z = -l$), we have

$$T = G\cos\theta - n^2a\cos^2\lambda = g, \quad (89)$$

where g is what may be called the *apparent gravity* of a unit particle along the vertical (see § 281). Thus, with ω written for $n\sin\lambda$, the equations are

$$\ddot{x} - 2\omega\dot{y} + \frac{g}{l}x = 0, \qquad \ddot{y} + 2\omega\dot{x} + \frac{g}{l}y = 0. \quad (90)$$

A suitable solution of (90) for the case of ordinary small vibrations in one plane is

$$x = a\cos mt\cos\omega t, \quad y = -a\cos mt\sin\omega t, \quad (91)$$

where $m = \sqrt{g/l}$. This satisfies the equations to terms involving ω^2. Here, when $t = 0$, $x = a$, $y = 0$; when $\omega t = \pi/2$, $x = 0$, and $y = -a\cos(m\pi/2\omega)$; and so on. The plane of oscillation of the pendulum will therefore seem to rotate about the vertical with angular velocity ω, that is $n\sin\lambda$, and will make a complete revolution in $24/\operatorname{cosec}\lambda$ hours.

This result was given by Foucault, who first proposed to illustrate the earth's rotation by the relative rotation of the plane of oscillation of a pendulum about the vertical. There is, however, a serious practical difficulty in carrying out the experiment owing to the

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influence of the mode of attachment of the pendulum-wire at the upper end. The result may, however, be taken as completely verified by the experiments that have been carried out.

It is clear that a possible mode of vibration of a freely suspended pendulum is one in which the bob moves in a circle. For this we must have a constant displacement $\sqrt{x^2+y^2}=a$, and the solution is

$$x=a\cos mt,\quad y=a\sin mt, \tag{91'}$$

with proper choice of the origin of time. (Here m is no longer g/l.) When these values are substituted in the differential equations (85) each equation gives

$$m^2+2\omega m-\frac{g}{l}+\omega^2=0.$$

The roots of this are $-\omega\pm\sqrt{g/l}$, and are real. In one case the pendulum goes round with the earth's rotation in the period $2\pi/\{\sqrt{g/l}-\omega\}$; in the other it revolves against the earth's rotation in the period $2\pi/\{\sqrt{g/l}+\omega\}$. These periods are $2\pi/(\sqrt{g/l}-n\sin\lambda)$ and $2\pi/(\sqrt{g/l}+n\sin\lambda)$, so that the angular velocity relatively to the earth in the former case is diminished by $n\sin\lambda$, in the latter is increased by $n\sin\lambda$, the angular velocity of the rotation of the earth.

This experiment and others analogous to it have been held to prove that the earth's rotation is absolute, and not merely relative to the fixed stars. On the other hand, the facts of the case may be expressed by saying that when the motion is referred to axes fixed relatively to the fixed stars, and all action on the body except that of the earth is neglected, the equations are of the simple form obtained by omitting from (90) the terms depending on ω, and that when it is referred to axes fixed to the earth these terms must be included. The question very much depends on what is meant by "absolute rotation" of the earth.

280. **Deviation of Falling Body from Vertical. Deviation of Projectile.**—As another example we calculate the deviation from the vertical of a body let fall from rest at a point at distance h above the origin supposed on the earth's surface. The equations of motion are by (88)

$$\ddot{x}-2\dot{y}n\sin\lambda=0,\ \ddot{y}+2\dot{z}n\cos\lambda+2\dot{x}n\sin\lambda=0,\ \ddot{z}-2\dot{y}n\cos\lambda=-g. \tag{92}$$

Integrating each of these we obtain, since $x=y=0$, $\dot{x}=\dot{y}=0$, and $t=0$,

$$\left.\begin{aligned}\dot{x}-2yn\sin\lambda=0,\quad \dot{y}+2zn\cos\lambda+2xn\sin\lambda&=2hn\cos\lambda,\\ \dot{z}-2yn\cos\lambda&=-gt.\end{aligned}\right\} \tag{92'}$$

The first and third of these give by elimination of y and integration

$$x=(\tfrac{1}{2}gt^2+z-h)\tan\lambda.$$

But clearly to a high degree of approximation $h-z$, the distance fallen parallel to the vertical in time t, is $\frac{1}{2}gt^2$, so that x, the

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southward deviation, is practically zero. Hence the third term in the second of the integrated equations may be taken as zero. Substituting for z the value $h - \frac{1}{2}gt^2$ in the second term of this equation, and integrating for the eastward deviation in time t, we get $y = \frac{1}{3}gt^3 n\cos\lambda$. When the particle has fallen to the surface $t = \sqrt{2h/g}$, and therefore $y = \frac{1}{3}2\sqrt{2h^3/g}.n\cos\lambda$. If, for example, h be 200 feet, the deviation y is about a quarter of an inch.

It is interesting to consider the case in which the velocity of the moving body is nearly horizontal. From the approximate equations (92) we can calculate the deviation due to the rotation of the earth produced in a shot fired with a velocity V at a given small elevation a, and in a direction inclined at an angle β with the positive (southward) direction of x. The reader may go through the details of calculation according to the following steps:

Equations (92) integrated once from 0 to t give, since initially $\dot{x} = V\cos a\cos\beta$, $\dot{y} = V\cos a\sin\beta$, $\dot{z} = V\sin a$,

$$\left.\begin{aligned} \dot{x} - 2yn\sin\lambda &= V\cos a\cos\beta \\ \dot{y} + 2zn\cos\lambda + 2xn\sin\lambda &= V\cos a\sin\beta \\ \dot{z} - 2yn\cos\lambda &= -gt + V\sin a. \end{aligned}\right\} \quad (93)$$

Neglecting the terms in n in the first and third of these, integrate from 0 to t, and find values of x and z. Substitute these in the second, and integrate from 0 to t, and find y. Substitute for y in the first and third of (93), and neglecting the terms in n^2 integrate from 0 to t. This gives x and z. The results are

$$\left.\begin{aligned} x &= Vt\cos a\cos\beta + Vt^2 n\sin\lambda\cos a\sin\beta \\ y &= Vt\cos a\sin\beta - Vt^2 n(\cos\lambda\sin a + \sin\lambda\cos a\cos\beta) + \tfrac{1}{3}gt^3 n\cos\lambda \\ z &= -\tfrac{1}{2}gt^2 + Vt\sin a + Vt^2 n\cos\lambda\cos a\sin\beta. \end{aligned}\right\} \quad (93')$$

The terms in n in these are the deviations.

For $V = 2400$ feet per second, $a = 4°$, $\lambda = 56°$ N., $\beta = 0$, and for the full horizontal range we obtain approximately

$$\delta y = -\frac{4V^3}{g^2} n\sin\lambda\sin^2 a\cos a = -15{\cdot}8$$

in feet. The deviation is therefore, allowing for the other terms, about 15 feet to the west. The projectile is supposed unresisted, and the range is nearly five miles. A 100 lb. shot fired from a service 6-inch gun with the velocity and elevation stated would have a range of about 5000 yards.

281. **Apparent and True Gravity.**—The theory just explained illustrates the resolution of rotations, and the fact that the attraction of the earth on a particle at any place has to furnish the force necessary to produce the acceleration required by the motion of the particle with the earth. Thus the bob of the pendulum has a mean motion round the circle of latitude at the place at which it is suspended. The acceleration due to gravity is thus altered both in

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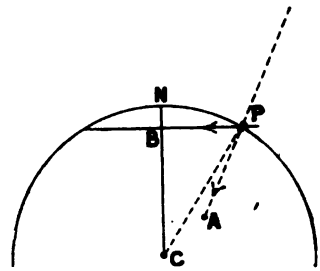
magnitude and direction from this cause. G denotes, as stated, the true gravitational force per unit mass on the particle P (Fig. 133). Regarding the earth as a sphere, we take PC, the line drawn to the centre, as the direction of G, and resolve it into two components, one PB along the radius at P of the circle of latitude, the other in a direction PA making a small angle with PC. The latter it is which gives the *initial* acceleration g of a body falling under the action of gravity and revolving with the earth, and is not toward the point C. Thus PA, the vertical in which the pendulum hangs when not oscillating, and about which the plane of oscillation revolves, is not directed to the earth's centre.

FIG. 133.



The acceleration along PB is $n^2R\cos\lambda$, if R be the earth's radius. Hence if θ be the angle PA makes with PC we have, since θ is very small, $n^2R\cos\lambda/g = \theta/\cos BCP = \theta/\sin\lambda$. Also

$$g/G = \sin\lambda/\sin(\lambda+\theta) = \sin\lambda/(\sin\lambda + \theta\cos\lambda).$$

Thus we have

$$\left.\begin{aligned} &G = g + n^2R\cos^2\lambda \quad\text{or}\quad g = G - n^2R\cos^2\lambda, \\ \text{and}\quad &\theta = \frac{n^2R\sin 2\lambda}{2g} = \frac{n^2R\sin 2\lambda}{2(G - n^2R\cos^2\lambda)}, \end{aligned}\right\} \quad (94)$$

The value of n^2R/g is about $\frac{1}{289}$, so that at the equator G is diminished by this fraction of itself, and by less at other places, according to the latitude, as stated in (94).

282. **Euler's Dynamical Equations.**—Returning to the example of a rotating rigid body, let the rotating axes be fixed in the body, and coincide with the principal axes of moment of inertia. Let the moment of inertia of the body about OA be A, about OB be B, and about OC be C. We shall denote the angular velocities of the axes, which are now also the angular velocities of the rigid body about the same axes, in this case by ω_1, ω_2, ω_3. Then $F = A\omega_1$, $G = B\omega_2$, $H = C\omega_3$. Equations (86) then become

$$\left.\begin{aligned} A\dot{\omega}_1 - (B-C)\omega_2\omega_3 &= L \\ B\dot{\omega}_2 - (C-A)\omega_3\omega_1 &= M \\ C\dot{\omega}_3 - (A-B)\omega_1\omega_2 &= N \end{aligned}\right\} \quad (95)$$

As already stated L, M, N are the moments about the axes of the forces applied to the body.

These equations are known as Euler's equations of a rigid body rotating, with one point fixed, about the principal axes for that point. Use has already been made of an equation equivalent to the second

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of these in § 264 above. If the body have an axis of symmetry (as a top for example) about which the moment of inertia is C, then $A = B$, and the equations become

$$\left.\begin{aligned} A\dot{\omega}_1 - (A - C)\omega_2\omega_3 &= L \\ A\dot{\omega}_2 - (C - A)\omega_3\omega_1 &= M \\ C\dot{\omega}_3 \qquad\qquad &= N \end{aligned}\right\} \quad (96)$$

These equations might have been made the starting-point of the discussion of the motion of the top, and the reader may so apply them as an exercise. It is to be noticed that we cannot take the axes OD, OE (Fig. 121) as the principal axes of equal moment to be used in Euler's equations, inasmuch as although OD and OE are always at right angles to one another and to OC, they do not in the general case fulfil the condition of being fixed in the body. Let an axis OA (Fig. 134) at right angles to OC and fixed in the body make an angle ϕ with OE' (EO produced) at the instant under consideration. Let OB be another axis in the same plane DOE and at right angles to OA. The axes OA, OB are revolving at the instant with angular velocity $\omega + \dot{\psi}\cos\theta$ about OC, the axis of the top. But OD is turning with angular velocity $\dot{\psi}\cos\theta$ in the same plane, and therefore OA is separating from OD with angular velocity ω. Hence $\dot{\phi} = \omega$.

The angular velocities ω_1 about OA, ω_2 about OB, and ω_3 about the axis OC are respectively $\dot{\theta}\sin\phi - \dot{\psi}\sin\theta\cos\phi$, $+\dot{\theta}\cos\phi + \dot{\psi}\sin\theta\sin\phi$, $\omega + \dot{\psi}\cos\theta$. The applied couple is $mgh\sin\theta$ about OD, and therefore the component couples are $mgh\sin\theta\sin\phi$ about OA, $mgh\sin\theta\cos\phi$ about OB, and zero about OC. These are L, M, N respectively.

The reader should substitute the values in (95), and verify that the equations of motion (58), § 259, are obtained. This can easily be done by eliminating ϕ between the first and second equations of motion, and putting ω for $\dot{\phi}$.

283. **Kinetic Energy in Terms of Angular Velocities of Body about Principal Axes.**—It will be noticed that the kinetic energy T is given by the equation

$$T = \tfrac{1}{2}(A\omega_1^2 + B\omega_2^2 + C\omega_3^2). \quad (97)$$

This expression for T cannot be used to obtain Euler's equations by the Lagrangian method, and the reason affords a good illustration of the circumstances in which that method is legitimate. The angular velocities ω_1, ω_2, ω_3 about the principal axes are not velocities of co-ordinates fixing the position of the system, as specified in § 229; and before an attempt is made to find the equations of motion from the expression for the kinetic energy that expression must be transformed to one involving such co-ordinates.

284. **Angular Velocities about Principal Axes expressed in Terms of Position Co-ordinates.**—Co-ordinates of the kind here referred to are those, θ, ψ, ϕ, used in the discussion of the motion of

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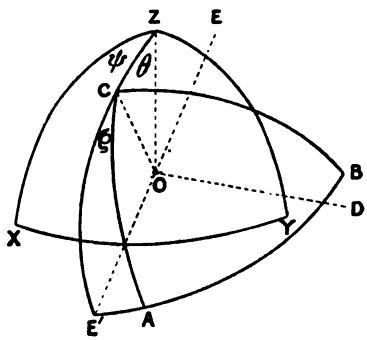
a top; and the relations between ω_1, ω_2, ω_3 and θ, ψ, ϕ, $\dot\theta$, $\dot\psi$, $\dot\phi$ have been given above. We collect here, however, for convenience of reference, the values ω_1, ω_2, ω_3 in terms of $\dot\theta$, $\dot\psi$, $\dot\phi$ and of $\dot\theta$, $\dot\psi$, $\dot\phi$ in terms of ω_1, ω_2, ω_3.

$$\left.\begin{aligned}
\omega_1 &= \dot\theta\sin\phi - \dot\psi\sin\theta\cos\phi, & \dot\theta &= \omega_1\sin\phi + \omega_2\cos\phi \\
\omega_2 &= \dot\theta\cos\phi + \dot\psi\sin\theta\sin\phi, & \dot\psi &= (\omega_2\sin\phi - \omega_1\cos\phi)\frac{1}{\sin\theta} \\
\omega_3 &= \dot\psi\cos\theta + \dot\phi, & \dot\phi &= \omega_3 - (\omega_2\sin\phi - \omega_1\cos\phi)\frac{1}{\tan\theta}
\end{aligned}\right\} \quad (98)$$

285. **Direction Cosines of Principal Axes.**—In Fig. 134 are represented three points X, Y, Z and three points A, B, C lying in the surface of a sphere of unit radius the centre of which is O, the fixed point of the body. X, Y, Z are the extremities of three rectangular axes which are fixed; A, B, C are the extremities of the axes fixed in the moving body. The sides XY, YZ, ZX and AB, BC, CA of the two spherical triangles drawn on the sphere are all quadrants, and the θ, ψ, ϕ are as indicated. The nine direction cosines of OA, OB, OC, with reference to X, Y, Z, are equal to the cosines of the arcs described on the sphere joining the three points X, Y, Z with A, B, C. They are easily obtained by the elementary formulæ of spherical trigonometry. If α_1, α_2, α_3 be the direction cosines of OA, β_1, β_2, β_3 those of OB, and γ_1, γ_2, γ_3 those of OC, we have

FIG. 134.



$$\left.\begin{aligned}
\alpha_1 &= \cos\theta\cos\psi\cos\phi - \sin\psi\sin\phi \\
\alpha_2 &= \cos\theta\sin\psi\cos\phi + \cos\psi\sin\phi \\
\alpha_3 &= -\sin\theta\cos\phi \\
\beta_1 &= -\cos\theta\cos\psi\sin\phi - \sin\psi\cos\phi \\
\beta_2 &= -\cos\theta\sin\psi\sin\phi + \cos\psi\cos\phi \\
\beta_3 &= \sin\theta\sin\phi
\end{aligned}\right\} \quad (99)$$

$$\gamma_1 = \sin\theta\cos\psi, \quad \gamma_2 = \sin\theta\sin\psi, \quad \gamma_3 = \cos\theta.$$

286. **Consideration of Terms in Euler's Equations.**—The first of Euler's dynamical equations may be written

$$A\dot\omega_1 = L + (B - C)\omega_2\omega_3.$$

Hence $A\dot\omega_1$ may be regarded as due to *two* couples, L and $(B - C)\omega_2\omega_3$. The latter has been called the *centrifugal couple* about OA due to

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the rotation of the body round the axes OB, OC; and similarly the corresponding quantities in the other equations have been named. There is no difficulty in making out this interpretation analytically, but it seems less intuitive than the following, by which indeed the equations can be inferred or recalled at any time.

The couples L, M, N acting on the body produce moment of momentum about OA, OB, OC in two ways—(1) by the generation of angular velocity about the axes; (2) by altering the position of the body. Imagine OA fixed for the moment, the rotation of OC about OB will turn OC nearer to OA in time dt by an angle $\omega_2 dt$, and this would produce about the fixed line OA moment of momentum $C\omega_3 \sin(\omega_2 dt) = C\omega_3\omega_2 dt$. Thus at the instant the moment of momentum about OA is increasing at rate $C\omega_3\omega_2$ in consequence of the motion about OB.

In the same time dt the rotation about OC turns OB away from the fixed position of OA through the angle $\omega_3 dt$, and so the moment of momentum about OA is diminishing at the instant at rate $B\omega_2\omega_3$.

Thus the total rate of increase of moment of momentum about OA at the instant considered is $A\dot{\omega}_1 - (B-C)\omega_2\omega_3$, and this must be equal to the moment, L, of the applied couple. Similarly the other equations may be explained.

It is to be noted that this process is only a particular case of one which may be applied to any set of rotating axes, OA', OB', OC' *fixed in the body*. Let h_1, h_2, h_3 be the moments of momentum round any three such axes turning with angular velocities θ_1, θ_2, θ_3 about themselves. Then, as we have seen, the rate of change of momentum about OA' is $\dot{h}_1 - h_2\theta_3 + h_3\theta_2$. The parts of this are capable of the same explanation.

287. **Motion of Rigid Body about a Fixed Point under No Forces.**—The motion of a rigid body unacted on by external forces and turning about a fixed point, that is the case in which L, M, N are all zero, is very interesting. Since no forces act the axis of maximum moment of momentum, as the body moves, remains unaltered, and at right angles to the invariable plane through the origin. The moment of momentum about it, H say, is constant, so that we have

$$A^2\omega_1^2 + B^2\omega_2^2 + C^2\omega_3^2 = H^2. \tag{100}$$

Also the kinetic energy of the body is constant, so that we can write

$$A\omega_1^2 + B\omega_2^2 + C\omega_3^2 = K^2. \tag{101}$$

These results are easily derivable from Euler's equations. For multiplying the first by $A\omega_1$, the second by $B\omega_2$, the third by $C\omega_3$, adding and integrating, we get (100). Again, multiplying the first by ω_1, the second by ω_2, the third by ω_3, adding and integrating, we obtain (101).

The angular velocity ω about the instantaneous axis is

$$\sqrt{\{\omega_1^2 + \omega_2^2 + \omega_3^2\}}.$$

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But the angular velocities are connected by the relation (101), so that if we take distances x, y, z parallel to the axes, and such that $\omega_1 = Kx/k$, $\omega_2 = Ky/k$, $\omega_3 = Kz/k$, we shall obtain instead of (101)

$$Ax^2 + By^2 + Cz^2 = k^2, \tag{101'}$$

the equation of an ellipsoid, which may be taken as the momental ellipsoid (see § 164). Thus the angular velocity about the instantaneous axis is proportional to the length of the coincident radius-vector of the momental ellipsoid.

As the body moves the direction OI of the instantaneous axis changes both in the body and in space. Its direction cosines with reference to the principal axis of the momental ellipsoid are ω_1/ω, ω_2/ω, ω_3/ω. As ω_1, ω_2, ω_3 vary it changes its position in the body.

*288. **Motion under No Forces. Poinsot's Representation by a Momental Ellipsoid rolling on a Fixed Plane. Sylvester's Measure of the Time of Motion.**—The direction OI of the instantaneous axis also changes in space. To find its motion we notice first that the axis of resultant moment of momentum, that is the perpendicular to the invariable plane at the origin, remains fixed in space. But the components of moment of momentum with reference to the axes of the momental ellipsoid are $A\omega_1$, $B\omega_2$, $C\omega_3$, and therefore the direction cosines with reference to these axes of the axis of resultant moment of momentum are $A\omega_1/H$, $B\omega_2/H$, $C\omega_3/H$. But these are the direction cosines, relatively to the principal axes of the ellipsoid, of the perpendicular let fall from the centre on the tangent plane to the momental ellipsoid at the extremity of the radius-vector coinciding with the instantaneous axis. The component of angular velocity along this perpendicular is therefore

$$(A\omega_1^2 + B\omega_2^2 + C\omega_3^2)/H = K^2/H,$$

and remains constant as the body moves.

Since this perpendicular is fixed in space the motion of the body is such that the momental ellipsoid which moves with it is always in contact with a plane the direction of which is fixed in space.

The plane just specified is also fixed in position. Let p be the length of the perpendicular (OL say) let fall from O on the tangent plane which touches the momental ellipsoid at the point at which it is intersected by the instantaneous axis. The cosine of the angle between this perpendicular and the instantaneous axis is p/r, and this has also the value

$$(Ax^2 + By^2 + Cz^2)/r\sqrt{A^2x^2 + B^2y^2 + C^2z^2} \text{ or } (A\omega_1^2 + B\omega_2^2 + C\omega_3^2)/\omega H.$$

Hence $p^2 = K^4r^2/H^2\omega^2 = k^2K^2/H^2$. Thus p is of constant length, and the plane is fixed in position.

It will be observed that ω^2p^2/r^2 is K^4/H^2, and is therefore constant, that is the angular velocity $\omega p/r$ of the body about the perpendicular is constant.

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Thus as the body turns, the momental ellipsoid, which turns with it, is always in contact, at its intersection with the instantaneous axis, with a plane fixed parallel to the invariable plane at the origin. The point of contact is always instantaneously at rest, and so the motion may be described as one in which the momental ellipsoid rolls on a plane, fixed parallel to the plane of the resultant impulsive couple which would generate from rest the actual motion at any instant. The time-integral of this couple is of course H.

Now consider an ellipsoid of which the equation referred to the principal axes of the momental ellipsoid is

$$\frac{A}{1+Ah}x^2+\frac{B}{1+Bh}y^2+\frac{C}{1+Ch}z^2=k^2, \qquad (101'')$$

This is an ellipsoid confocal with the momental ellipsoid and moving like the latter with the body. A plane touching this ellipsoid at the point x, y, z and parallel to the invariable plane at the origin has its normal in the direction OL. Hence for such a plane

$$\frac{A}{1+Ah}x=RA\,\omega_1, \quad \frac{B}{1+Bh}=RB\,\omega_2, \quad \frac{C}{1+Ch}=RC\,\omega_3,$$

where R is a constant. These give $R^2=k^2/(K^2+hH^2)$.

The co-ordinates of the point of contact, Q say, are thus

$$x=R(1+Ah)\omega_1, \quad y=R(1+Bh)\omega_2, \quad z=R(1+Ch)\omega_3.$$

where R has the value just obtained. The projection of the radius-vector OQ on the line OL is therefore

$$\frac{R}{H}\{(1+Ah)A\,\omega_1^2+(1+Bh)B\omega_2^2+(1+Ch)C\omega_3^2\}=\frac{k}{H}\sqrt{(K^2+hH^2)},$$

a constant. Hence this tangent plane is fixed in space.

If r' be the length of OQ, and θ denote the angle QOI, the velocity of Q is $\omega r'\sin\theta$. But if ϕ denote the angle QOL, the distance of Q from OL is $r'\sin\phi$, and therefore the angular velocity of Q round OL is $\omega\sin\theta/\sin\phi$. By calculating the values of $\sin\theta$, $\sin\phi$ we easily obtain

$$\omega^2\sin^2\theta/\sin^2\phi=H^2\{\omega^2r'^2-R^2(\omega^2+hK^2)^2\}/\{H^2r'^2-R^2(K^2+hH^2)^2\},$$

which reduces to h^2H^2. Hence the angular velocity of Q round OL is hH.

Hence if the plane touching the ellipsoid (101″) at Q turn round OL with this uniform angular velocity, the time during which the body has performed any part of its motion is equal to the angle turned in that time by this plane divided by hH. This mode of representing the time is due to Sylvester (*Phil. Mag.*).

*289. **Polhode and Herpolhode.**—The points of contact of the fixed plane and the momental ellipsoid give two loci, one on the

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surface of the ellipsoid, the other on the plane. The former is called the *polhode*, the latter the *herpolhode*.

To find the polhode we have only to determine the locus of points in the body the tangent planes at which are at a constant distance from the fixed point O. Let p be the constant length of the perpendicular: we have the equations

$$Ax^2 + By^2 + Cz^2 = k^2, \quad A^2x^2 + B^2y^2 + C^2z^2 = k^2\frac{H^2}{K^2} \tag{102}$$

These equations give the single relation

$$A(K^2A - H^2)x^2 + B(K^2B - H^2)y^2 + C(K^2C - H^2)z^2 = 0, \tag{103}$$

which is the equation of a cone the vertex of which is the origin. This cone is fixed in the body and turns with it. Its intersection with the surface of the momental ellipsoid is the locus of the points in which the ellipsoid touches the fixed plane. We may call this the body-cone. It rolls on a fixed or space-cone the intersection of which with the fixed plane is the locus of the points of contact of the momental ellipsoid with the plane.

By eliminating z between the two equations (102), we obtain

$$A(A - C)x^2 + B(B - C)y^2 = k^2\left(\frac{H^2}{K^2} - C\right) = k^2\left(\frac{k^2}{p^2} - C\right), \tag{104}$$

which is the equation of the projection of the polhode on a plane perpendicular to OC. If C be the greatest moment of inertia, p^2 has its least value when the ellipsoid touches the plane at either extremity of the axis of z, for then $z^2 = p^2 = k^2/C$. The value of k^2/p^2 is C in this particular case, and is always less in others so long as $C > B > A$. Therefore the quantity on the right of (104) is negative, and the co-efficients of x^2 and y^2 on the left are also negative. The projection of the polhode is therefore an ellipse. When C is the least principal moment, the quantity on the right of (104) and the co-efficients on the left are all positive: hence the curve is an ellipse in this case also.

If C be the intermediate moment the co-efficients of x^2 and y^2 have opposite signs, and the curve is a hyperbola. If A be the greatest moment, B the least, the equation may be written $ax^2 - by^2 = c$ if $H^2/K^2 > C$, and $by^2 - ax^2 = c$ if $H^2/K^2 < C$, where a, b, c are all positive. In the former of these two cases the axis of x is the axis of the hyperbola, in the latter the axis of y is the axis of the hyperbola.

If $H^2/K^2 = C$, the intermediate moment, the equation takes the form $ax^2 - by^2 = 0$, which represents the two straight lines $x\sqrt{a} - y\sqrt{b} = 0$, $x\sqrt{a} + y\sqrt{b} = 0$. These lines cross on the axis of mean moment, and are the projections on the plane of x, y of the polhodes round the axes of greatest and least moment. These are two closed curves, the projections of which on planes at right angles to the axes they surround are ellipses, and which meet in the axis of mean moment.

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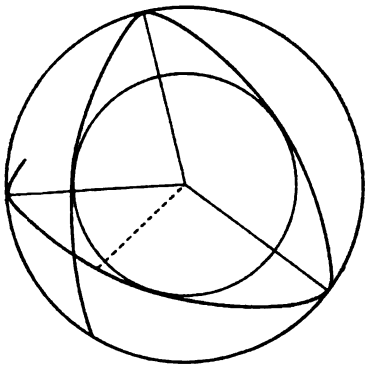
The straight lines separate those hyperbolic projections of the polhodes which have their axis parallel to OA from those which have their axis parallel to OB, and the polhode is therefore called the *separating polhode*.

FIG. 135.



With regard to the herpolhode we have not space to say more than that, since the polhodes are closed curves round the axes of greatest and least moment, the herpolhode must lie between two circles, of which the perpendicular let fall from O on the fixed plane is the common axis. For the square of the distance of the point of contact at any instant from the foot of this perpendicular is $r^2 - p^2$, and we have for this the value $p^2(\omega^2 H^2/K^4 - 1)$. By giving to ω its greatest and least values we obtain from this the radii of the two circles, and clearly the herpolhode lies between them, and as the body rolls must touch them alternately. Since the herpolhode is a curve in the fixed plane, if we join each point of it with the origin we obtain a fixed or space-cone on which the body-cone rolls as the body turns about the fixed point. The form of the herpolhode is shown in Fig. 135, which is taken from Routh's *Advanced Rigid Dynamics*, chap. iv. It will be observed that the curves are always concave to the centre of the circles.

The reader may prove from (100) and (101) that, if $A > B > C$, $\omega^2 (= \omega_1^2 + \omega_2^2 + \omega_3^2)$ lies between $\{K^2(C + A) - H^2\}/CA$ and

$$\{K^2(B + C) - H^2\}/BC \quad \text{or} \quad \{K^2(A + B) - H^2\}/AB,$$

according as H^2/K^2 is less or greater than B.

*290. **Stability of Motion of Body under No Forces.**—We have given this slight sketch of Poinsot's representation of the motion by the rolling of the momental ellipsoid on a fixed plane, mainly on account of the light which the discussion throws on the stability of the motion of a rotating body unacted on by forces. When the body rotates about any one of the principal axes the momental ellipsoid merely spins on the fixed plane; if a small disturbing impulse be applied the body will begin to rotate about a slightly different axis.

If the axis deviated from is that of greatest or least moment of inertia C, the polhodes are closed curves enclosing that axis: and, since in this case H^2/K^2 is very nearly equal to C, the elliptic projection on the fixed plane is very small in every dimension, unless either $k^2(H^2 - K^2)/K^2 A(A - C)$ or $k^2(H^2 - K^2)/K^2 B(B - C)$ be great. Hence for a *very* small disturbance the axis of rotation changes position in the body, but only slightly, and the motion may be said to be stable. For a moderate disturbance, however, if the ellipse is

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very eccentric, that is if $A(A-C)/B(B-C)$ is very small or very great, the axis may deviate much from the principal axis, and though the instantaneous axis may return again to proximity to the principal axis, the motion may be said to be unstable. Thus for any wide limits of stability it is necessary that the neighbouring polhodes should be approximately circular.

If the axis is that of intermediate moment, the polhodes in the neighbourhood are convex toward the axis, and so the instantaneous axis, if disturbed from the principal axis, will deviate greatly from its original position, unless the disturbance displaces it along one of the straight lines which form the separating polhode. Since $H^2/K^2=B$, this case requires separate discussion. Investigation shows that the motion is stable if the displacement is along the separating polhode.*

The preceding discussion of the top will show the reader clearly that the effect of a given disturbance is smaller the greater the angular velocity with which the body is rotating about, or nearly about, a principal axis. To take the case most favourable to the disturbance, let it suddenly generate a velocity, ω say, about an axis perpendicular to that about which the body is spinning with angular velocity Ω. The deviation of the instantaneous axis is then $\tan^{-1}\omega/\Omega$. The greater Ω the smaller is this angle, and if Ω is very great ordinary slight disturbances hardly affect the direction of the principal axis, which remains nearly fixed in space. As we have seen, the axis of a rapidly rotating top has only a slow precessional motion combined with a slight motion of nutation, though the top is considerably inclined to the vertical, and the direction of the earth's axis only changes with an extremely slow conical motion, under the influence of the couple due to solar attraction.

The effect of the spinning motion given to plates, knives, &c., thrown by a juggler, in preserving the directions of the body in space is also explained in this way. The bodies are generally symmetrical round an axis about which they are made to spin. Without this spin a disturbance would cause the body to move in a way impossible to foresee, and there would be no certainty as to the position in which the bodies would return to the performer.

A quoit well thrown is made to spin about its axis of figure, and preserves the direction of its axis nearly unchanged during its flight. A rifle bullet also is given a very rapid spin about its axis of least moment by the helical grooving of the barrel, and the direction of this axis remains approximately unchanged. In such cases the action of the air on the body moving through it is rendered definite, and repeated shots can be made under practically the same conditions.†

* See Routh's *Advanced Rigid Dynamics*, p. 99.

† For the motion of a rifle bullet, see Greenhill, *Proc. Royal Artillery Institution*, 1879.

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CHAPTER VI.*

STATICS OF A MATERIAL SYSTEM.

291. **Conditions of Equilibrium.**—We shall now consider a little more in particular the equilibrium of a material system. It follows from equations (26) of chap. iii. that if the centroid of the system be without acceleration we have the three equations

$$\Sigma X = 0, \ \Sigma Y = 0, \ \Sigma Z = 0. \tag{1}$$

If, moreover, the rate of change of moment of momentum of the system about all axes be zero, we must have likewise

$$\Sigma(Zy - Yz) = 0, \ \Sigma(Xz - Zx) = 0, \ \Sigma(Yx - Xy) = 0. \tag{2}$$

It will be observed that these equations are not inconsistent with the possibility of accelerations of individual particles. The forces X, Y, Z involved in the sums are the externally applied forces, since, as explained in §§ 153, 155, internal forces between the particles cannot enter into the summations.

If the system be a rigid body the equations still hold. Equations (2), however, can then be transformed to

$$\Sigma R_x r_x = 0, \ \Sigma R_y r_y = 0, \ \Sigma R_z r_z = 0 \tag{3}$$

where R_x is the component of any force X, Y, Z applied at the point x, y, z taken parallel to the plane yz, and at right angles to the axis of x, and r_x is the distance, $\sqrt{(y^2+z^2)}$, of the point x, y, z from the axis of x; similarly $R_y r_y$ are the corresponding quantities for the plane of zx and the axis of y, and so on. The sums $\Sigma R_x r_x$, ... are those of the moments of all the forces about rectangular axes drawn through the origin. It will be seen from equations (32) of chap. iii. that if each of these sums be zero, the body is without angular acceleration about the axes. When a rigid body is thus without linear acceleration of its centroid and angular acceleration about any axis it is said to be in equilibrium, and the equations (1) and (2) are then called equations of equilibrium.

It will be observed that *motion* of the system is not excluded. The centroid of the body may be in motion with uniform velocity in

* Sections 293–297, which are marked with an asterisk, are intended for advanced students.

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a straight line; and the body may be rotating with uniform angular velocity about an axis through the centroid. The accelerations in the latter case of the particles towards the axis of rotation are then produced entirely by internal forces.

We may remark that if (1) holds the sum of the component forces in any direction will vanish, and conversely that if the sum of the resolved components along each of three non-coplanar non-parallel lines vanish then (1) must hold. The first proposition is obvious, the second is proved in the following way: let l_1, m_1, n_1, l_2, m_2, n_2, l_3, m_3, n_3, be the direction cosines of three lines which are not coplanar, then for the sum of components along the first, say, ΣF_1, we have

$$\Sigma F_1 = l_1 \Sigma X + m_1 \Sigma Y + n_1 \Sigma Z$$

and similarly

$$\Sigma F_2 = l_2 \Sigma X + m_2 \Sigma Y + n_2 \Sigma Z$$
$$\Sigma F_3 = l_3 \Sigma X + m_3 \Sigma Y + n_3 \Sigma Z.$$

Now if $\Sigma F_1 = \Sigma F_2 = \Sigma F_3 = 0$ we must have identically $\Sigma X = 0$, $\Sigma Y = 0$, $\Sigma Z = 0$, since by the condition that the lines are not coplanar and not parallel the determinant $(l_1 m_2 n_3)$ is not zero.

It can also be proved that the sum of moments round any axis whatever is zero if (1) holds, and the sum of the moments round any three axes which are not coplanar and not parallel is zero. Let, as before, l_1, m_1, n_1, l_2, m_2, n_2, l_3, m_3, n_3, be the direction cosines of the three axes, let the first pass through the point a_1, b_1, c_1, the second through the point a_2, b_2, c_2, the third through the point a_3, b_3, c_3; and let, as before, x, y, z be the co-ordinates of a point through which the force X, Y, Z passes, and let λ, μ, ν, be the direction cosines of any axis whatever passing through the origin; for the sum L, say, of the moments round the axis $(\lambda\mu\nu)$ we have

$$L = \lambda \Sigma(Zy - Yz) + \mu \Sigma(Xz - Zx) + \nu \Sigma(Yx - Xy),$$

but for the sum of moments round the axis $(l_1 m_1 n_1)$

$$L_1 = l_1 \Sigma\{Z(y - b_1) - Y(z - c_1)\}$$
$$+ m_1 \Sigma\{X(z - c_1) - Z(x - a_1)\} + n_1 \Sigma\{Y(x - a_1) - X(y - b_1)\}$$

with similar expressions for the moments round the other two axes. But by hypothesis $L_1 = L_2 = L_3 = 0$, and since $\Sigma(X) = 0$, $\Sigma(Y) = 0$, $\Sigma(Z) = 0$, all such sums as $\Sigma Z b_1$, $\Sigma Y c_1$, $\Sigma X b_1$, ..., must be each zero. Hence the equations for L_1, L_2, L_3 reduce to

$$l_1 \Sigma(Zy - Yz) + m_1 \Sigma(Xz - Zx) + n_1 \Sigma(Yx - Xy) = 0$$

and two similar equations for the other two axes. But since the axes are not coplanar the determinant $(l_1 m_2 n_3)$ is not zero, and therefore we must have identically

$$\Sigma(Zy - Yx) = 0, \ \Sigma(Xz - Zx) = 0, \ \Sigma(Yx - Xy) = 0$$

from which it follows that the moment round every axis through the

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origin is zero, that is the moment round every axis is zero, since the origin may be taken at any point whatever.

292. **Energy Criterion of Equilibrium.** — Now consider a perfectly general system of n particles which are subject to kinematical conditions expressed by m equations connecting the co-ordinates, we have the equation of work [(37), § 213 above]

$$\delta W = \Sigma(X\delta x + Y\delta y + Z\delta z)$$

where X, Y, Z represent the forces impressed on any particle the co-ordinates of which are x, y, z.

If, as we here suppose, the impressed forces do no work on the whole, this expression is zero, and we have

$$\Sigma(X\delta x + Y\delta y + Z\delta z) = 0. \tag{4}$$

If $X = -\partial E/\partial x$, &c., this becomes

$$\Sigma\left(\frac{\partial E}{\partial x}\delta x + \frac{\partial E}{\partial y}\delta y + \frac{\partial E}{\partial z}\delta z\right) = 0 \tag{5}$$

This equation asserts that for the small displacements δx, δy, δz the variation of the potential energy is zero, that is to say, when no work is done by the impressed forces in a set of possible displacements of the particles, the potential energy V has for the configuration then possessed by the system a stationary value.

*293. **Derivation of the Internal Forces from Kinematical Conditions.**—If the co-ordinates of the particles of the system are connected by m kinematical relations as expressed in (41) § 215 above, we obtain from the results there set forth the $3n$ equations of equilibrium

$$X_1 + \lambda_1\frac{\partial f_1}{\partial x_1} + \lambda_2\frac{\partial f_2}{\partial x_1} + \ldots = 0, \qquad X_2 + \lambda_1\frac{\partial f_2}{\partial x_2} + \lambda_2\frac{\partial f_2}{\partial x_2} + \ldots = 0. \tag{6}$$

These with the m equations of condition make up $3n + m$ equations by which the $3n$ co-ordinates of the particles in the configuration of equilibrium, and the m undetermined multiples may be found.

The conclusions obtained at § 215, as to the nature of the terms $\lambda_1\partial f_1/\partial x_1$, $\lambda_2\partial f_2/\partial x_1$, ... hold here with some slight modification of statement. Thus $\lambda_1\partial f_1/\partial x_1$, $\lambda_1\partial f_1/\partial y_1$, $\lambda_1\partial f_1/\partial z_1$ are the components of the force applied to m_1 in consequence of the condition $f_1 = 0$, in order that under the given applied forces and the forces due to the other $m - 1$ conditions, the particle m_1 may be without acceleration. This, as explained also at § 215, is a force tending to vary the condition $f_1 = 0$, that is acts at right angles to the surface expressed by $f_1 = 0$ when all the variables except x_1, y_1, z_1 are supposed maintained constant.

When the equations $f_1 = 0$, $f_2 = 0$, ... involve the time explicitly as in (45), § 215, we obtain from (6) by (46), § 215, the equation

$$\Sigma(X\dot{x} + Y\dot{y} + Z\dot{z}) - \Sigma\left(\lambda\frac{\partial f}{\partial t}\right) = 0 \tag{7}$$

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which shows that in the case of equilibrium the activity of the forces of constraint is not zero, as it is in the case of invariable kinematical conditions, but has the value $\Sigma(\lambda\partial f/\partial t)$.

For further information on the subject of the equilibrium of a system of connected particles the reader should consult the *Mécanique Analytique* of Lagrange, to whom the method described above of finding the forces produced by the constraints is due.

*294. **Derivation of the Equations of Equilibrium from the Principle of Work.**—Suppose now the system to be a rigid body. The most general displacement the body can have is made up of displacements of the centroid parallel to three rectangular axes, and rotations round parallel axes through the centroid.

Let the co-ordinates of a point P be x, y, z and a, b, c be those of the centroid. Then

$$x = a + x',\ y = b + y',\ z = c + z'$$

where x', y'. z' are the co-ordinates of P relatively to the centroid. Now let the centroid be displaced so that a, b, c are increased by δa, δb, δc, and let the body be turned through angles $\delta\xi$, $\delta\eta$, $\delta\zeta$ about axes parallel to Ox, Oy, Oz through the centroid. Then, as shown in effect at § 280 above,

$$\begin{aligned}\delta x &= \delta a + z'\delta\eta - y'\delta\zeta\\ \delta y &= \delta b + x'\delta\zeta - z'\delta\xi\\ \delta z &= \delta c + y'\delta\xi - x'\delta\eta.\end{aligned}$$

Hence if X, Y, Z represent the components of applied force on the particle P we have for the whole body

$$\begin{aligned}\Sigma(X\delta x + Y\delta y + Z\delta z) = \delta a\Sigma X + \delta b\Sigma Y + \delta c\Sigma Z + \delta\xi\Sigma(Zy' - Yz')\\ + \delta\eta\Sigma(Xz' - Zx') + \delta\zeta\Sigma(Yx' - Xy'). \qquad (8)\end{aligned}$$

For equilibrium it is necessary by (40) of chap. iv. that the work done in these displacements should vanish.

The displacements represented here correspond to the six freedoms of a rigid body, and must be independent. Hence we may equate the co-efficients of δa, δb, δc, $\delta\xi$, $\delta\eta$, $\delta\zeta$ to zero, and we obtain

$$\Sigma X = 0,\ \ldots,\ \ldots,\ \Sigma(Zy' - Yz') = 0,\ \ldots,\ \ldots,$$

the six equations of equilibrium for a rigid body.

*295. **Screw Displacement of a Body. Degrees of Freedom.**—It is shown at § 115 above, that if a body be free to move as a nut upon a fixed screw-shaft it has one degree of freedom; that this is the case is evident from the fact that the angle through which the nut is turned, and the distance through which the centroid of the nut moves parallel to the axis of the screw, are connected by an invariable relation, so that a motion of this kind involves change of only one independent variable. If δa, δb, δc be variations of a, b, c and $\delta\xi$, $\delta\eta$, $\delta\zeta$ be rotations about the three axes, then if the body is

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perfectly free these six quantities are independent. The introduction of a single constraint is effected when they are connected by a single equation, for example

$$A\delta a + B\delta b + C\delta c + G\delta\xi + H\delta\eta + I\delta\zeta = 0. \tag{9}$$

The imposition of one such relation removes one of the degrees of freedom of the body, the imposition of five such conditions would leave the body with only one degree of freedom, and it is easy to see that the freedom would be of the nature just explained. For let five such equations hold, and be independent in the sense that the determinant given by the co-efficients of any group of four of the variations is not identically zero. We can find any five of the quantities δa, δb, ... in terms of the sixth—that is to say, the whole displacement is settled when the angular displacement $\delta\zeta$, say, round the axis of z is assigned. Thus the displacement is one of the centroid in a perfectly definite direction, and a rotation of corresponding amount about a determinate axis through the centroid. This can be shown, as at § 114 above, to be equivalent to a linear displacement, parallel to this axis, together with a rotation about a definite parallel axis, that is the displacement can be reduced to that of a nut on a definite screw.

*296. **Condition that a Wrench should produce no Displacement about a Given Screw. Reciprocal Screws.**—In § 177 we have considered the system of forces called a wrench, and defined the *pitch of the wrench* as the ratio of the resultant moment of the forces about the axis of the wrench to the intensity of the wrench, that is to the resultant of the force along the axis. We may consider the axis of the wrench as that of a screw, and speak of the wrench either as a wrench or as a screw.

Let now the body be acted on by a wrench of intensity F on a screw β. The resultant couple being G, the pitch of the wrench is G/F: denote this by p_β. Now let the body be free to move about a screw α of pitch p_α at a perpendicular distance h from p_β, and consider the work done by the wrench in a twist of amplitude $\delta\phi$ about α. The couple of the wrench resolves into two couples, one round the screw α of moment $G\cos\alpha\beta$, where $\alpha\beta$ denotes the angle between the two screws, and the other of moment $G\sin\alpha\beta$, the axis of which is at right angles to α. The force F resolves into two forces, one, $F\cos\alpha\beta$, along α, the other, $F\sin\alpha\beta$, at right angles to α, and at the perpendicular distance h from it. The displacement along the axis of α is $p_\alpha\delta\phi$. Hence the work δW done in the displacement is

$$G\cos\alpha\beta.\delta\phi + F\cos\alpha\beta.p_\alpha\delta\phi - Fh\sin\alpha\beta\delta\phi.$$

This may be written more briefly as the equation

$$\delta W = F\{(p_\alpha + p_\beta)\cos\alpha\beta - h\sin\alpha\beta\}\delta\phi. \tag{10}$$

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If δW is zero for the displacement $\delta\phi$, we must have the condition

$$(p_\alpha + p_\beta)\cos\alpha\beta - h\sin\alpha\beta = 0 \tag{11}$$

This expression may be called the *work co-efficient* of the displacement round the screw α. It will be observed that if the wrench acted on the screw α, and the body were free to turn round the screw β, the work co-efficient would not be altered. We shall denote it in future by $\varpi_{\alpha\beta}$. Two screws are said to be reciprocal if $\varpi_{\alpha\beta}$ vanishes, that is when a wrench on one screw does no work in a displacement of the body on the other, and also when a wrench on one screw can produce no displacement of the body on the other.

If two screws intersect, that is if $h=0$, they are reciprocal if $p_\alpha + p_\beta = 0$, or if $\alpha\beta = \pi/2$. If two screws are at right angles, and do not intersect, the value of h is not zero, and the value of $\varpi_{\alpha\beta}$ will be $-h\sin\alpha\beta$, unless p_α or p_β is infinite. When $p_\alpha = \infty$ the body is incapable of rotation, when $p_\beta = \infty$ the intensity of the wrench is zero; since $\alpha\beta = \pi/2$, the wrench cannot move the body in the former case, in the latter case the couple has no component about α, and the body is not moved. Thus the screws in this case are reciprocal.

If α, β be coincident $h=0$, and the body will not be moved if p_α and p_β are both zero or both infinite. In the former case there is no couple, and the body is capable only of rotation; in the latter case there is no force, and the body is capable only of motion of translation. Hence there is no work done in any displacement, and the screws are reciprocal. This is expressed by saying that if the pitch of a screw be zero or infinite it is reciprocal to itself.

We have seen that a cylindroid is determined by two screws. It follows that if a screw be reciprocal to two screws θ, ϕ on a cylindroid it is reciprocal to every screw on the cylindroid.

It has been proved above that a twist on any other screw ψ on the cylindroid can be expressed as a twist on each of the screws θ, ϕ. Hence, since a wrench on β does no work on a body free to move in a twist round θ or ϕ, it can do no work in a twist round ψ. Thus the proposition is proved. It is stated shortly by saying that the screw β is reciprocal to the cylindroid θ, ϕ.

*297. **Conditions of Equilibrium given by Theory of Screws.**—Now returning to the question of equilibrium, we have for each degree of freedom of the body an equation of the form (10). Thus if α, β, γ,... denote the various screws about which the body can move, and θ denote the wrench to which the forces reduce, we have for the equations of equilibrium

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$$\begin{aligned}(p_\alpha + p_\theta)\cos\alpha\beta - h\sin\alpha\beta = 0\\ (p_\beta + p_\theta)\cos\beta\theta - h\sin\beta\theta = 0\end{aligned} \qquad (12)$$

. . . . . .

For a rigid body there are six such independent equations if the body is perfectly free. The freedom of the body is limited by the introduction of constraints, which are each expressed by an equation of the form (9). Let these equations be in number $m(m < 6)$, then the body has $6 - m$ degrees of freedom.

Thus if $m = 4$ it can be proved that for equilibrium the wrench to which the forces reduce must be reciprocal to each of *two* screws. Thus there are in this case only two freedoms, that is in reality only two independent equations of the form (10) hold for the body.

If two screws about which the body is free to move be determined, the body is free to move about any screw on the cylindroid expressed by these two screws. For as stated above a twist about any screw on a cylindroid may be expressed in terms of twists about any two axes on the same cylindroid.

A cylindroid contains all the screws (infinite in number of course) which are related to two screws, and is called a *screw-complex of the second order;* all the screws which can be constructed from three given screws form a screw-complex of the third order; and so on. A screw-complex of the sixth order comprises all the screws which can be formed from six given screws. A body capable of turning round six given screws is a perfectly free rigid body, and can turn about every line in space. A screw-complex of the sixth order therefore comprises every line in space.

For further information on the Theory of Screws the reader is referred to Sir Robert Ball's treatise, and also to Professor Minchin's "Treatise on Statics," vol. ii. The subject is intimately related to the mathematical theory of *Linear Complexes*, which has been treated by Plücker, Klein, and others.

298. **Reduction of Applied Forces to Two.**—The applied forces on the body may of course be reduced in an infinite number of ways to two forces. It is only necessary to take any one of the resolutions to a force F and a couple G in a plane inclined at some definite angle ϕ to the force. Then suppose the couple represented by two equal, and along the projection of F upon the plane of the couple, and opposite forces, one G/a at a point in the line of action of F, and the other G/a at a distance a from the plane of F and the former force. The two intersecting forces F and G/a give a resultant $R = (F^2 + G^2/a^2 + 2FG/a.\cos\phi)^{\frac{1}{2}}$, in a plane at right angles to that of the couple, inclined at an angle $\cos^{-1}\{(F + G/a.\cos\phi)/R\}$ to the direction of F. This resultant R and the other force G/a form a system of two forces equivalent to the applied system.

A particular case of reduction to two forces which is worth noticing is the following. Let Oz (Fig. 136) be the central axis of the system. The force F is at right angles to the plane of the couple. Replace the latter by two opposite forces each G/a acting parallel to

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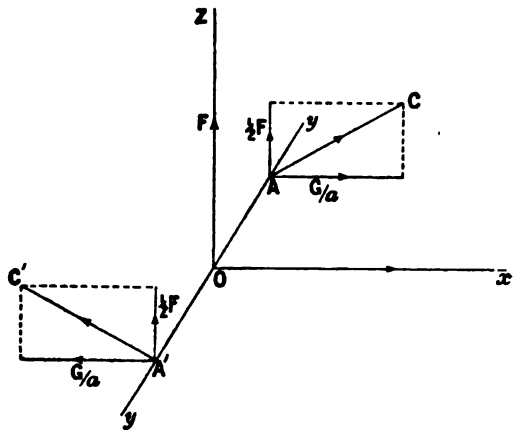
the axis Ox, and at points A, A', $\frac{1}{2}a$, $-\frac{1}{2}a$ from O along Oy, and replace F by two forces each $\frac{1}{2}F$ acting parallel to Oz at A, A'. The forces $\frac{1}{2}F$ and G/a at A give the force $\frac{1}{2}(F^2 + 4G^2/a^2)^{\frac{1}{2}}$ acting along AC, and similarly those at A' give a force of the same amount at A'. These forces act towards the same side of the plane xOy, and are inclined to it as shown in Fig. 136 at angles each $\tan^{-1}(Fa/2G)$.

299. **Further Discussion of Work-Criterion of Equilibrium.**—Returning to the more general case of a material system of any kind, we see that the condition of invariability of any configuration of the

FIG. 136.



system, that is of equilibrium of the system in that configuration, is that the impressed forces on the system should not do any work in *any* infinitely small possible displacement of the system from that configuration, so far at least as quantities of the first order of small quantities are concerned. In other words, if δx, δy, δz be *any* infinitely small changes of the co-ordinates x, y, z, the sum $\Sigma(X\delta x + Y\delta y + Z\delta z)$ must vanish for the changes of configuration of the particles of the system. It is to be remembered that X, Y, Z represent work-forces defined as at § 214 above, and that the sum is taken for all points to which these forces are applied, and which move so that the forces do work.

To see that this condition holds, we take the case of a system the configuration of which is specified by m independent co-ordinates, p, q, ... Here the work equation for infinitely small changes, δp, δq, ..., of these co-ordinates is

$$\delta W = P\delta p + Q\delta q + \ldots \tag{13}$$

If for *all possible modes of variation* of the configuration this sum of works vanish, the forces acting on every part of this system must be

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a balancing system of forces, otherwise work would be done in one or more of the possible modes of displacement. The condition is therefore sufficient. To see that it is necessary we may limit the freedom to any one co-ordinate, p, say. Then the work done in any displacement will be $P\delta p$. If this is not zero, it can be made zero by applying a force P' such that for a displacement δp, $(P+P')\delta p=0$, or $P = -P'$. The system is now in equilibrium and cannot have been in equilibrium before.

It is always possible to limit the freedom to any one of the possible freedoms of the system by means of frictionless constraints. Hence we see that in order that there may be no change of configuration of any kind in a system the configuration of which is fixed when the co-ordinates p, q, ..., are fixed, it is necessary and sufficient that the work done by the forces in every possible displacement should vanish. The equations of equilibrium obtained for the separate particles of the system by the introduction of the constraints have been given in § 293 above.

Since the co-ordinates $p, q, \ldots$, are independent, we must have moreover for equilibrium the relations

$$P=0, \quad Q=0, \quad R=0, \ldots. \tag{14}$$

300. **Stable, Unstable, and Neutral Equilibrium.**—In the case of a conservative system, in which also no work is consumed in overcoming frictional resistances, the work δW has the value

$$\delta W = -\left(\frac{\partial E}{\partial p}\delta p + \ldots\right),$$

that is, it is the amount of diminution of the potential energy involved in the displacements δp, ..., and we have seen that for equilibrium the value of E must be stationary. In general, though not necessarily in all cases, E is a maximum or a minimum when $\partial E/\partial p = 0, \ldots$, that is in a configuration of equilibrium. In the case of a minimum of E any small displacement from the configuration of equilibrium would (when small quantities of higher order than the first are taken into account) involve an increase of the potential energy, that is any forces applied to produce the displacement would on the whole have to work against the forces depending on E called into play by the displacement. The system, if displaced and then left to itself by the removal of the displacing forces, would under the forces depending on the value of E now existing return towards the position of equilibrium. The equilibrium is then said to be *stable*.

If on the contrary the forces applied to produce the displacement from the configuration of equilibrium act in the same direction as those called into play by the variation of E, which will be the case if E is a maximum, the system if left to itself after any small displacement will move further away from the equilibrium configuration. The equilibrium is then said to be *unstable*.

If finally the forces depending on E are zero after any small dis-

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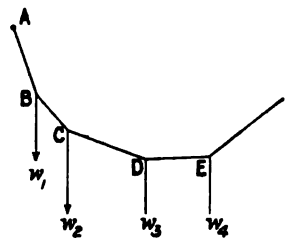
placement there is equilibrium also in the new configuration, and the equilibrium is said to be *neutral*.

We have examples in the equilibrium of a body under gravity, for example a pendulum. In every such case the equilibrium is stable only when the body is in the lowest position which it can assume. Take the case of a ball on a surface. If it is at rest at the bottom of a spherical cup, the equilibrium is essentially stable. If the ball is at the top of a spherical cap the equilibrium is essentially unstable. If it is at rest on a plane it is in neutral equilibrium.

A body may, however, be stable for some displacements and unstable for others. For example, let the ball be at rest on a saddle-shaped surface, so placed that one principal section of it is convex upwards, the other concave upwards. The ball is stable for displacements along the concave section, and unstable for displacements along the other.

301. **Example. The Funicular Polygon.**—A usual but very instructive example of the conditions of equilibrium is the funicular polygon represented in Fig. 137. A number of weights, $w_1, w_2, \ldots, w_n$, are suspended at points $B, C, D, \ldots, M$ at different distances apart along a cord which is secured at its ends to fixed points A, N. It is required to find the inclinations of the successive parts of the cord to the horizontal and the stretching forces in those parts.

FIG. 137.



We suppose the mass of the cord itself to be negligible, so that no action of gravity on it has to be considered. Also we suppose that the cord is without rigidity, in other words, that a portion of it cannot be in equilibrium under forces applied to its ends, unless those forces are equal and opposite, and every part of the cord between the points of application is in the line of action of the forces. The applied forces are the weights $w_1, w_2, \ldots, w_n$, and it is clear that under their action the cord will lie in the vertical plane through A, N. Let the weights be arranged in the order shown in the figure from the extremity A of the cord, and let $a_1, a_2, \ldots, a_n, a_{n+1}$ be the lengths of the portions of cord $AB, BC, \ldots, MN$. Further let the distances of the points $B, C, D, \ldots, N$ below the horizontal through A be $y_1, y_2, y_3, \ldots, y_{n+1}$, and their horizontal distances from A be $x_1, x_2, x_3, \ldots, x_{n+1}$. Then for any possible displacements of the system the equation of work is

$$\delta W = w_1 \delta y_1 + w_2 \delta y_2 + \ldots + w_n \delta y_n.$$

For equilibrium the expression on the right must be zero.

The geometrical equations are

$$x_1^2 + y_1^2 = a_1^2, \ \ldots\ldots\ldots\ldots, \ (x_{n+1} - x_n)^2 + (y_{n+1} - y_n)^2 = a_{n+1}^2,$$

which give the $n+1$ relations

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$$x_1\delta x_1 + y_1\delta y_1 = 0, \ldots, (x_{n+1} - x_n)(-\delta x_n) + (y_{n+1} - y_n)(-\delta y_n),$$

since $\delta x_{n+1} = 0$, $\delta y_{n+1} = 0$.

Multiplying the first of the last set of equations by λ_1, the second by λ_2, ..., and the last by λ_{n+1}, adding to the expression for the work and equating separately to zero the co-efficients of δy_1, δy_2, ..., δy_n thus formed, we obtain

$$\left.\begin{array}{l} w_1 + \lambda_1 y_1 \qquad\qquad - \lambda_2(y_2 - y_1) = 0 \\ w_2 + \lambda_2(y_2 - y_1) - \lambda_3(y_3 - y_2) = 0 \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ w_n + \lambda_n(y_n - y_{n+1}) - \lambda_{n+1}(y_{n+1} - y_n) = 0 \end{array}\right\} \quad (15)$$

Besides these we have from the terms in δx_1, δx_2, ... the equations

$$\lambda_1 x_1 - \lambda_2(x_2 - x_1) = 0,\ \lambda_2(x_3 - x_1) - \lambda_3(x_3 - x_2) = 0, \ldots,$$
$$\lambda_n(x_n - x_{n-1}) - \lambda_{n+1}(x_{n+1} - x_n) = 0,$$

or

$$\lambda_1 x_1 = \lambda_2(x_2 - x_1) = \lambda_3(x_3 - x_2) = \ldots\ldots\ldots = \lambda_{n+1}(x_{n+1} - x_n) = \tau, \text{ say.}$$

These give values of λ_1, λ_2, ... in terms of the constant τ. Substituting in (15) we obtain the n equations,

$$\left.\begin{array}{l} w_1 = \tau\left(\dfrac{y_2 - y_1}{x_2 - x_1} - \dfrac{y_1}{x_1}\right) \\ w_2 = \tau\left(\dfrac{y_3 - y_2}{x_3 - x_2} - \dfrac{y_2 - y_1}{x_2 - x_1}\right) \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ w_n = \tau\left(\dfrac{y_{n+1} - y_n}{x_{n+1} - x_n} - \dfrac{y_n - y_{n-1}}{x_n - x_{n-1}}\right). \end{array}\right\} \quad (16)$$

If θ_1, θ_2, ..., θ_{n+1} be the angles the parts a_1, a_2, ..., a_{n+1} of the cord make with the horizontal, $y_1/x_1 = \tan\theta_1$, $(y_2 - y_1)/(x_2 - x_1) = \tan\theta_2$, ..., so that the equations just found may be written

$$\left.\begin{array}{l} w_1 = \tau(\tan\theta_2 - \tan\theta_1) \\ w_2 = \tau(\tan\theta_3 - \tan\theta_2) \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ w_n = \tau(\tan\theta_{n+1} - \tan\theta_n) \end{array}\right\} \quad (17)$$

The obvious interpretation of these equations is that τ is the horizontal component of the stretching force in each cord. Evidently the same horizontal pull must be exerted towards each side on every individual particle, and the quantities on the right of (15) and (17) are the sums of the upward vertical pulls exerted on each particle by the cords attached to it. The stretching forces in the cords AB, BC, ... are respectively $\lambda_1 a_1$, $\lambda_2 a_2$, ..., $\lambda_{n+1} a_{n+1}$.

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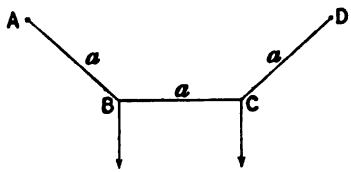
For the complete determination of $\theta_1, \theta_2, \ldots, \theta_{n+1}$ we have the two conditions

$$\left.\begin{aligned} a_1 \sin\theta_1 + a_2 \sin\theta_2 + \ldots + a_{n+1} \sin\theta_{n+1} &= h \\ a_1 \cos\theta_1 + a_2 \cos\theta_2 + \ldots + a_{n+1} \cos\theta_{n+1} &= l \end{aligned}\right\} \quad (18)$$

where h, l are respectively the vertical and the horizontal distance of N from A. Equations (17) serve to find $\tan\theta_2, \tan\theta_3, \ldots, \tan\theta_{n+1}$ in terms of $\tan\theta_1$ and τ. Substitution in (18) will then give θ_1 and τ, and the problem is completely solved.

As a particular example we may take the simple case of two equal weights connected as in Fig. 138 by three equal strings with two

FIG. 138.



points on the same level and at a distance l apart ($l > a$). The equations are in this case,

$$\begin{aligned} w &= \tau(\tan\theta_2 - \tan\theta_1) = \tau(\tan\theta_3 - \tan\theta_2) \\ \sin\theta_1 + \sin\theta_2 + \sin\theta_3 &= 0 \\ \cos\theta_1 + \cos\theta_2 + \cos\theta_3 &= 0. \end{aligned}$$

If in these equations we put $\theta_2 = 0$ we obtain

$$\begin{aligned} \tan\theta_1 + \tan\theta_3 &= 0 \\ \sin\theta_1 + \sin\theta_3 &= 0 \\ \cos\theta_1 + \cos\theta_3 &= \frac{l}{a} - 1. \end{aligned}$$

The first two of these are satisfied by $\theta_1 = 2\pi - \theta_3$. The values $\theta_1 = -\theta_3 = \frac{1}{2}\pi$ might be taken as a solution, but they would give $l = a$, a case which is excluded by the question. The other values give

$$\theta_1 = \cos^{-1}\frac{l-a}{2a}.$$

This is the solution represented in Fig. 138, and it is evidently the only solution.

302. **Example. Biflar Suspension.**—As an example of stability of equilibrium we consider here an arrangement which forms part of several instruments of great importance in Physics. A body of given mass M is suspended by two threads L, L of given lengths attached at their upper ends to two fixed points not necessarily in a horizontal line. The points of attachment of the lower ends of the threads to

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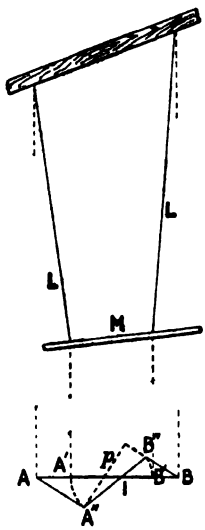
the suspended body are not assumed to be on the same level, or at equal distances on the two sides of the vertical through the centroid of the body.

In Fig. 139 A, B are the projections on a horizontal plane of the points to which the upper ends of the threads are attached, A', B' are those of the lower ends of the threads or points at which they are attached to the body, A'', B'' are those of the latter points when the body is deflected from the position indicated by $A'B'$. As we shall see, the body is in stable equilibrium when $A'B'$ are in line with AB.

FIG. 139.



That the body is in equilibrium in this position is evident from the fact that then the threads are in one plane. For let the body be deflected from that position to that shown by AB, $A''B''$. Each thread becomes inclined to the vertical plane through the two upper points of attachment, and a couple is brought into play tending to bring the body back to the former position. The horizontal components of the tension of the threads act along AA'', $B''B$, and are equal and parallel since we suppose the centroid of the body to remain in the same vertical in all positions of the body. The sum of the vertical components of the stretching forces in the threads is Mg, the gravity of the body. Hence if one of these vertical components, that in AA' say, be denoted by $\frac{1}{2}Mg(1+c)$, the other must be $\frac{1}{2}Mg(1-c)$. Also if the mean of the vertical heights of A above A', and B above B' be h, and the former height be $h(1+f)$, the other must be $h(1-f)$. If c be positive, f will be positive or negative according as the greater or less stretching force acts in the thread of greater vertical height.

Let L be the restoring couple called into play when the body is deflected, and p be the perpendicular distance between the parallels AA'', BB''. Then the horizontal forces along these lines are each L/p. This is the horizontal force which must be applied at A'', B'' in each of these directions to keep the body deflected. Hence, since for each thread the vertical, the horizontal, and the direction of the thread represent the directions of three forces in equilibrium, we have the two equations

$$\frac{AA''}{h(1+f)} = \frac{L/p}{\frac{1}{2}Mg(1+c)}, \quad \frac{BB''}{h(1-f)} = \frac{L/p}{\frac{1}{2}Mg(1-c)}.$$

But if $AB = 2a$, $A'B' = 2b$, the sum of the areas of the two triangles AIA'', BIB'' is $\frac{1}{2}(AA'' + BB'')p$, and this is evidently $2ab\sin\theta$, if θ be

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the angle of deflection AIA''. Hence we obtain from the equations just written

$$\tfrac{1}{2}(AA'' + BB'')p = \frac{2Lh}{Mg}\,\frac{1-cf}{1-c^2} = 2ab\sin\theta,$$

and therefore

$$L = Mg\frac{ab(1-c^2)}{h(1-cf)}\sin\theta. \tag{19}$$

This shows that for a given deflection L is smaller the greater h, that is the *sensibility* is greater the greater the mean length of the threads. Also L is greater the greater a or b, and vanishes with either of these quantities. The sensibility therefore is less the greater a or b, and is infinite when a or b is zero.

In general the arrangement is made symmetrical, that is each pair of points of attachment, the pair at the top and the pair at the bottom, is in a horizontal line, and I lies midway between AB, and also between $A'B'$. Hence

$$L = Mg\frac{ab}{h}\sin\theta.$$

The value of h is now $\sqrt{l^2 - AA''^2}$ and $AA''^2 = a^2 + b^2 - 2ab\cos\theta$. Therefore $h^2 = l^2 - (a-b)^2 - 4ab\sin^2\tfrac{1}{2}\theta$. Hence

$$L = \frac{Mgab\sin\theta}{\{l^2 - (a-b)^2 - 4ab\sin^2\tfrac{1}{2}\theta\}^{\frac{1}{2}}} \tag{20}$$

If the fibres be nearly at the same distance apart at top and bottom, and l be great in comparison with a and b, we have

$$L = Mg\frac{ab}{l}\sin\theta. \tag{21}$$

That the potential energy of the body is a minimum when the threads are in the same plane is obvious from the fact that when the body is deflected from that position each thread becomes inclined at a greater angle to the vertical, and so the body is raised.

303. **Stability of Equilibrium of Bodies at Rest or in Steady Motion. Centre of Gravity. Vehicles Moving in Curves.**—For a body resting under gravity on a plane, whether horizontal or inclined—for example, a table supporting a load—the condition of stable equilibrium is simply that the vertical through the centroid of the body shall fall within the *base* or area of support of the body. To define this area, suppose a closed thread placed on the horizontal plane of support so as to surround every point of contact between the supports of the body and the plane, and then drawn tightly up to the outer supports. It will mark out an area on the plane, and that area is called the *base* of the body.

If the vertical fall outside this area the resultant force of gravity

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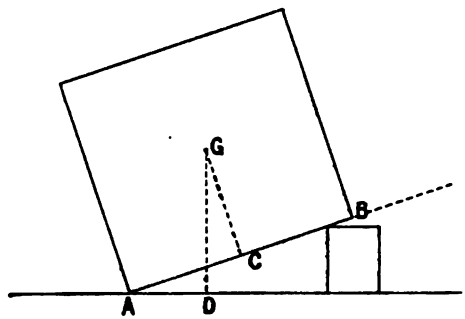
on the body and the resultant of the vertically upward actions on the body of the points of support will give a couple turning the body about a horizontal axis. Hence is produced the overturning of a cart or carriage when it is drawn along a slope so that one wheel is higher than the other, or in other cases when one wheel passes over a sufficiently high obstacle.

The higher the centroid of the vehicle the more danger there is of overturning—that is, the smaller is the angle through which the vehicle can be tilted from equilibrium on a horizontal plane without capsizing. This angle measures the margin of stability of the body. For let a perpendicular be let fall from the centroid to the plane of support and meet the latter in C (Fig. 140). Then if A be one of

FIG. 140.



the points of support the angle through which the body can be turned in the plane GCA is $\tan^{-1}CA/CG$, which is less the greater CG. The figure shows a block in stable equilibrium tilted over through a smaller angle CGD or BAD. The block would just be on the point of overturning if it were tilted so far that D coincided with A.

We may remark here that it is usual to refer in this connection to the "centre of gravity" of the body instead of to the centroid. The centre of gravity of a body is, strictly speaking, that point, *if any*, fixed relatively to the body through which the resultant of the forces of gravity on the particles of the body always passes, in whatever position the body may be placed relatively to the earth. Strictly speaking, no such point exists for bodies in general, but only for distributions of matter fulfilling certain conditions. We shall discuss these under *Gravitational Attraction;* but it may be stated that since the forces of gravity on the particles of bodies of such moderate dimensions as those that have ordinarily to be considered are very approximately parallel, the resultant of gravity always passes through a small space about the centroid, and hence the centroid may be regarded as approximately fulfilling the definition of the centre of gravity for the body. In this approximate sense, perhaps it may be referred to as the centre of gravity, but it is better to use the latter term only where it is strictly applicable.

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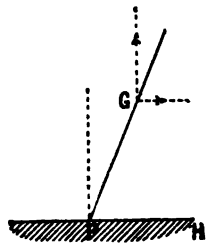
When a body—for example, a railway carriage—moves with uniform speed round a curve, the forces acting on the body, over and above those required to give the acceleration towards the centre of curvature, must be in equilibrium. For this reason a line drawn at right angles across the rails is inclined to the horizontal, so that the carriage as it runs is tilted inwards towards the centre of curvature.

Let M be the mass of the carriage, v its speed, and R the radius of the curve. The force towards the centre of curvature of the path (supposed plane) is Mv^2/R. The reaction of the rails on the carriage must be such as to give a resultant through the centroid of the carriage which can be resolved into two components, one vertical and equal to the weight of the carriage, the other horizontal and equal to Mv^2/R. Hence the inclination of the line across the rails to the horizontal is $\tan^{-1}(Mv^2/RMg)$ or $\tan^{-1}(v^2/gR)$.

FIG. 141.



Of course, in practice the tilt of the rails is arranged for some convenient speed which should not be exceeded. The passage from the straight to the curve on railways is also made gradual according to certain practical rules, which will be found set forth in books on Civil Engineering.

Again, let a pendulum be hung in such a carriage. The bob will move outwards from the vertical until the cord is so far inclined as to give a horizontal pull equal to that required to give it acceleration equal to v^2/R. The tangent of the inclination of the bob to the vertical is then v^2/gR—that is, the thread is perpendicular to the line across the rails.

In turning a corner quickly a bicyclist inclines the machine over towards the corner. Let PH in Fig. 141 represent the horizontal, and let PG be a line representing the inclination of the frame of the bicycle to the vertical. Owing to the grip each tire has of the ground, a horizontal force is applied to the wheels in the direction PH, besides a vertical component, the sum of which for the two wheels is equal to the weight of the machine and rider, with a slight modification due to the rotation of the wheels. The resultant acts in the direction represented by PG. On bicycle racecourses, which are circular, the track is inclined inwards to give, without danger of slipping, a reaction along the sloping frame, as here indicated. If G be the centroid of the rider and machine, the resultant may be there resolved into two components, a horizontal and a vertical. The former furnishes the inward force Mv^2/R towards the centre of the circle of turning, where v is the speed, M the total mass, and R the radius. The tangent of the inclination of the machine to the vertical has the approximate value v^2/gR.

304. **Equilibrium of a Pendulum hung in Vehicle under Acceleration.**—Also as another case of relative equilibrium consider a pendulum hung in a railway carriage which is undergoing constant

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acceleration in the direction of motion. The position of the pendulum is one inclined backwards just enough to give a forward pull ma on the bob. Hence the inclination of the thread to the vertical is $\tan^{-1}(a/g)$. If the carriage is under a retarding acceleration of the same amount, the inclination of the thread is in the direction of motion forward, and has the same value.

When either acceleration is imposed from uniform motion the pendulum is deflected and oscillates about the position just indicated, finally coming to rest in the inclined position if the acceleration is uniformly maintained.

305. **Simple Conical Pendulum.**—As an example of relative equilibrium we may consider the arrangement called the *Conical Pendulum.* A particle P of mass m is suspended by a cord of negligible mass from a fixed point A (Fig. 142), and describes uniform circular motion under the action of gravity round the vertical through the point support. The horizontal component of pull T applied by the thread gives the necessary acceleration of the particle towards the centre C of the circular orbit; the vertical component balances the weight of the particle. Thus if v be the velocity of the particle, and r the radius of the path, l the length of the thread, and θ its inclination to the vertical, we have $T\sin\theta = mv^2/l\sin\theta$. But the vertical component of T is $T\cos\theta$, and therefore $T\cos\theta = mg$. Hence we obtain, substituting in the former relation $mg/\cos\theta$ for T,

$$v^2\cos\theta = gl\sin^2\theta.$$

Now if τ be the period of revolution of the particle in the circle, $\tau = 2\pi l\sin\theta/v$, and $v^2 = 4\pi^2 l^2\sin^2\theta/\tau^2$. Substituting this value of v^2 in the equation last obtained we find $g\tau^2 = 4\pi^2 l\cos\theta$, or

$$\tau = 2\pi\sqrt{\frac{l\cos\theta}{g}} \tag{22}$$

that is, the period is that of a simple pendulum of length equal to the vertical distance of A above the plane of the circle.

306. **Compound Conical Pendulum.**—Now let the pendulum be a rigid body free to move in a vertical plane about a point A in a vertical spindle round which it turns in azimuth, like one of the arms with attached weight of the centrifugal governor of a steam-engine. Let it be symmetrical about a line through A and the centroid G of the pendulum, and let it revolve with constant angular velocity ω at a constant inclination θ to the vertical (Fig. 143). It is required to find the period of revolution τ in terms of ω and θ.

This problem may be solved at once by observing that it is precisely the problem of a top, the motion of which is limited to steady precession about the vertical (§ 264), or its solution may be deduced at once from Euler's equations of motion, § 282 above. It is instructive, however, to solve it from first principles.

We observe first that the sum of the moments, about an axis

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through A perpendicular to the paper, of the mass-accelerations of the particles towards the vertical, must be equal to the sum of the moments about the same axis of the forces applied by gravity, that is to $Mgl\sin\theta$, if M be the total mass of the pendulum, and l be the distance of G from A.

Let P (Fig. 143) be a particle of mass m, and let it lie in a plane PQR perpendicular to the axis AG. Let fall a perpendicular PR from P on the plane containing the axis of symmetry and the vertical,

FIG. 142.

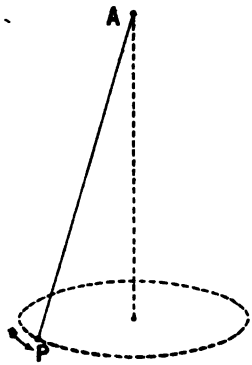
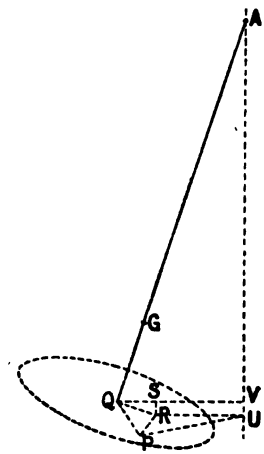


FIG. 143.



and let x be the distance QR from the foot R of this perpendicular to the axis of symmetry, l' be the distance AQ and y the distance PR. The distance PU of the particle from the vertical is then $\sqrt{\{(l'\sin\theta - x\cos\theta)^2 + y^2\}}$. Hence the force necessary to give the acceleration along PU required for the rotation is

$$m\omega^2\{(l'\sin\theta - x\cos\theta)^2 + y^2\}^{\frac{1}{2}}.$$

The moment of this force about the line through A perpendicular to the paper is the component along the line RU multiplied by the distance AU. The component is $m\omega^2(l'\sin\theta - x\cos\theta)$ and AU is $l'\cos\theta + x\sin\theta$. Hence the moment required is

$$m\omega^2\{(l'^2 - x^2)\sin\theta\cos\theta - l'x(\cos^2\theta - \sin^2\theta)\}.$$

If now we find the sum of moments for all particles in the plane, we see that, since l' and θ are the same for all such points, the terms $m\omega^2 l'x(\cos^2\theta - \sin^2\theta)$ contribute nothing to the result. Hence summing for the whole pendulum we have $\omega^2\sin\theta\cos\theta(\Sigma ml'^2 - \Sigma mx^2)$, for the total moment. If A be the moment of inertia of the pendulum about the axis through A perpendicular to the paper, and C that about the axis of symmetry, the foregoing sum is

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$$\omega^2(B-C)\sin\theta\cos\theta.$$

since $\Sigma ml'^2 - \Sigma mx^2 = \Sigma ml'^2 + \Sigma my^2 - \Sigma mx^2 - \Sigma my^2 = B - C$.

Equating this to the moment applied by gravity, we obtain $\omega^2 = Mgl(B-C)\cos\theta$, or since $\omega = 2\pi/\tau$.

$$\tau = 2\pi\sqrt{\frac{(B-C)\cos\theta}{Mgl}}. \tag{23}$$

The same result may be deduced from Euler's equations thus. We have, putting ω_3, ω_2 for the angular velocities respectively about the axis of symmetry, and about an axis perpendicular to the latter through A in the plane GAU, $A\dot{\omega}_1 - (C-B)\omega_3\omega_2 = L$, where $L = Mgl\sin\theta$, $\omega_1 = 0$, $\dot{\omega}_1 = 0$, $\omega_3 = \omega\cos\theta$, $\omega_2 = \omega\sin\theta$. Thus we obtain at once $\omega^2 = (B-C)\cos\theta/Mgl$, the same result as before. When the pendulum is a particle of mass m suspended by a massless thread $B = ml^2 C = 0$, and $M = m$. Hence we fall back on the case discussed in § 302.

307. **Equilibrium of a Particle on a Surface and on a Curve.**—If the surface is smooth it is only necessary for equilibrium that the resultant of the forces applied to the particle should be perpendicular to the surface. The resultant will then be balanced by the reaction of the surface. Let direction-cosines of the normal to the surface be l, m, n, and X, Y, Z be total components along the axis of the applied forces. For equilibrium we must have

$$\frac{X}{l} = \frac{Y}{m} = \frac{Z}{n}. \tag{24}$$

The co-ordinates x, y, z of any point on a surface are connected by a relation which is expressed by $\phi(x, y, z) = 0$, where ϕ denotes a function of x, y, z. If dx, dy, dz be increments of x, y, z giving new values $x + dx$, $y + dy$, $z + dz$, then

$$\phi(x+dx,\ y+dy,\ z+dz) - \phi(x,\ y,\ z) = 0.$$

In the limit when dx, dy, dz are taken very small this gives

$$\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz = 0.$$

Hence $d\phi/dx$, $d\phi/dy$, $d\phi/dz$ must be proportional to the direction-cosines of a line perpendicular to the short line PQ on the surface of which the projections on the axes are dx, dy, dz. Since this holds for every direction in which PQ is taken the direction of this line is perpendicular to the surface at P, that is, $d\phi/dx$, $d\phi/dy$, $d\phi/dz$ are proportional to the direction-cosines of the normal to the surface, we have therefore

$$\frac{1}{X}\frac{d\phi}{dx} = \frac{1}{Y}\frac{d\phi}{dy} = \frac{1}{Z}\frac{d\phi}{dz}. \tag{25}$$

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These give two equations, which with $\phi(x, y, z) = 0$ determine the co-ordinates of P, the position of equilibrium.

A curve is given by two independent equations connecting the co-ordinates x, y, z of points on the curve, that is the curve is regarded as the intersection of two surfaces. Let l, m, n be the direction-cosines of the tangent to the curve at any point P, then if a particle be in equilibrium on the curve at P the component of the applied forces taken along the curve must be zero, that is, we must have

$$lX + mY + nZ = 0. \tag{26}$$

This with the two equations of the curve determines the position of equilibrium.

Now let the surface be rough. Resolve the resultant of the applied forces into two components, one along the normal and the other in the tangent plane. The former is $\lambda X + \mu Y + \nu Z$, if we put λ, μ, ν for the direction-cosines of the normal. We shall denote this by N. The cosine of the angle θ say between the direction of the resultant of the applied forces and the normal is this quantity divided by $\sqrt{X^2 + Y^2 + Z^2}$. The cosine of the complementary angle is $\sqrt{1 - \cos^2\theta}$, and therefore the component in the tangent plane is $\sqrt{X^2 + Y^2 + Z^2}\sqrt{1 - \cos^2\theta}$, and has the value

$$\{(\nu Y - \mu Z)^2 + (\lambda Z - \nu X)^2 + (\mu X - \lambda Y)^2\}^{\frac{1}{2}}$$

We shall denote this by T. Hence for equilibrium

$$\mu N \pm T = 0. \tag{27}$$

No force, however great, which makes an angle θ with the normal to the surface less than $\tan^{-1}\mu N/N$, that is than $\tan^{-1}\mu$, will produce motion of the particle. For any such force will have a normal component N' and a tangential component T' such that $N' \tan\theta = T'$, and since $\tan\theta < \mu$, $T' < \mu N'$. But in consequence of N' friction of amount $\mu N'$ can be called into play to prevent motion, and this is greater than the tangential force T'.

Now imagine a curve drawn on the surface to mark out a region on the surface near P so that at every point Q on the curve the angle which the direction of the normal at P makes with that at Q is just equal to $\tan^{-1}\mu$. That curve will enclose a space round P within which the angle θ is less than $\tan^{-1}\mu$, and at every point of which therefore the particle is in equilibrium.

In the same way for equilibrium of a particle on a rough curve, if as before l, m, n be the direction-cosines of the tangent we have the tangential component $lX + mY + nZ$. The component at right angles to the tangent and in the plane of the resulant force and the tangent is $\{(mZ - nY)^2 + (nX - lZ)^2 + (lY - mX)^2\}^{\frac{1}{2}}$. Hence for equilibrium

$$lX + mY + nZ \pm \mu\{(mZ - nY) + (nX - lZ) + (lY - mX)\}^{\frac{1}{2}} = 0, \tag{28}$$

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which with the two equations of the curve determine the position of equilibrium.

Clearly equilibrium will exist for any point for which the angle θ between the direction of the resultant force and the line perpendicular to the tangent in the plane of the latter and the resultant force is less than $\tan^{-1}\mu$. The angle $\tan^{-1}\mu$ is called the *angle of repose*.

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CHAPTER VII.

GRAPHICAL STATICS.

308. **Funicular Polygon and Polygon of Forces.**—In engineering the forces acting on the different parts of a structure or machine are for the most part determined by graphical processes. Of these we give a short account here, with a few examples; but many other applications will be found in what follows.

As an example to introduce the subject we take the problem of the funicular polygon, of which we have considered (§ 301) the analytical solution in a certain case. At each point **1**, **2**, **3**, ... of the polygon (Fig. 144) three forces may be regarded as acting,

FIG. 144.

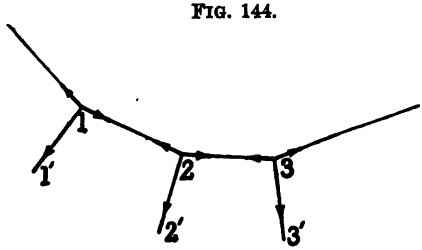
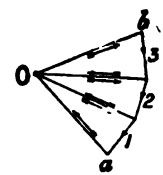


FIG. 154.



namely, two along the adjacent parts of the chain, and the third the applied force **11′**, **12′**, ... as the case may be. In each part **12**, **23**, ... of the chain the stress along the part pulls on the terminals with equal and opposite forces.

Since the three forces at each point **1**, **2**, ... must be in equilibrium they must be capable of being represented by the three sides of a triangle taken in order (§ 142). Thus they can be represented by any triangle to the sides of which their directions are parallel. Taking, then, **11′** we draw a triangle the sides of which are parallel to the forces at **1**, we denote by *O* the point in which the two lines representing the forces in the two parts of the chain meet, and indicate by arrows drawn within the triangle the directions in which these forces act on the point **1**. By an arrow on the third side and the number 1 placed opposite the middle of that side the applied force which acts at **1** in the funicular polygon is indicated.

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Now the side of this triangle which represents the force from **1** to **2** in the funicular polygon will with a reversed arrow represent the opposite force which acts at **2** from **2** to **1**. Then drawing a line from the extremity of 1 (Fig. 145) parallel to the applied force at **2** (Fig. 144), to meet a line from *O* drawn parallel to the force in the part of the chain from **2** to **3**, we add in Fig. 145 the triangle the sides of which and the arrows within it represent the magnitudes and directions of the forces acting at 2 in the funicular polygon. The line representing the applied force is marked 2. So the force-diagram (Fig. 145) is constructed by adding a triangle for successive points at which forces are applied to the chain; and may be either confined to any partial sequence of the points **1**, **2**, in Fig. 144, or made complete by the inclusion of all such points together with the terminal points at which the chain is supported. [In the figures only three forces are shown, but there may of course be any number.]

For distinction the numbers which designate points in the funicular polygon and corresponding sides of the polygon of forces are printed in clarendon type in the former case and in ordinary type in the latter.

309. **Forces equivalent to Given System obtained from Force-polygon.**—The lines 1, 2, ... Fig. 145, form the sides of a polygon which represents the applied forces and which we shall call the *force-diagram* or *force-polygon*; and we shall prove that the two forces which join the extremities of this polygon to *O*, if taken as shown by the arrows in Fig. 145, represent two forces which properly applied to a rigid body would just equilibrate the forces 1, 2, ... if these were applied without change of magnitude or line of action as given in Fig. 144 to the same body.

For these two forces have, when the forces are all transferred to one point of the body (as specified in § 177) a resultant equal and opposite to that of 1, 2, ... Further the lines *bO*, *Oa* represent the stretching forces in the terminal parts of the chain (Fig. 144), and when the chain is in equilibrium the sum of the moments of these forces round any point in the plane of the forces must be equal to the sum of the moments of the applied forces about the same point, since those in the parts **12**, **23**, ... of the chain being pairs of equal and opposite forces have no moments round any point. Thus the equality of moments must still exist when the forces **11′**, **22′**, ... are applied as above to a rigid body. Thus the proposition is proved.

From the proposition just established we deduce at once the conclusion that if any forces **11′**, **22′**, ... act in any directions in one plane on a rigid body they are equivalent to two forces represented by *Oa*, *bO* of Fig. 145, acting in these directions in the terminal parts of the funicular polygon of Fig. 144; that is the graphical construction reduces the system to two forces.

But, moreover it is obvious that for the given system of coplanar forces acting on a rigid body we can construct any number of funicular polygons each of which will give a system of two forces

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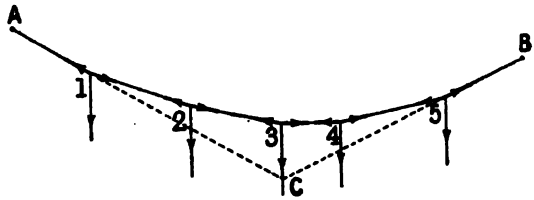
equivalent to the given system. It is only necessary for the construction that the origin O of the force-diagram should not lie in the direction produced of any of the sides 1, 2, ... of this diagram, or that if it is so situated a terminal point of the funicular polygon should be taken on the line of action of the applied force represented by the side of the force-diagram on which O lies.

310. **Case of Vertical Forces.**—When the applied forces are vertical the sides 1, 2, 3, ... of the force-polygon become a vertical straight line, and the lines drawn from the origin O in Fig. 147 give

FIG. 146.



the directions and magnitudes of the forces in the sides of a funicular polygon (Fig. 146) which correspond to these lines. We may apply this to find the funicular polygon in which would hang a weightless chain loaded with equal weights at successive equal distances apart,

FIG. 147.

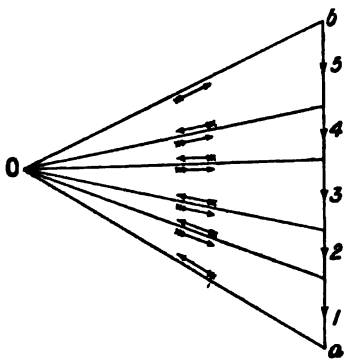
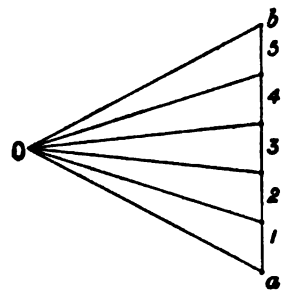


FIG. 148.



(1) when these distances are measured along the polygon, (2) along the horizontal line drawn across the applied forces. We shall suppose first the terminal points A, B of the funicular polygon given and to be on the same level.

Draw the force-polygon, that is, draw a vertical straight line at (Fig. 147) divided into as many equal parts as there are applied forces. Through the middle point m of this line draw a perpendicular to it, and take any point on this perpendicular as origin O.

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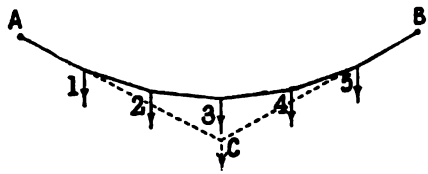
From O draw lines to a b and each of the points of division of the line a b. These are the directions of the sides of a funicular polygon in which the chain will hang. For case (1) take any distance r and lay it off from O along Oa, and take its projection p, on Om. Lay the same distance along $O12$, and find the projection p_1 on Om, and so on for each of the lines Oa, $O12$, $O23$, ... Ob. Take the sum u of the projections, and if it is greater or less than AB, diminish or increase r in the proper proportion to give the proper sum of projections. Having obtained the projections draw vertical lines at successive distances apart equal to the successive projections.

FIG. 149



Then from a point A (Fig. 148) representing the first terminal point, draw a line parallel to the line Oa, meeting the first vertical in 1, from 1 draw a line parallel to O 12, meeting the second vertical in 2, and so on until at the point B, the polygon, is completed.

For case (2), draw the verticals at the proper equal distances apart. From A draw a line parallel to Oa, meeting the first vertical in 1, from 1 a line parallel to $O12$, meeting the second vertical in 2, and so on until B is reached.

311. **Given System of Forces Reduced to Two Forces through Two Given Fixed Points and Internal Forces.**—In the applications generally made of the methods of graphical statics the applied forces are given in magnitude and line of action. We have seen that given coplanar forces, if acting on a rigid body, are equivalent to two forces represented by Oa, bO of the force-polygon, which act along lines parallel to these directions from the extreme points of a funicular polygon. When the forces act on any connected system of particles, each force may be regarded as applied to some particle of the system. They may be replaced by two forces applied at two arbitrary points G, H of the system together with pairs of equal and opposite forces acting between G and H, and the points of application of the forces (Fig. 150). For take any force applied at a point A to the system. It can be resolved into two components P, Q acting at A, acting along AG, AH respectively. Introduce at G two opposite forces each equal to P in the line AG, and at H two opposite forces each equal to Q and acting in the line AH. Thus we represent the force at A by a force P at G, and a force Q at H, together with the pair consisting of P in the direction AG at A and P in the direction GA at G, and the pair consisting of Q at A acting from A to H and Q at H acting from H to A.

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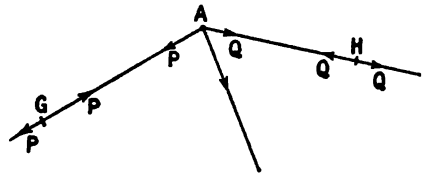
The same process can be followed for all the applied forces, and we obtain, therefore, a resultant force at G, and a resultant at H, which, with the pairs of internal forces between G and H and the

FIG. 150.



points of application of forces of the system, are equivalent to the given forces.

312. **System of Forces on Rigid Body Reduced to Two Forces through Two Given Points.**—If now the points of the system including G, H (Fig. 150) are rigidly connected the introduction of the pairs of equal and opposite forces is by the connections rendered unnecessary, as the two forces acting through G and H are in their action on the body equivalent to the given applied forces. These two forces must therefore with the given applied forces form a force-polygon, that is, if the applied forces be laid down in order as the sides of a polygon beginning with a and ending with b, it must be possible to find an origin O such that the lines Oa, bO represent the forces at G and H. The lines aO, Ob represent the equilibrant of the system.

The forces at G and H, which, with the pairs of internal forces specified, are equivalent to the given forces on the rigid system, are therefore the forces in the terminal sides of a funicular polygon of the given applied forces.

It is clear also that in the case of a rigid system, it is possible to draw a funicular polygon so that the terminal sides in which the pair of equivalent forces act may pass through two points G, H chosen arbitrarily in the body. Though only one pair of forces through G, H is obtainable by the process described above, it is clear that more than one funicular polygon can be drawn by which the system of forces is reduced to a pair of forces through G, H. For let the single resultant force (if one exist) of the system of forces be found from the pair of forces equivalent to the given system, and its direction be laid down in the funicular diagram. Two forces in lines drawn from any point in the resultant to G, H respectively can be made to give that resultant.

313. **Conditions of Equilibrium of Coplanar Forces.**—We may now consider the conditions of equilibrium of a system of coplanar applied forces acting on a rigid body.

The necessary and sufficient conditions of equilibrium are: (1) that the forces should have a zero resultant; (2) that the sum of their moments about any point in their plane should vanish.

The first condition is evidently fulfilled only when the force-

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polygon of the applied forces is closed, that is when the points a, b coincide: the second is fulfilled only when the funicular polygons are closed. [The first condition is sufficient by itself when the forces are all applied at one point.]

It will be sufficient to show that one funicular polygon is closed, for then it will follow that all are closed. Since a, b coincide in the force-polygon the two forces Oa, bO to which the given forces are equivalent are equal and opposite. They must also be in the same line, otherwise the sum of the moments of the system round every

FIG. 151.

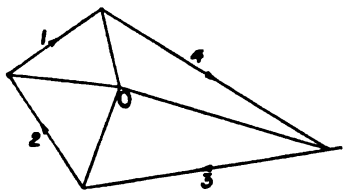
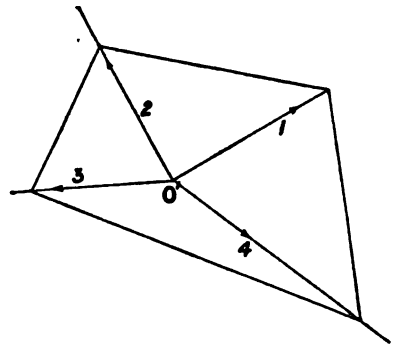


FIG. 152.



point in the plane would not be zero. But different pairs of these equivalent forces are given by the different positions of O, and each force of every pair must pass through both the terminal points A, B of the funicular polygon. Hence A, B coincide.

This is obviously true for every funicular polygon that can be drawn.

If the lines of action of the forces all pass through one point the polygon of applied forces is closed, and, as remarked above, that is the single condition necessary and sufficient for equilibrium. From this we obtain the following geometrical theorem. If the polygon of forces be drawn, and from any point O lines be drawn to the points of meeting 12, 23, 34, ... of the sides, and if then from a point chosen arbitrarily, in the line of action of, say, the force **1**, a line be drawn parallel to $O\overline{12}$ to meet the line of action of **2**, then from the latter point of meeting a line be drawn parallel to $O\overline{23}$ to meet the line of action of **3**, and so on, these lines form a closed polygon.

314. **Reciprocal Figures.**—In fact the following geometrical theorem holds for the two polygons. The force-polygon (Fig. 151) drawn in a plane has its angular points joined with any point O in the plane. Then a second closed polygon (Fig. 152) is drawn with its sides parallel to the lines drawn from the point O in the first, and lines are drawn through the angular points of the second polygon parallel to the sides of the first, and these lines meet

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in a point O' (Fig. 153). The polygons are said to be reciprocal figures and the points O, O' are called their poles.

Each polygon is a force-polygon for a system of equilibrating forces acting along the lines drawn from the pole of the other, and a funicular polygon for a system of forces acting along the lines through its own pole, and represented by the sides of the other polygon. Hence the names *reciprocal figures*. The discussion of systems of forces by means of reciprocal diagrams is due to Clerk Maxwell, in his Memoir on *Reciprocal Figures, Frames, and Diagrams of Forces.* (Trans. R.S.E., vol. xxvi., or Rep. of Papers, vol. ii. 161.)

315. **Given System of Coplanar Forces represented by Single Force.**—In order that the forces in their action on a rigid body may be equivalent to a single force it is only necessary that the force-polygon should not be closed. Then the system of forces is equivalent, as we have seen, to the two forces represented by Oa, bO of the force-polygon, and acting as shown in Fig. 145. Let 1, m, be the points of application of the terminal forces of the funicular polygon, and the two lines of action $A1$, Bm be produced to meet in C, then the two forces in these lines may be replaced by their resultant, a single force through C. The same reduction to a single force may be obtained by any funicular polygon.

If the two forces $A1$, Bm (Fig. 153), to which the system reduces in the first instance, are parallel the construction fails. This will be

FIG. 153.

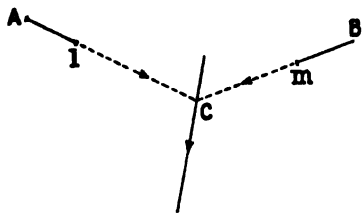
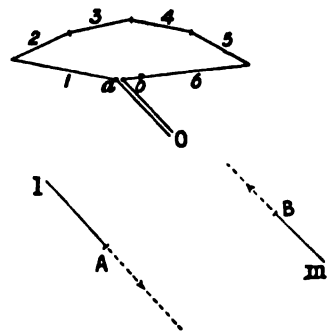


FIG. 154.



the case when the pole O of the force-polygon is in line with the terminal points a, b of that polygon. It is only necessary then to take another pole not so situated, and the reduction is given at once by the corresponding funicular polygon.

When the force-polygon is closed the single resultant is zero. If one of the funicular polygons is not closed, no funicular polygon is closed, and the sum of the moments of the forces round any point in the plane is not zero. The forces reduce to two equal and opposite forces represented by Oa, bO in the force-polygon (as shown in Fig. 154 by the two close parallel lines) or the parallel forces $A1$, Bm

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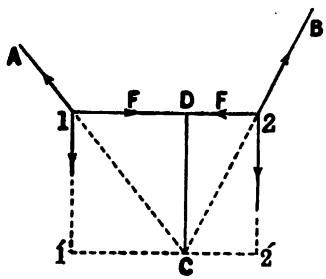
of the funicular polygon. Hence these forces form a couple, the sum of the moments of the forces of which round any point in their plane—that is, the moment of the couple—is equal to the product of either force by the distance between them. Since the force-polygon is closed this result is obtained for every position of O, and every corresponding funicular polygon. Different pairs of forces are obtained for the couple for the different positions O, but all have the same magnitude and direction of moment.

Any group of forces of the applied system may obviously have its resultant determined by treating it as a separate system by the methods above described.

316. **Case of Parallel Applied Forces.**—When the applied forces are parallel the force-polygon obviously reduces to a straight line, and the lines of different forces may in whole or in part coincide, though their directions may be opposed. For example, when the system has a zero resultant the polygon is still to be regarded as closed—that is, the terminal points a, b still coincide; the straight line represents the different parts of the polygon brought together in line.

FIG. 155.



If the funicular polygons are not closed the forces are reducible to a single force parallel to the given forces. For example, in the system of forces shown in Fig. 146 the force-polygon is represented by the line a, 1, 2, 3, 4, 5, b, in Fig. 147, and the resultant is represented by the line ba. To find its line of action it is only necessary to draw a single funicular polygon corresponding to any pole O of the force-polygon. Two forces in the directions $A\mathbf{1}$, $B\mathbf{5}$ (Fig. 146) meeting in C will be thus obtained which are equivalent to the given forces, and the line through C is the line of action of the resultant.

317. **Graphical Process for Resultant of Parallel Forces.**—From this we can prove the rule for finding the resultant of two parallel forces. For let $\mathbf{11}'$, $\mathbf{22}'$ (Fig. 155) be the two given forces, and draw the force-polygon. Then, choosing a pole, draw a funicular polygon as shown. Two forces in the directions $A\mathbf{1}$, $B\mathbf{2}$ are equivalent to the two given forces. These are respectively the resultants of $\mathbf{11}'$ and F and $\mathbf{22}'$ and F acting at $\mathbf{1}$ and $\mathbf{2}$, as shown in the diagram. Produce $A\mathbf{1}$ and $B\mathbf{2}$ to meet in C, and draw lines through C parallel to $\mathbf{12}$, and to the given forces respectively. Let the latter line meet $\mathbf{12}$ in D. Then in the triangle $\mathbf{1}DC$, $\mathbf{1}D$ represents F and DC represents the force $\mathbf{11}'$, and in the triangle $2DC$, $D\mathbf{2}$ and DC represent respectively F and $\mathbf{22}'$. Thus $\mathbf{1}D/DC = F/\mathbf{11}'$ and $D\mathbf{2}/DC = F/\mathbf{22}'$. Hence $\mathbf{1}D/D\mathbf{2} = \mathbf{22}'/\mathbf{11}'$, which is the well-known result for two parallel forces acting on a rigid body. Plainly all funicular polygons lead to the same result.

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Clearly from this it is possible to replace a single force by two forces in given lines parallel to the given force. For let the given force act at C and be represented in magnitude and direction by CE, and let A, B be points in the given lines. Through E draw a line parallel to AB meeting the given lines in F and G. Join BF, cutting CE in H. CH and HE represent the force which must act

FIG. 156.

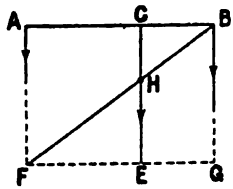
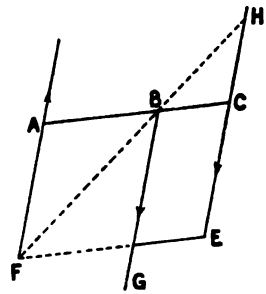


FIG. 156′.

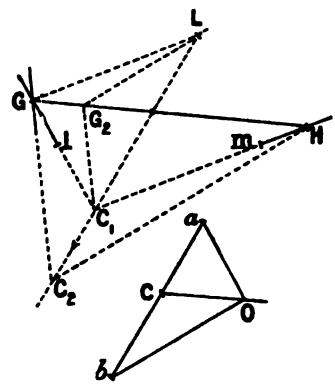


along AF and BG respectively. This follows at once from the similarity of the triangles CHB, FHE.

The diagrams (Figs. 156, 156′) show the possible cases, and the description applies to both.

318. **Locus of Pole of Funicular Polygons when Terminal Forces pass through Fixed Points.**—The position of O may be taken arbitrarily, and there is a funicular polygon for every chosen position. When, however, conditions are imposed on the terminal forces this choice of O is restricted. Let, for example, the terminal forces pass through two fixed points G, H.

FIG. 157.



It is clear that as the pole of the funicular polygons is displaced the point of meeting of any two of the sides of the polygon moves along a straight line. For in Fig. 157 let 1, m be points on two of the applied forces from which the sides 1A, mB of a funicular polygon are drawn; then, as we have seen, forces in these lines in the directions A1, Bm and of amount given by Oa, bO are equivalent to the applied forces included between the points 1, m. Hence the resultant acts at C_1 the point of meeting of these forces, and this point of meeting, as O is varied in position, must always lie on the line of the resultant through C.

We can now prove that if the terminal lines A1, Bm of the

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funicular polygon in which act the forces represented by Oa, bO of the force-polygon pass through two fixed points GH, the locus of the pole O is a straight line parallel to that joining the fixed points. The resultant of these forces is represented by the line ab, and the point of meeting of the two lines must lie on a line parallel to ab. The lines drawn from ab to the pole O must therefore be parallel to lines drawn from the fixed points to the different points of intersection C of $G1$, Hm on the line of the resultant.

Let C_1, C_2, C_3 be different points on this line, and join these with G and H. From G draw a line parallel to C_1H intersecting C_2C_1 in L, then GC_1, LG may be taken to represent the forces given by Oa, bO, and on the same scale LC_1 will represent ab the resultant. Now through C_1, L draw lines respectively parallel to C_2G, C_2H. These will meet in a point G_2 on GH. Similarly lines drawn through C_1, L parallel to C_3G, C_3H will meet on GH. The points G, G_2, ... represent the poles O of the force-polygon. The locus of O is a straight line parallel to GH.

Or suppose the terminal force $G1$ resolved into two components at G, one along GH, the other parallel to the single resultant, and resolve the other terminal force in the same manner. The two forces parallel to the resultant must together be precisely equal to the resultant, and the forces along GH must be equal and opposite, otherwise the resultant could not have the direction it has. Therefore, the former pair of components is always the same. Let ab be divided in C into two parts representing these components, then O must be so situated that the components of aO, Ob parallel to GH may be equal and opposite. Hence CO is parallel to GH.

Lines drawn through GH parallel to aO, Ob respectively are terminal sides of the funicular polygon, and the remaining sides are given by the other lines drawn from the pole O.

319. **Frames. Stiffness of Frames.**—The reader will now easily see how to apply the principles which have been explained to the solution of various problems regarding funicular polygons. For example, to draw a funicular polygon the terminal sides AA', BB' of which shall pass through two fixed points G, H, while any other side shall pass through a fixed point I, it is only necessary to find in the diagram of the force-polygon the locus of the poles of funicular polygons which pass through G, I and I, H. The point of intersection of these two loci is the pole of the funicular polygon required, which can then be constructed.

We have not space to deal with such problems in detail; we pass on to give one or two applications to structures of connected bars After these we can only take examples as they arise in the discussion of physical results; but many most interesting applications of graphical statics to continuous bodies will be afforded by the study of the elastic properties of bodies.

A *Frame* is an arrangement of bars connected where they meet by joints, round which each bar connected is otherwise free to turn. Since here we consider only plane structures, the axes round which

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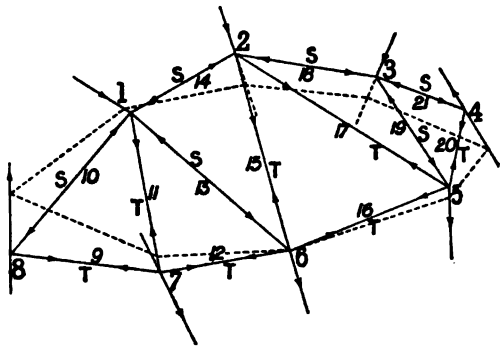
the bars turn at the joints are at right angles to the plane of the bars. The applied forces are to be regarded as applied to pins uniting the bars and forming the joints, and not to the ends of the bars. The only forces in the bars are then stresses along their lengths. In structures, however, the bars are often fixed at the ends, and this has an important effect on the strength of those bars

FIG. 158.



which act as struts, which will be dealt with in the chapter on Elasticity.

A frame is said to be stiff when no change of relative configuration of the bars is possible except that produced by extension or shortening of the bars. Consider any bar of such a frame. Forces are applied to it by the other bars at its extremities, and forces from without the frame may also be there applied to it by the pins of the joints. Thus, if the weight of the bar be neglected, the applied forces under which it is in equilibrium are two resultant forces applied at its ends. Since the forces internal to the bar form by themselves so far as the bar is concerned an equilibrating system, the forces applied to the ends must be in equilibrium. They must therefore be equal and opposite forces, and must act along the bar if it be straight.

If the forces on the ends be pulling forces, the bar is said to be under tension, if they are pushing forces the bar is said to be under thrust. In the former case, as already stated, it is called a tie, in the latter a strut.

In order that a closed frame in which the bars meet in n points may be stiff, it is necessary and sufficient that there should be $2n-3$ connecting bars.

320. **Application of Graphical Methods to find Stresses in Bars of Frame.**—Fig. 158 shows a frame formed of bars jointed at the points **1**, **2**, **3**, ..., **8**, with external forces applied to it at these points as shown by the arrows external to the frame. The system of external forces is in equilibrium and the force-polygon is shown

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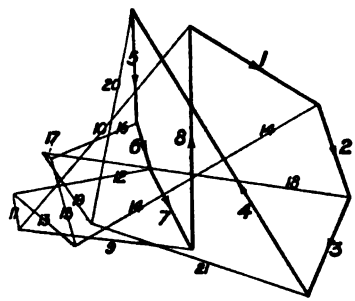
below in Fig. 159. *O* is the pole of a funicular polygon indicated by the dotted lines in Fig. 158. This polygon is of course also closed. The forces in the connecting bars are found for each point, 1, 2, 3, ... of meeting the force-polygon.

We begin at the point **8**, where only three forces act. The polygon for these is evidently the triangle the sides of which are the lines numbered 8, 9, 10 in Fig. 159. This order gives also the direction of the forces at **8**.

FIG. 159.



Then we pass to **7**, where four forces act; there the polygon of forces is a quadrilateral of which the two sides 7 and 9 are known, since the force applied by the bar **9** is of the same amount but opposite in direction according as it acts at **7** or **8**.

The other two sides are parallel to the bars **11** and **12**. The four forces are thus in the order indicating their direction, 7, 9, 11, 12, at **7**.

Next we take the forces at **1**. There the polygon is a pentagon of which three sides are known, those given by the applied force at 1,

FIG. 160.

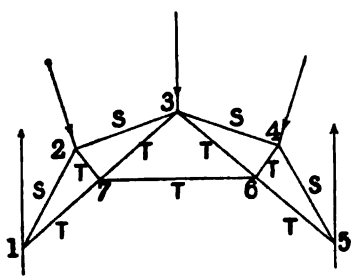
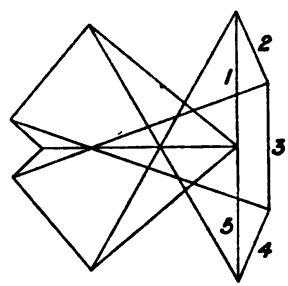


FIG. 161.



and the forces applied at that point by the bars 10 and 11. These latter forces have at **1** the opposite directions to those applied by the same bars at the points 8 and 7 respectively. Thus the polygon is to be completed by sides parallel to 13 and 14, and taken so that the pentagon is closed by the sides in the order 1, 14, 13, 11, 10, which gives the direction of the forces at **1**. The sides 10 and 13 cross.

Similarly, the forces applied by the other bars are obtained. The nature of the stress in each bar is shown in Fig. 158, by the letters *S*, *T*. Thus 10, 14, 18, 21, with the cross-bars 13 and 19, are in

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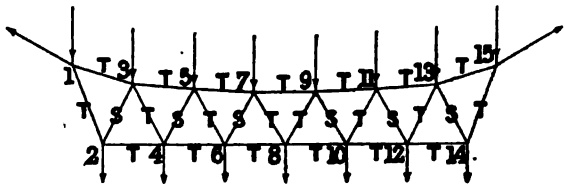
compression and act as struts, the bars 9, 12, 16, 20, with the cross-bars 11, 15, 17 are in tension, that is act as ties.

321. **Example. Roof-Girder.**—Another example, showing a roof-girder, is given in Figs. 160, 161. The reader should go through the exercise of finding the force-polygons for the points of meeting,

FIG. 162.

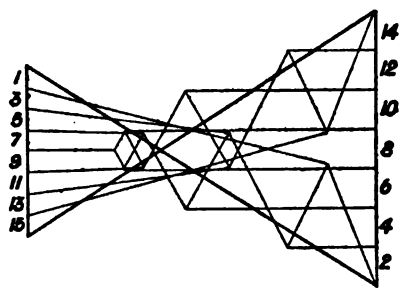


1, 2, 3, 4, 5, 6, 7, observing that there are no applied forces at the two latter points. It will be verified that the bars **12, 23, 34, 45** are struts, and the remaining bars, **17, 27, 73, 36, 46, 65, 67,** are ties. Each bar is marked in the diagram with *S* or *T*, according as it is a strut or a tie. The force-polygons are exhibited in Fig. 161.

322. **Example. Bridge-Girder.**—Fig. 162 shows a bridge-girder with two sets of vertical applied forces. The polygon of forces consists of two sets of parallel forces arranged as indicated by the numbers. The bars in the frame are marked *T* or *S*, according as they are ties or struts.

The diagrams show the forces acting in bars which occur in any section of the girder, and of course a girder may be regarded as cut

FIG. 163.



across, provided forces are supposed to be applied to maintain equilibrium. These forces are those applied to the sections of the bars cut across by the parts removed.

When the frame contains bars in excess of those required for stiffness, the determination of stresses becomes more troublesome, and we do not here enter into the question. The student may seek further information in practical works on Applied Mechanics and Mechanical Drawing, or in the treatises on Graphical Statics of Cremona, Maurice Lévy, and Culmann.

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CHAPTER VIII.

EQUILIBRIUM AND MOTION OF A CHAIN OR FLEXIBLE CORD.

323. **Ideal Flexible Chain.**—The lengths of flexible comparatively massless cord in the funicular polygon suggest an ideal, flexible, and massless connection between particles. We shall consider the equilibrium of a system of small massive particles thus connected, first when the particles are under the action of gravity, then for any field of force whatever. By increasing the number of particles and diminishing the distance between them, we can arrive at an exact idea of the behaviour of a flexible cord or chain of sensible mass.

FIG. 164.

A chain of great flexibility and considerable mass per unit length is made for certain purposes not requiring great strength by connecting small hollow spherical pellets of metal by short links of thin wire, as shown in Fig. 164. Each link is of a kind of dumb-bell shape formed by two small knobs at its ends and the connecting piece of wire. The adjoining links of each pellet are secured by these knobs, which have been passed at the ends of a diameter of the pellet through holes the edges of which have then been pressed down round the wire. This is an approximate realisation of the ideal chain referred to above. The alternation of perfectly flexible links with loading spherules will help to make the dynamics of a chain clearer, while leaving the results applicable to continuous strings and cords which are capable of being regarded as flexible.

324. **Chain under Gravity. The Catenary.**—Let the particles under the action of gravity be n in number, of equal gravity w, and separated by successive flexible links each of length a. Then instead of (17) of § 301 we have

$$w = \tau(\tan\theta_2 - \tan\theta_1) = \tau(\tan\theta_3 - \tan\theta_2) = \ldots = \tau(\tan\theta_{n+1} - \tan\theta_n).$$

Now consider the curve touching each link at its middle point, and let there be a horizontal link to coincide with the horizontal tangent to the curve. The equations just written express that the

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tangent of the inclination of the curve to the horizontal varies directly as the distance of each point of contact measured along the polygon from any chosen point of contact at which $\tan\theta = 0$. If a_T/w be denoted by c, and distance along the polygon from the middle of the horizontal link (evidently the lowest point of the curve) be denoted by s, the property just stated may be expressed by

$$\tan\theta = \frac{s}{c} \tag{1}$$

Let there be a very large number of particles at very small distances apart loading the chain, then the polygon approximately coincides with the curve, and can be made to do so without limit of closeness, by taking the distance between each pair of successive particles infinitely short. The distance s along the polygon thus agrees in the limit with the distance along the curve between the two points, and (1) may be applied to every point of the curve. It may now be written in the form

$$\frac{dy}{dx} = \frac{s}{c} \tag{2}$$

where dy/dx is the rate of increase of y with increase of horizontal distance x from the lowest point.

Now let δs be an element of length of the curve thus obtained, and let δx, δy be corresponding changes in x and y. We have

$$\delta s^2 = \delta x^2 + \delta y^2 \text{ or } (\delta y/\delta s)^2 = \delta y^2/(\delta x^2 + \delta y^2) = (\delta y/\delta x)^2/\{1 + (\delta y/\delta x)^2\}.$$

In the limit when δx, δy are made infinitely small this becomes

$$\left.\frac{dy}{ds} = \frac{\dfrac{dy}{dx}}{\sqrt{\left\{1 + \left(\dfrac{dy}{dx}\right)^2\right\}}}\right\} \tag{3}$$

Hence by (2)

$$\frac{dy}{ds} = \frac{s}{\sqrt{(s^2 + c^2)}} \tag{4}$$

In the same way we obtain

$$\frac{dx}{ds} = \frac{c}{\sqrt{(s^2 + c^2)}}. \tag{5}$$

In these expressions we suppose the positive sign always given to the radical.

It may be verified by the reader that the last equation is derivable from the integral relation,

$$x = c\log\{s + \sqrt{(s^2 + c^2)}\} + C, \tag{6}$$

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where C is a constant. Since s is measured from the lowest point of the curve, and we take $x=o$ when $s=o$, we have $C=-c\log c$. Thus

$$x=c\log\frac{s+\sqrt{(s^2+c^2)}}{c}.$$

We may write the last equation in the form

$$e^{\frac{x}{c}}=\frac{s+\sqrt{(s^2+c^2)}}{c},$$

where e is the base of the Napierian system of logarithms. If we multiply both numerator and denominator of the quantity on the right by $-s+\sqrt{(s^2+c^2)}$ and take the reciprocals of both sides we obtain

$$e^{-\frac{x}{c}}=\frac{-s+\sqrt{(s^2+c^2)}}{c};$$

so that

$$s=\frac{c}{2}\left(e^{\frac{x}{c}}-e^{-\frac{x}{c}}\right). \tag{7}$$

This is the equation in terms of s and x of the curve in which the particles hang. It is known as the *catenary* since it is the curve in which a uniform flexible chain fixed at its two ends always hangs under gravity. To find the equation in terms of x and y we have, since $dy/dx=s/c$,

$$\frac{dy}{dx}=\frac{1}{2}\left(e^{\frac{x}{c}}-e^{-\frac{x}{c}}\right), \tag{8}$$

and therefore

$$y=\frac{c}{2}\left(e^{\frac{x}{c}}+e^{-\frac{x}{c}}\right)+C', \tag{9}$$

where C' is a constant. If the origin be taken at the lowest point of the curve $C'=-c$, and the equation is

$$y+c=\frac{c}{2}\left(e^{\frac{x}{c}}+e^{-\frac{x}{c}}\right). \tag{10}$$

This is slightly simplified if the origin be taken at a point distant c vertically below the lowest point of the curve. Then

$$y=\frac{c}{2}\left(e^{\frac{x}{c}}+e^{-\frac{x}{c}}\right). \tag{11}$$

A line through this origin parallel to the tangent at the lowest point will in what follows be taken as the axis of abscissæ, and the equation of the curve will be (11).

Since (7) gives $y^2=s^2+c^2$, equations (4) and (5) can be written

$$\frac{dy}{ds}=\frac{s}{y},\quad \frac{dx}{ds}=\frac{c}{y} \tag{12}$$

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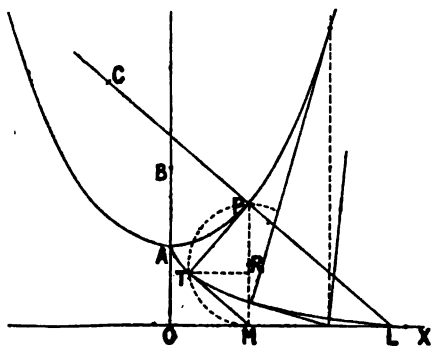
These results are shown in Fig. 165. A tangent is drawn to the catenary at any point P, and a length $PT = s$ is taken from P along the tangent. The point T is a point in that involute of the catenary which starts from the lowest point. From P draw the ordinate of the curve meeting the axis OX in M. From T draw a perpendicular to the ordinate meeting it in R. Then

$$dy/ds = \sin PTR = \cos TPR.$$

But this is s/y, that is PT/PM. Hence PTM is a right angle, and the tangent at T to the involute meets the axis OX in the foot of the ordinate drawn to the point of contact P.

Again, $dx/ds = \cos PTR = \sin TPR = TM/PM = TM/y$, since PTM

FIG. 165.



is a right angle. But $dx/ds = c/y$, hence $TM = c$, that is, the point of contact T of a tangent to the involute starting from the lowest point of a catenary is at a constant distance from the point of intersection of the tangent with a fixed straight line in the plane of the curve. The curve possessing this property is called the *tractory*.

We have thus a simple rule for drawing a normal and a tangent at any point P of a catenary, and at the same time finding the corresponding point on the involute already referred to. Draw the ordinate MP to the point P, and on this line as diameter describe a circle. In the circle lay off from M a line MT of length c, and draw the line PT. PT touches the catenary at P, and a line PN perpendicular to P is a normal. T is the corresponding point on the involute. This gives of course a mode of describing the tractory from the corresponding catenary.

A catenary photographed from one actually formed by the flexible chain described in § 323 is shown in Fig. 166. There is, of course, a little distortion where the chain passed over the two supports.

325. **Geometrical Description of Catenary.**—The catenary can be described by the method of successive small arcs drawn from the centres of curvature as explained in § 105. The radius of curvature

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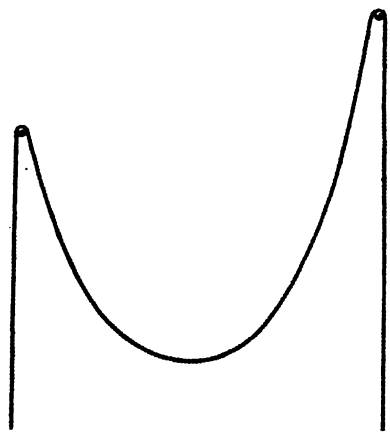
can be found as follows. Since $\tan\theta = s/c$, $(1+\tan^2\theta)d\theta = ds/c$, and therefore if R be the radius of curvature,

$$R = \frac{ds}{d\theta} = c(1+\tan^2\theta) = \frac{y^2}{c}.$$

Hence to find the centre of curvature for any point P of the catenary we describe a circle touching PM at P and passing through the point T. If then MT be produced to meet the circle in another point U the line MU will give the length R of the radius of curvature. If this length is laid off from P along the normal it will give as its other extremity the centre C of curvature.

FIG. 166.



Or still more simply thus: Produce the normal through P backwards to meet the axis OX in L, LP is equal to R, which can then be laid off from P to C. For the two triangles PTM, LMP are similar and $MT = c$. Hence $LP/y = y/c$ or $LP = y^2/c$.

For example, to draw a catenary for which the value of c and the lowest point A are given we may proceed as follows. From the value of c find the origin O. Through O draw OA (Fig. 167), and produce it so that $AB = OA$. From B as centre draw a short circular arc through A. Join B to one extremity A_1 of this and produce BA_1 to meet the axis OX in M_1. Then on A_1B take a point B_1 such A_1B_1 is equal to M_1A_1. From this point as centre draw a second arc continuing the former short arc beyond A_1 to A_2. Join B_1A_2 and produce to M_2. Repeat the process just described, and so on, till the catenary has been drawn as far as may be desired.

The curve may of course also be constructed by calculating for given abscissæ the corresponding ordinates, and laying the curve down on squared paper in the usual way.

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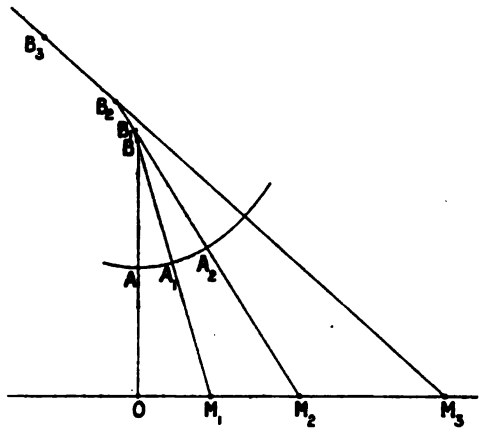
326. **Stretching Force in Catenary.**—The stretching force, T, in the chain at any point is easily found. The horizontal component of T is τ and therefore $T = \tau ds/dx$, that is by (12) and the value $w/a\tau$ of $1/c$.

$$T = \tau\frac{y}{c} = \frac{w}{a}y. \tag{13}$$

Clearly w, the gravity of a particle, is the action of gravity on the matter corresponding to the length a of the chain. When the number of particles is made very great a becomes correspondingly

FIG. 167.



small, and then we call w/a the force of gravity on the chain (now regarded as continuous) per unit of length. The stretching force at any point is thus equal to the downward force of gravity on a portion of the chain of length equal to the ordinate of the point as given by (11).

327. **General Theory of the Equilibrium of a Flexible String or Chain.**—The conception of a chain as a succession of small massive particles connected by short flexible links enables the general conditions of equilibrium to be at once written down. Let the chain be placed in a field of force in virtue of which there is exerted on a particle of mass μ situated at the point P (x, y, z), a force the co-ordinates of which are $\mu X, \mu Y, \mu Z$. Let s denote a distance along the chain from any fixed point P, and let δs denote the distance between the particle at P and the particle next it on either side. Also let T denote the tensile (stretching) force in the link before P, and $T + \delta T$ that in the link after P, that is before and after as marked by a point traversing the chain in the sense of s increasing. We easily arrive at the following results :

a. The component of the applied forces X, Y, Z, on the particle in

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the direction of either link must balance the difference of pulls dT between the two links. That is, if S be the component in question

$$mS + dT = 0. \tag{14}$$

b. If this component S is zero, the stretching forces are the same in every link. The applied forces on the particle of mass m bisect the angle (infinitely nearly 180° for a chain of infinitely short links) between the links on the two sides of the particle; in other words the applied forces are everywhere normal to the chain. It is evident that the pulls in the links can furnish no component of force at right angles to the plane defined by two successive links. Hence the resultant of the applied forces on each particle lies in the plane of the pair of links connected with it, or, as it is generally expressed, there is no component of applied force at any point perpendicular to the osculating plane of the curve of the chain at that point.

c. Also it can be proved, as at § 205, that the normal component N of the applied forces on each particle is equal to the product of the stretching force T in either link at the particle into the curvature of the chain at the particle, that is, if R be the radius of curvature and the direction of N be taken as positive when across the curve from the convex to the concave side,

$$-N = \frac{T}{R}. \tag{15}$$

328. **Analytical Conditions of Equilibrium.**—These results can be expressed more analytically as follows. Take rectangular axes of x, y, z. The direction-cosines of the link on the negative side of the particle at P are those of the tangent to the curve along which s is taken, and are dx/ds, dy/ds, dz/ds. Those of the next link (that on the positive side) are $dx/ds + \delta(dx/ds)$, $dy/ds + \delta(dy/ds)$, $dz/ds + \delta(dz/ds)$. The components of T along the axes of x, y, z are therefore Tdx/ds, Tdy/ds, Tdz/ds, and of $T + dT$ are $(T + dT)\{dx/ds + \delta(dx/ds)\}$, &c. Hence for the equilibrium of the particle at P we have for the direction parallel to the axis of x

$$\mu X + (T + \delta T)\frac{dx}{ds} + \delta\frac{dx}{ds} - T\frac{dx}{ds} = 0,$$

or, if σ denote the mass of the chain per unit of length, $\mu = \sigma ds$, and the equation just written becomes

$$\left.\begin{aligned} &\frac{d}{ds}\left(T\frac{dx}{ds}\right) + \sigma X = 0; \\ \text{similarly,}\quad &\frac{d}{ds}\left(T\frac{dy}{ds}\right) + \sigma Y = 0; \\ &\frac{d}{ds}\left(T\frac{dz}{ds}\right) + \sigma Z = 0. \end{aligned}\right\} \tag{16}$$

These three equations are really all expressed in (14) and (15)

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above, which can be deduced from them as follows. Taking the axis of x along the curve at the particle, that is, taking $dx/ds = 1$, $dy/ds = dz/ds = 0$, we have $X = S$ and $d^2x/ds^2 = 0$, and obtain

$$\frac{dT}{ds} + \sigma S = 0. \tag{17}$$

The reader may verify that the same result is got by multiplying the first of (15) by dx/ds, the second by dy/ds, and the third by dz/ds, adding the products, and observing that since $(dx/ds)^2 + (dy/ds)^2 + (dz/ds)^2 = 1$, $dx/ds \,.\, \delta(dx/ds) + \ldots = 0$.

When the forces X, Y, Z are derivable from a potential V so that $X = -\partial V/\partial x$, $Y = -\partial V/\partial y$, $Z = -\partial V/\partial z$, the equations of equilibrium have the form

$$\frac{d}{ds}\left(T\frac{dx}{ds}\right) - \sigma\frac{\partial V}{\partial x} = 0, \text{ \&c.} \tag{18}$$

Equation (16) becomes then also

$$\frac{dT}{ds} - \sigma\frac{dV}{ds} = 0. \tag{19}$$

It is shown at § 55 that if ρ be the step from the origin to an element of a curve, $d\rho/ds$ is a unit step along the tangent, while $Rd^2\rho/ds^2$ (R being the radius of curvature) is a unit step at right angles to the tangent and in the plane of two consecutive elements of the curve. Thus if dx, dy, dz be the projections of ds on the axes and i, j, k be unit steps parallel to the axes,

$$d\rho = i\,dx + j\,dy + k\,dz,$$

and

$$\frac{d\rho}{ds} = i\frac{dx}{ds} + j\frac{dy}{ds} + k\frac{dz}{ds}.$$

Hence also

$$\frac{d^2\rho}{ds^2} = i\frac{d^2x}{ds^2} + j\frac{d^2y}{ds^2} + k\frac{d^2z}{ds^2}.$$

Thus $d^2\rho/ds^2$ is a step along the radius of curvature of which the projections on the axes of x, y, z are respectively d^2x/ds^2, d^2y/ds^2, d^2z/ds^2. The components of any step along the radius of curvature are proportional to these quantities.

The direction-cosines of the radius of curvature are thus

$$\left(\frac{d^2x}{ds^2}, \frac{d^2y}{ds^2}, \frac{d^2z}{ds^2}\right) \Big/ \sqrt{\left\{\left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2z}{ds^2}\right)^2\right\}},$$

and since $Rd^2\rho/ds^2$ is a step of unit length, we have

$$\frac{1}{R^2} = \left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2z}{ds^2}\right)^2 \tag{20}$$

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Denote the direction-cosines just found for the normal by λ, μ, ν for brevity, multiply the first of (16) by λ, the second by μ, the third by ν, and add. Since $\lambda dx/ds + \mu dz/ds + \nu dz/ds$ is zero, being the cosine of 90°, we obtain

$$\frac{T}{R} + \sigma(\lambda X + \mu Y + \nu Z) = 0. \tag{21}$$

The expression $(\lambda X + \mu Y + \nu Z)$ is what we have denoted by N in (15), which is therefore identical with (21). Hence statement (*c*) is proved.

The reader may easily construct, if necessary, a formal analytical proof that the force normal to the osculating plane is zero. This follows from the fact that the direction-cosines of the binormal (that is, a normal to the osculating plane at the point considered) are $\nu dy/ds - \mu dz/ds$, &c.

He may also as an exercise find the equation of the common catenary from (16) by putting $X=0$, $Y=0$, $Z=g$; and as another find the equation of the curve in which the chain would hang if its mass per unit length at each point were proportional to the tensile stress there existing. The latter curve is called the *catenary of uniform strength*. The equation connecting ρ the radius of curvature at any point in this catenary, and the distance s of the point from the lowest point is

$$\rho = \frac{a}{2}\left(e^{\frac{s}{c}} + e^{-\frac{s}{c}}\right).$$

329. **Analogy between Equilibrium of String and Curvilinear Motion of a Particle.**—The equilibrium of a chain bears a remarkable analogy to that of the motion of a particle along a curve under applied forces, and from solutions of problems of the one kind, those of problems of the other kind can be inferred. To see the nature of this analogy, imagine a particle to move along the curve of equilibrium of the chain in such a manner that its velocity at each point is numerically equal to T, the stretching force in the chain at the point. Then we get for (17) $d\dot{s}/ds + \sigma S = 0$, or $\dot{s}d\dot{s}/ds + \sigma ST = 0$, that is since $\dot{s}d\dot{s}/ds = \ddot{s}$

$$\ddot{s} + \sigma ST = 0, \tag{22}$$

the equation giving the acceleration of a particle of unit mass under a tangential force of amount $-\sigma ST$ in the direction of motion.

Again (21) clearly gives in like manner

$$\frac{\dot{s}^2}{R} = -\sigma NT, \tag{23}$$

for the acceleration at right angles to the tangent. The specified motion of the particle is thus given by the applied tangential and normal forces S, N per unit length of the chain, each multiplied by T.

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The same result is, of course, at once given by equations (16), which become

$$\ddot{x}+\sigma TX=0, \quad \ddot{y}+\sigma TY=0, \quad \ddot{z}+\sigma TZ=0. \tag{24}$$

Each of these is of the same form as (22), but the three together give also (23).

For example, a particle falling along any smooth curve in a vertical plane under the action of gravity has at each point velocity proportional to the square root of the depth of the point considered below a certain level. In the particular case of a free particle which has a horizontal component of velocity, the curve is a parabola with its axis vertical and its vertex upward. The velocity at any point is that which would have been acquired by a particle in falling freely from the directrix (§ 50) of the curve.

The form of a chain hanging under the action of gravity will, therefore, be a parabola with its axis vertical and vertex downward if the horizontal component of stretching force represented by $\dot{x}$ be constant. This will be the case if $X=0$, and $\ddot{y}=g$. But in the circumstances supposed $Y=g$, so that $\sigma T=1$. But $\dot{s}=T=\sqrt{2gy}$ where y is the height of the point at which T is taken above the directrix of the parabola. Hence $\sigma=1/\sqrt{2gy}$, that is the mass of the chain per unit of length at different cross-sections is inversely proportional to the square root of the height of the point above the directrix.

The equation of the parabola referred to the directrix and the axis as axes of x and y is $x^2=4a(y-a)$, where a is a constant. Thus we find for the length of an element ds the value $dx\sqrt{1+x^2/4a^2}$ $=dx\sqrt{y}/a$. The weight of the element is proportional to $ds/\sqrt{y}$, and is therefore proportional to dx. Thus the weight of any arc of the parabola is proportional to the horizontal projection of the arc.

The analogy just discussed is given in Thomson and Tait's *Natural Philosophy*, §§ 581, 582, to which the reader may refer for further examples.

The theory of a string in the field of a *central force* is very interesting, but we have not here space to deal with it.

330. **Motion of an Inextensible Chain. Condition of Inextensibility.**—We now consider the motion of an inextensible string or chain, leaving questions regarding elastic strings, so far as it is necessary to deal with them apart from the physical theories in which they arise, to the chapter on Elasticity. First of all it is clear that a condition is imposed on the variation of velocity along the chain by the fact that no element can alter in length. Let $\dot{x}$, $\dot{y}$, $\dot{z}$ be the component velocities at one point P of the chain, and $\dot{x}+d\dot{x}$, $\dot{y}+d\dot{y}$, $\dot{z}+d\dot{z}$ the component velocities at another point Q, distant ds along the chain from the former. Then the tangential velocities at the two ends of ds are

$$\dot{x}\frac{dx}{ds}+\dot{y}\frac{dy}{ds}+\dot{z}\frac{dz}{ds},$$

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$$(\dot{x}+d\dot{x})\left(\frac{dx}{ds}+d\frac{dx}{ds}\right)+(\dot{y}+d\dot{y})\left(\frac{dy}{ds}+d\frac{dy}{ds}\right)+(\dot{z}+d\dot{z})\left(\frac{dz}{ds}+d\frac{dz}{ds}\right).$$

If the length ds is to remain unchanged these velocities must be the same. Hence neglecting small quantities of the second order—namely, the products $d\dot{x}d(dx/ds)$, &c.—we obtain the necessary condition.

$$\frac{dx}{ds}\frac{d\dot{x}}{ds}+\frac{dy}{ds}\frac{d\dot{y}}{ds}+\frac{dz}{ds}\frac{d\dot{z}}{ds}=0. \tag{25}$$

This may also be obtained at once by differentiating with respect to t the equation $(dx/ds)^2+(dy/ds)^2+(dz/ds)^2=1$, *on the supposition that ds is constant.* The same equation differentiated with respect to t without this supposition gives the rate of stretching of the cord, which it will be found is represented by the expression on the left of (25).

This equation is sometimes called the *equation of continuity* of an inextensible chain.

331. **Equations of Motion.**—To obtain the equations of motion we have only to express the fact that the component forces on an element of the chain as reckoned in (16) above are no longer zero, but are equal to the mass-accelerations of the element in the direction of the axes. Taking as in (328) axes of x, y, z, fixed in space, and putting u, v, w for the velocities at any instant of the element in the directions parallel to these axes, we have for the equations of motion

$$\left.\begin{aligned}\sigma\dot{u}&=\frac{d}{ds}\left(T\frac{dx}{ds}\right)+\sigma X\\ \sigma\dot{v}&=\frac{d}{ds}\left(T\frac{dy}{ds}\right)+\sigma Y\\ \sigma\dot{w}&=\frac{d}{ds}\left(T\frac{dz}{ds}\right)+\sigma Z.\end{aligned}\right\} \tag{26}$$

These, with the kinematical equation (25), are sufficient to enable the co-ordinates x, y, z of an element of the chain, and the stretching force T, to be expressed in terms of s and t.

332. **Tangential and Normal Resolution. Conditions of Inextensibility.**—When the motion of the chain is in one plane it is convenient to refer the motion to moving axes consisting of the tangent and normal to an element of the chain. The appropriate kinematical condition, or rather, conditions, may be obtained from (25) or investigated separately. We shall choose the latter course.

Let u, v now denote the component velocities along the tangent in the direction of s increasing, and towards the centre of curvature respectively. These are not to be confounded with the u and v used above, which are velocities relatively to fixed axes of x and y In a short interval of time, dt, the end P (Fig. 168) will have moved a distance

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udt along the tangent at P and a distance vdt along the normal there; in the same interval Q has moved a distance $(u+du)dt$ along the tangent at Q and a distance $(v+dv)dt$ along the normal at the same point. Apply to both points displacements udt, vdt equal and opposite to the displacements of P. P will be brought back to its original position; Q will have then been displaced relatively to P through the two component displacements,

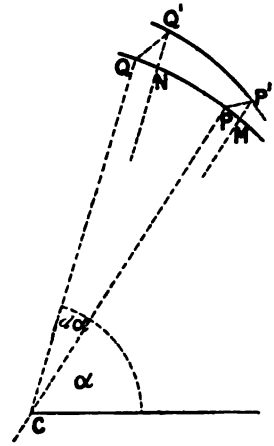
$$\{(u+du)\cos da - u - (v+dv\sin da\}dt$$

parallel to the tangent at P, and

$$\{(v+dv)\cos da - v + (u+du)\sin da\}dt$$

parallel to the normal at P. This can be easily verified from Fig. 168.

FIG. 168.



If ds is inextensible these displacements cannot have altered the length of ds. If, therefore, $d\phi$ be the angle between the directions of the element at the beginning and end of dt, we have, supposing da very small, so that $\cos da = 1$, and $\sin da = da$

$$\frac{1}{2}d\phi = \frac{u+du-u-(v+dv)da}{v+dv-v+(u+du)da},$$

and

$$d\phi = \frac{v+dv-v+(u+du)da}{ds}dt.$$

By making dt very small we can make $d\phi$ as small as we please. Hence in the limit the first of these equations gives $u - vda = 0$, or since $da = ds/R$

$$\frac{du}{ds} - \frac{v}{R} = 0. \tag{27}$$

The second equation gives in the limit also

$$\frac{dv}{ds} + \frac{u}{R} = \omega, \tag{28}$$

where $\omega(=d\phi/dt)$ is the angular velocity of the element in the plane of motion.

Since u, v are velocities referred to moving axes, the accelerations of the element are to be calculated as explained in § 272 above. Thus the accelerations along fixed axes coinciding with the positions of the tangent and normal at the instant under consideration are

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respectively $\dot{u} - \omega v$, $\dot{v} + \omega u$. The equations of motion are thus by § 273 above

$$\left.\begin{aligned} \sigma(\dot{u} - \omega v) &= \sigma S + \frac{dT}{ds}; \\ \sigma(\dot{v} + \omega u) &= \sigma N + \frac{T}{R}. \end{aligned}\right\} \quad (29)$$

It will be remembered that N as defined in § 327 is the applied force along the normal towards the centre of curvature. The value of ω is $dv/ds + u/R$, as given in (28).

333. **Steady Motion of Chain. Stiffness due to Motion. Persistence of Indentation.**—From these equations we can at once draw an important conclusion regarding the steady motion of a chain. Let the chain be of uniform mass per unit length, and move uniformly without change of figure or change of position of its curve in space. Then $\dot{u} = \dot{v} = 0$, and $\dot{v} = 0$. Also by (28) $\omega/u = 1/R$. Hence the equations of motion become

$$\left.\begin{aligned} \sigma S + \frac{dT}{ds} &= 0; \\ -\sigma N + \frac{T - \sigma u^2}{R} &= 0. \end{aligned}\right\} \quad (30)$$

Thus the curve in which the chain moves is the equilibrium curve of the chain at rest, and the stretching force T is $\sigma(RN + u^2)$, that is T exceeds the value for stationary equilibrium by σu^2. When the chain is driven very rapidly the value of T becomes very great in comparison with the applied forces, and consequently these have only a comparatively slight effect in controlling the figure of the chain. Thus let a chain be hung over a pulley in a vertical plane, and let it be put into rapid motion by rotating the pulley. If then it be struck a blow with a hammer, the form of the chain will be altered at the part struck, and the hollow made by the blow will remain at the same position in space and gradually disappear under the action of gravity. The chain, in fact, when running yields to the blow like a bar of lead. This *quasi*-rigidity or stiffness conferred on the chain by its motion is very suggestive as to the possibility of explaining the mechanical properties of solid substances as the result of the inertia of a system of moving particles, or even by the inertia of a moving fluid (see § 336).

334. **Experimental Illustrations of Stiffness due to Motion.**—The experiment of the moving chain is illustrated by Fig. 169, which shows the chain (1) revolving in its figure of equilibrium, (2) as disturbed by a blow. It is well in making the experiment to use a somewhat large and deep grooved pulley, and a fairly massive chain. The kind of dimensions are shown by Fig. 168; if the pulley is taken as a foot or more in diameter it will serve very well. A curb chain such as might be used as a light trace-chain, say weighing about

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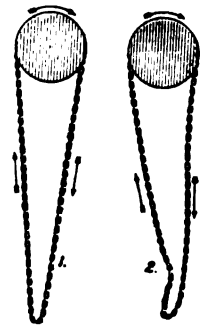
three or four ounces per foot, serves very well for an experiment on the scale here indicated. The pulley should be well secured: if the room is not too lofty the frame of the pulley may be bolted to the joists of the ceiling and driven by a cord from a driving apparatus, and the chain made long enough to be reached easily from below.

FIG. 169.



When striking the chain with the hammer, it is well, in order to avoid throwing the chain off, to direct the blow horizontally in the plane in which the chain is moving. Fig. 169 is taken from an address on *The Theory of Stream-Lines in Relation to the Resistance of Ships*, by the late Mr. William Froude (*Proc. B. A.*, 1875, *Nature*, Nov. 18, 1875).

Experiments on a smaller scale with lighter chains may be made with a vertical pulley attached to a whirling table. The pulley should overhang so that the chain may be jerked off the pulley while still spinning. It is then possible to illustrate the apparent or virtual solidity given to the flexible chain by the motion in the following manner. The pulley (not deeply grooved) is, say, about six or eight inches in diameter, and a closed chain a few inches longer than the circumference of the pulley is hung on it so that there is a part below the pulley which hangs in a nearly circular figure, really in a catenary. The chain is set into rapid motion in this figure by rapidly rotating the pulley, and is then jerked off, by a slight sidewise impulse, so that it is received by a horizontal board placed just under and just a very little below the lowest point of the chain. The chain continues to revolve after leaving the pulley, and preserves its shape while it rolls like a hoop for some little distance along the board.

335. **Chain in Steady Motion moves in Equilibrium Figure. Velocity of Pulse of any Form.**—The dynamical explanation of the result, as stated above, lies in the increased stretching force in the chain, which enables the applied forces to be neglected. If there are no applied forces any form of the chain is an equilibrium curve, and uniform motion of the chain along the curve, whatever it may be, causes no distortion. The value of T is then σu^2 by (30).

To make the matter clearer, consider a uniform chain moving in a plane through a smooth tube of any form under no applied forces except those due to the reaction of the tube. This reaction is everywhere normal to the direction in which the chain is travelling relatively to the tube, and cannot affect the stretching force in the chain, which is therefore the same at every point, however the curvature may vary, and is equal to the pull applied at and in the direction of its ends if these are free. Let now its value T be equal to σu^2. The force σN applied to the chain per unit of length will by (30) be zero, and there is no action whatever between the tube and chain. The velocity of the chain is given by the equation

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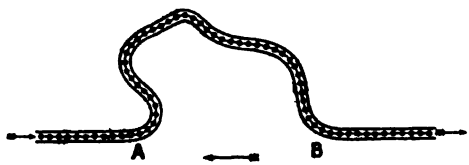
$$u = \sqrt{\frac{T}{\sigma}}. \tag{31}$$

Let the ends of the tube be straight and in line as shown in Fig. 170. The chain enters at A and emerges at B with velocity u. Now impose on the whole system of tube and chain a uniform volocity u in the direction opposite to that in which the ends are moving. This will not disturb the relative motion, but will bring

FIG. 170.



The chain in tube is a series of small spheres at equal distances apart, connected by fine threads.

the straight ends of the chain to rest and produce a motion of the shape AB of the chain with velocity u towards the left. The mutual action between the tube and chain remains zero, and therefore the tube may be removed. Hence we get for the velocity of propagation of a bend of any shape along a chain or flexible cord under a stretching force T, the value $\sqrt{T/\sigma}$. This is the result arrived at in the ordinary theory of the propagation of waves of transverse displacement along cords. In that theory, however, the motions are generally supposed to be small: here no limit is imposed on the size or shape of the bend in the chain. This simple proof of the formula for the velocity of a transverse wave along a cord seems to have been first given by Thomson and Tait.*

336. **Quasi-Rigidity of Tube produced by Fluid flowing in it.**—The same kind of rigidity may be given to a flexible tube by running through it a fluid of considerable density, such as water. If the radius of curvature at any cross section be R and the velocity of the fluid be u, the normal force applied by the tube towards the centre of curvature will be $\sigma u^2/R$ if σ be the mass of the fluid per unit of length of the tube. This will be applied by the tube in consequence of the existence in it of a stretching force T of amount σu^2, since the normal force per unit of length due to T is T/R. Thus T is independent of the curvature of the tube and is the same throughout.

337. **Example. Chain Falling under Gravity.**—As an example of the equations of motion we may consider a chain falling under

* "Elements of Natural Philosophy." Appendix.

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the action of gravity, and so that every part of the chain lies in one vertical plane; a chain, for example, hung in a vertical plane, and then suddenly released at one end. Let ϕ be the angle the tangent to the chain at any element makes with the horizontal. The equations of motion are

$$\left.\begin{aligned} \sigma(\dot{u} - \omega v) &= -\sigma g \sin\phi + \frac{dT}{ds}. \\ \sigma(\dot{v} + \omega u) &= -\sigma g \cos\phi + \frac{T}{R}. \end{aligned}\right\} \quad (32)$$

Differentiate the first of these with respect to s, and multiply the second by $1/R$, that is by $d\phi/ds$, and subtract the second result from the first. The right-hand side of the resulting equation is $d^2T/ds^2 - T/R^2$. The left-hand side reduces by the kinematical relations $[du/ds = v/R,\ dv/ds + u/R = \omega(=\dot{\phi}),\ 1/R = d\phi/ds]$ to $-\sigma\dot{\phi}^2$. Hence we obtain

$$\dot{\phi}^2 = \frac{1}{\sigma}\left(\frac{T}{R^2} - \frac{d^2T}{ds^2}\right), \quad (33)$$

which gives the rate at which each element of the chain is changing its direction.

At the instant of release the chain has no motion, and consequently the tensile stress at different points just after release satisfies the equation

$$\frac{d^2T}{ds^2} - \frac{T}{R^2} = 0, \quad (34)$$

which we shall presently arrive at in a different manner.

The integral of this equation is (see § 340)

$$T = -T_0 \frac{e^{(s-l)/R} - e^{-(s-l)/R}}{e^{l/R} - e^{-l/R}}, \quad (35)$$

in which s is supposed to be measured from the point at which $T = T_0$, and T is zero when $s = l$.

338. **Mode of Solving Problems on Chains. Examples.**—In solutions of the problem of finding T, it is necessary generally to solve a differential equation, and the constants of the solution must be determined to suit the case in hand. For instance, if the chain is cut at any point, the value of T is there zero; if it is fixed at any point, the tangential velocity and acceleration are there zero, and so on. If one extremity of the chain is constrained to be always perpendicular to a surface, the velocity of the extremity along the normal to the surface is always zero.

If both ends of the chain were released at once, it is evident that from that instant there would be no tensile force anywhere in the chain, which would fall freely in the curve in which it hung before release. This also follows from (35) above.

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The result (34) obtained in § 337 may be made to apply to any case in which the stretching force in a uniform chain is suddenly altered at any point—for example, the case in which the chain given in equilibrium or in motion under applied forces is suddenly freed or fixed at some point, or has the stretching force in it suddenly altered at one or more points. The solution is obtained by superimposing upon the stretching force T, formerly existing in the chain, a stretching force T' fulfilling equation (34). The stretching force at any point is thereby changed from T to $T+T'$. We take here a few examples.

[In the more general case of a non-uniform chain, the equation for the variation of the stretching force T to be applied is (41) below.]

(1) A uniform chain is given in the form of a circle of radius R on a smooth horizontal table, under the action of a repulsive force F from the centre of the circle: it is required to find the change of stretching force in the chain just after it is cut at one point.

The stretching force at the point cut is suddenly reduced to zero, and thus the tensile force T' suddenly applied there is $-FR$. We have, therefore, by (34), at any point at an angular distance θ from the point at which the chain is cut (since $s=\theta k$, $l=2\pi R$):

$$T'=FR\frac{e^{\theta-2\pi}-e^{-(\theta-2\pi)}}{e^{2\pi}-e^{-2\pi}}.$$

The tensile force in the chain at this point thus becomes

$$T+T'=FR\left\{1-\frac{e^{2\pi-\theta}-e^{-(2\pi-\theta)}}{e^{2\pi}-e^{-2\pi}}\right\},$$

or the stretching force is diminished in the ratio of the factor in brackets to 1.

(2) A uniform chain is held in contact with the convex surface of the lower half of a vertical hoop by equal upward vertical forces just sufficient for the purpose applied at the two ends of the horizontal diameter. The hoop is suddenly removed without alteration of these forces: it is required to find the stretching force in the chain just after the removal of the hoop.

Let F be the amount of each applied vertical force and T the tensile force at any point when the hoop is in position. Then if ψ be the angle the radius to the point makes with the horizontal diameter, the first of (32) gives

$$T=F-\sigma gR\int_0^{\phi}\cos\psi\,d\psi=F-\sigma gR\sin\psi.$$

At the lowest point, therefore, $T=F-\sigma gR$.

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Also, if N be the force per unit length on the chain at the same point acting normally inward, and N' be the reaction of the hoop on the chain,

$$-N=\frac{T}{R}=N'+\sigma g\sin\psi,$$

by (32). Thus

$$N'=\frac{T}{R}-\sigma g\sin\psi=\frac{F}{R}-2\sigma g\sin\psi,$$

by the value of T just obtained. At the lowest point, therefore, $N'=F/R-2\sigma g$. If the forces F are just sufficient to support the chain, $N'=0$ at the lowest point, and therefore $F=2\sigma gR$.

Let now the hoop be removed. The chain just after is not in motion, and the equations fulfilled by the tensile force are:

$$\frac{dT}{ds}=-\sigma g\cos\psi,\quad \frac{T}{R}=\sigma g\sin\psi,$$

which give $d^2T/ds^2-T/R^2=0$, that is, since $ds=Rd\psi$

$$\frac{d^2T}{d\varphi^2}-T=0.$$

Hence

$$T=Ae^{\psi}+Be^{-\psi},$$

subject to the conditions that $T=F$ when $\psi=0$, and when $\psi=\pi$. Thus $F=A+B=Ae^{\pi}+Be^{-\pi}$. Determining A and B from these two equations we obtain

$$T=F\frac{e^{\psi}(1-e^{-\pi})-e^{-\psi}(1-e^{\pi})}{e^{\pi}-e^{-\pi}}.$$

For the lowest point this gives

$$T=2F\frac{1}{e^{\frac{1}{2}\pi}+e^{-\frac{1}{2}\pi}},$$

so that the value of T is there changed by the removal of the hoop in the ratio $4/(e^{\frac{1}{2}\pi}+e^{-\frac{1}{2}\pi})$, since as we have seen $T=F-\sigma gR$ as the lowest point, and $F=2\sigma gR$.

(3) A uniform chain is hung vertically from its two ends, which are close together. One of the ends is then let go: it is required to find the tensile force at the bight, where the chain passes over from the free side to the other.

This cannot be taken as an example of (34), since at the bight $R=0$, but the solution may be obtained from first principles. It is clear that there is no stretching force in the chain on the side that is let go, for each portion on that side at any instant is then falling

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freely under gravity. When the free end has fallen for a time t, it has descended a distance $s = \frac{1}{2}gt^2$, and acquired a velocity $v = gt$, which at that instant is the velocity of the falling part of the chain. But since the whole length of the chain is constant, $2l$, say, the stationary portion is now of length $l + \frac{1}{2}s$, and is therefore increasing in length at rate $\frac{1}{2}\dot{s} = \frac{1}{2}v$.

The rate at which mass is passing over to the stationary side is $\frac{1}{2}\sigma v$, and since as it passes over each element is brought to rest, the rate of destruction of momentum at the bight is $\frac{1}{2}\sigma v^2 = \frac{1}{2}\sigma g^2 t^2$. This is the value of the tensile force at the lower end of the non-moving portion.

The rate of exhaustion of kinetic energy at the bight is $\frac{1}{4}\sigma g^3 t^3$, and hence the whole energy exhausted from the beginning up to time t is

$$E = \tfrac{1}{4}\sigma g^3 \int t^3 dt = \tfrac{1}{16}\sigma g^3 t^4.$$

As the reader may verify, the loss of potential energy in the time t is $\sigma g s(l - \frac{1}{4}s)$, and the kinetic energy of the falling portion is $\sigma g s(l - \frac{1}{2}s)$. The energy exhausted is thus $\frac{1}{8}\sigma g^2 t^2 s = \frac{1}{16}\sigma g^3 t^4$ as already found. This energy is wholly dissipated if the chain be inextensible.

Another example involving discontinuity in the stretching force due to impulsive setting into motion of successive elements of the chain is the following:—

(4) A fine chain lies on a table in a small heap close to the edge. One end is carried vertically up from the table, over a smooth horizontal rod at distance a above the table, and down again until the free end hangs below the edge of the table: it is required to determine the motion and the stretching force in the chain at the rod.

If x be the distance below the table of the lower extremity of the descending portion of the chain at any instant, the momentum on the descending side is then $\sigma(a+x)\dot{x}$, and the rate of increase of momentum of the part *already on that side* is $\sigma(a+x)\ddot{x}$. If, then, T_1 be the stretching force at the top of that portion, we have

$$\sigma(a+x)\ddot{x} = \sigma g(a+x) - T_1.$$

The rate of increase of momentum of the chain on the other side of the rod, the side of constant length of chain, is made up of two parts—the rate at which momentum is given to the chain formerly at rest in the heap, and the rate of increase of upward momentum of the upward moving chain. The upward velocity is numerically $\dot{x}$, since the chain is inextensible. In time dt mass $\sigma\dot{x}dt$ of chain is set moving with velocity $\dot{x}$, and the rate of generation of momentum at the lowest point is $\sigma\dot{x}^2$. The rate at which the portion of chain below a cross-section very near the top (and moving with the chain), where the stretching force is T_2, is gaining momentum, is thus $\sigma a\ddot{x} + \sigma\dot{x}^2$. Thus

$$\sigma a\ddot{x} + \sigma\dot{x}^2 = T_2 - g\sigma a.$$

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In time dt a length $\dot{x}dt$ of chain is carried from the ascending side to the other and has its momentum changed from $\sigma\dot{x}^2dt$ upwards to $\sigma\dot{x}^2dt$ downwards. If, however, the length of chain on the rod is infinitesimal, this by (29) cannot cause any finite difference in T between the two sides. Thus $T_1 = T_2$, and we obtain, by adding the equations formed,

$$\sigma(2a+x)\ddot{x} + \sigma\dot{x}^2 = \sigma g x$$

or

$$\frac{d}{dt}\{(2a+x)^2\dot{x}^2\} = 2gx(2a+x)\dot{x}.$$

Hence, integrating and determining the constant of integration from the fact that $\dot{x} = 0$ when $x = x_0$, we get

$$(2a+x)^2\dot{x}^2 = \tfrac{2}{3}g\{x^2(3a+x) - x_0^2(3a+x_0)\}.$$

If we divide both sides of this equation by $2(2a+x)$ and multiply by σ, we obtain on the left the kinetic energy of the moving chain. The reader may prove that the expression obtained on the right is less than $\frac{1}{2}\sigma g(x-x_0)^2$ the amount of potential energy lost from the beginning of the motion.

The time of falling may also be obtained from the value found for $\dot{x}$. Of course, if $x_0 = 0$, the time is infinite.

339 **Chain on Rough Surface. Friction of Rope round Cylinder.** —When a string or chain rests on a surface, as for example when a rope is wound round a cylinder, or if the chain is in motion under applied forces, the action of the surface including the friction must be taken into account. Thus for the equations of motion of an element of the chain we must write by (29)

$$\left.\begin{aligned} \sigma(\dot{u} - \omega v) &= \frac{dT}{ds} - F, \\ \sigma(\dot{v} + \omega u) &= \sigma N + N' + \frac{T}{R}, \end{aligned}\right\} \quad (36)$$

where N' and F are the normal force and the friction applied to the element per unit of its length by the surface. Let the chain be at rest on the surface and be on the point of motion under the tangential pull T_0 applied at one extremity. Further let $N = 0$, and $F = \mu N'$, where μ is a constant. We have $N' = -T/R$, and since $dT/ds = F$, we have also

$$\frac{dT}{ds} = -\mu\frac{T}{R}, \quad \text{or} \quad \frac{dT}{T} = -\mu\frac{ds}{R}. \quad (37)$$

Hence for the value of T at distance s from the end at which the pull is T_0,

$$\log T = -\mu\int_0^s \frac{ds}{R} \quad (38)$$

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If the surface be a circular cylinder round which the chain is wound in spires at right angles to the generating lines, we have

$$T = T_0 e^{-\mu s/R}. \tag{39}$$

Thus the pull which must be applied at distance s from the end to which T_0 is applied is only the fraction $e^{-\mu s/R}$ of the whole pull T_0. The remainder $1 - e^{-\mu s/R}$ of the applied pull is balanced by frictional resistance. For each turn the ratio of diminution is $e^{-2\pi\mu}$.

To form a numerical estimate of the diminution of the required balancing pull, let the cylinder be, say, 1 foot in diameter, and let the co-efficient of friction μ be $\frac{1}{5}$. In a distance round the cylinder of six inches the value of T is diminished to $e^{-1/5}T$, and therefore in a distance of 30 inches to T/e, that is to $T/2{\cdot}71828....$ In a little less than four turns of the rope round the cylinder the value of T is brought down to T/e^5, that is, to $T/150$ nearly.

This shows how by passing a rope a few times round a post a great pulling force, for example that exerted by a steamer moored to a quay, can be balanced by a comparatively slight pull applied to the free end of the rope.

340. **Impulsive Motion of Chain.**—Now let the string given on any curve, whether plane or not, be set into motion by tangential impulses applied at its end-elements. The effect of the applied forces during the short interval τ of the impulse amounts to nothing, and thus if T denote the tensile impulse at any point of the chain, and u, v denote the tangential and normal velocities generated, we have, adapting (32) to the case of impulse,

$$\sigma u = \frac{dT}{ds}, \quad \sigma v = \frac{T}{R}. \tag{40}$$

The second of these by (27) is $\sigma\, du/ds = T/R^2$, and the first gives $\sigma\, du/ds + u\, d\sigma/ds = d^2T/ds^2$. Hence, eliminating u, we obtain the relation

$$\frac{d^2T}{ds^2} - \frac{1}{\sigma}\frac{d\sigma}{ds}\frac{dT}{ds} - \frac{T}{R^2} = 0, \tag{41}$$

which connects the values of T at different points along the curve.

That this result is not limited to plane curves for the initial form of the chain is obvious from the fact that the initial motion of each element, produced as it is by impulsive stretching force along the chain, can only take place in the osculating plane (the plane defined at each point by a pair of consecutive elements of the curve which there meet). There is no force anywhere, and no velocity generated, along the binormal, that is perpendicular to the osculating plane.

This equation can easily be solved when the variation of σ along the curve is known. In the particular cases of R a constant with

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(1) $(d\sigma/ds)/\sigma$, (2) σ, a constant, the solution is easy. In the first case in which $(d\sigma/ds)/\sigma$ is a constant, a say, we have $\sigma = \sigma_0 e^{as}$ where σ_0 is the value of σ where $s = 0$. This is therefore the case of a chain the mass of which varies exponentially with distance along the chain. The constancy of R limits the curve to the case of a circle or helix.

The solution in the case of σ constant is that of the equation

$$\frac{d^2T}{ds^2} - \frac{1}{R^2}T = 0, \tag{42}$$

or

$$T = Ce^{\frac{s}{R}} + C'e^{-\frac{s}{R}}. \tag{43}$$

If $T = T_0$ when $s = 0$, and $T = 0$ when $s = l$, we have

$$C = -Te^{-l/R}/(e^{l/R} - e^{-l/R}), \quad C' = T_0 e^{l/R}/(e^{l/R} - e^{-l/R}),$$

so that with these conditions

$$T = -T_0 \frac{e^{(s-l)/R} - e^{-(s-l)/R}}{e^{l/R} - e^{-l/R}}. \tag{44}$$

If l be infinite $C = 0$ and $C' = T_0$, and the solution is

$$T = T_0 e^{s/R}, \tag{45}$$

that is, the ratio of the value of the tensile impulse at any point to the value at a point at a distance R farther along the curve is e. The curve in this case must be in the form of a helix extending to infinity. This result is given in Thomson and Tait's *Natural Philosophy*, vol. i., Part I., § 317,* where the differential equation (41) is obtained in a different manner.

It may be noticed here that the differential equation may be solved also when the radius of curvature R is a quadratic function of s. The reader may refer to a paper by Stokes, *Camb. Phil. Trans.* 1849, or *Collected Papers*, vol. i., or to Routh's *Advanced Rigid Dynamics*, p. 306.

Equation (41), as has already been remarked in § 338, may be applied to all cases in which the tensile force T in the chain is suddenly altered to a new value: for example that in which the chain attached at fixed points is suddenly released at one point.

* From the passage here referred to a clause seems to have dropped out. In line 18 of p. 299 after "for instance," supply the words "and when the cord is given in the form of a helix, extending to infinity from the point acted on." The discussion is carried out by Lagrange's method of multipliers, and is very instructive.

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CHAPTER IX.

STATICS AND KINETICS OF FLUIDS.

341. **Distinction between a Solid and a Fluid.**—So far we have considered the Dynamics of a System of Particles or of a Rigid Body: we have now to consider the Equilibrium and Motion of Fluids.

The actual physical distinction between a solid and a fluid, or at all events between a solid and a liquid, is no doubt clearly apprehended by most people, but the expression of the distinction in words is a matter of some difficulty. The best definition perhaps of a fluid is that it is a substance which has no elasticity of shape. In a following chapter we shall consider the subject of elasticity, but one or two definitions must be given here to enable the distinction between a solid and a fluid to be understood.

342. **Elasticity of Bulk and Shape.**—Elasticity itself may be defined as that property of matter which renders necessary the application of external force to change the bulk or shape of a body, and the continued application of the external force to maintain the change effected, and which enables the body to return towards or to its former bulk or shape when the external force is removed.

When a body suffers change of bulk or of shape it is said to be *strained*, and the change of relative configuration of the particles of the body involved sets up within the body forces opposing the change, which depend only upon the amount of the change which has been effected. It is these internal forces that have to be overcome or balanced by the external forces which are applied to the bounding surface of the body, and the external forces required depend only upon the amount of change of bulk or shape which has taken place.

343. **Viscosity.**—There are however developed within almost all bodies forces of resistance to change of shape which do not depend upon the amount of change which has been effected, but upon the rate at which the change is proceeding. But this kind of action on the part of the body is no result of elasticity of the body. It is due in fact to tangential action between the portions of the body which always opposes their relative motion, and extinguishes vibrational motion, with conversion of the energy of the motion into heat within the substance. Such forces are called *viscous forces*, and also dissipative forces because they dissipate energy. They have nothing in common with the forces which are developed within elastic bodies

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when subjected to change of shape or bulk, which depend on the amount of change that has taken place, and which must be balanced by the applied forces, when the body remains in a state of equilibrium in the changed state. These latter forces are called *elastic forces*. They are forces concerned in the transference of energy from storage in one always available form to storage in another, for example from energy of motion of a vibrating body to energy of change of conconfiguration of the body in a state of strain. They have no tendency to dissipate energy although in the body in which they are developed dissipative forces may at the same time have play.

344. **Homogeneity and Isotropy of a Body.**—Before proceeding to the discussion of elasticity of bulk and elasticity of shape we take the two following definitions, first, of a homogeneous body, second, of an isotropic body. A body is said to be *homogeneous* when, if cubes of equal size similarly situated in the body (that is, with sets of parallel edges parallel) are taken, these cubes show similarity of properties, that is, when properties taken with respect to the direction parallel to each set of four parallel edges are identical in all. The properties however relative to one set of four parallel edges may not be the same as the corresponding properties relative to another set of four parallel edges of the same cube. If however the properties of any one cube with respect to a specified direction in the original body be the same for all the cubes, the body is homogeneous.

If the properties of the body be the same for all directions through the same point the body is not only homogeneous but *isotropic*. The body is said to possess *homogeneity* and *isotropy*.

A body is said to be homogeneous or isotropic as regards its elastic properties when the elastic properties are those by which is is tested. In general bodies which are homogeneous or isotropic with respect to one set of physical properties are homogeneous or isotropic with respect to others, but this is not invariably the case. For example, a crystal of sulphur is not isotropic as regards its electrical properties, but is very approximately isotropic in its magnetic properties.

345. **Elasticity of an Isotropic Body.**—Bodies which occur most commonly in nature may be regarded as approximately isotropic as regards their elastic properties, and in the case of an elastic body we have only two kinds of elasticity to deal with (1) *elasticity of bulk*, (2) *elasticity of shape*. Change of bulk without change of shape is easily imagined. Take a sphere of material and immerse it in water in the cylinder of a hydraulic press; equal pressure will be applied all over the surface of the sphere in the direction towards the centre. If the sphere be isotropic it will remain a sphere under the uniform pressure applied all over its surface, there will be no change of shape of the sphere or of any portion of it. It will have been however converted into a smaller sphere, and to maintain it a smaller sphere the continued application of pressure will be required. If the pressure be removed the sphere will return towards its former bulk. The strain to which the sphere is subjected by

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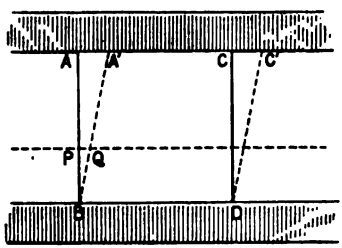
uniform pressure is measured by the ratio of the amount of diminution of volume to the original volume of the sphere, that is if δv be the positive amount of change of volume, and v the original volume, the measure of the strain is $\delta v/v$. If r be the original radius of the sphere and r' be the new radius the value of δv is $4\pi(r^3 - r'^3)$, and the original volume is equal to $4\pi r^3$, hence the strain of the sphere is $(r^3 - r'^3)/r^3$. This may be written $(r - r')(r^2 + rr' + r'^2)/r^3$ which, if r' be nearly equal to r, is very approximately $3(r - r')/r$. The quantity $(r - r')/r$, which is the ratio of diminution of the length of radius to the length of the original radius, may be called the strain that the radius is subjected to; thus the strain for a small alteration of volume of the sphere is approximately three times the strain of the radius. This relation will appear again frequently in the discussion of the elasticity of bodies.

346. **Specification of Shearing.**—In order to specify alteration of shape without alteration of volume let the reader imagine a cubical part of an isotropic body. Let one face of the cube be imagined cemented fast down to a horizontal bed and let the opposite face of the cube be cemented to a horizontal plate at a distance from the bed just equal to the length of edge of the cube. Now let the upper plate be moved *in its own plane* through a small distance in the direction of one set of parallel horizontal edges of the cube. There is thus no alteration of the vertical distance between the two faces of the cube, but the effect of the motion of the upper face in the direction specified will have been to render the four edges which were formerly vertical a set of four parallel lines no longer vertical. If x be the distance (AA' or CC') through which the upper face has been moved as stated, and l be the distance between two faces, that is the edge of the cube, the inclination of each of the formerly vertical lines to the vertical is now $\tan^{-1}x/l$. If x be small in comparison with l, this inclination is, very nearly, x/l.

FIG. 171.



By this displacement each horizontal section of the cube has been moved parallel to the direction of motion of the upper face of the cube through a distance proportional to the distance of the section from the lower face, that is, through a distance PQ, which is equal to xy/l if y be the distance of the section from the lower face. Obviously there has been no change of volume of the body or of any part of it, the body, and every part of it in a similar manner, has undergone change of shape.

The kind of strain here specified is called *shearing strain* and will engage our attention later to a considerable extent; the measure of the strain is the ratio x/l, that is, when x is small, the inclination to the vertical of the two inclined positions into

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which two of the formerly vertical faces of the cube have been thrown.

347. **Stress involved in Shearing Strain.**—When a solid body is thus subjected to change of shape without change of bulk, force must be applied in this manner to the plate to maintain change of shape of the body. This applied force is force tangential to the interface between the plate and the cube to which it is cemented, and has the same value per unit of area all over the interface. The applied force must be maintained if the state of strain on the body is to continue, and on the withdrawal of the force the body returns towards its former shape. Within the body, if it has elasticity of shape, is set up a system of internal forces in consequence of the state of strain, and tending to undo that state.

An isotropic solid has elasticity both of bulk and shape, a fluid is isotropic but has only elasticity of bulk, and has no elasticity of shape, at least it has not in ordinary circumstances. For it is not impossible that in certain special circumstances bodies which ordinarily behave as fluids may be found to possess elasticity of shape. At present however we define a solid as a body which has the two elasticities we have specified, and a fluid as a body which has only the first, namely elasticity of bulk.

348. **Viscous Resistance to Change of Shape.**—In all actual fluids however there is resistance to change of shape not depending upon the rate at which the change is proceeding, that is to say, all fluids are more or less viscous. This is sufficiently evident from the most ordinary observation. By tilting a glass of water and leaving it to itself the water can be set into oscillation in the vessel; continual changes of shape of every part of the water, though not of bulk, proceed and are resisted by the mutual action of the different parts of the water. This resistance depends upon the rate of change of shape and disappears when that rate is zero however great the change that has been effected may be. In the course of a short time the vibration has subsided and a sufficiently delicate thermometer would show that the temperature of the water had been slightly raised. Fluids differ very considerably in viscosity, that is in the amount of resistance which they offer to change of shape going on at a given rate; in fact this resistance may be so great that the body only very slowly changes its shape under ordinary forces and appears to behave as a solid. If however the body be a highly viscous fluid, a sufficiently long continuance of only a small force will suffice to produce a marked change of shape. Water, for example, has a small degree of viscosity, alcohol and some other liquids have still less; the various animal and vegetable oils have a higher viscosity of varying amounts, that of castor oil for example is very considerable. The viscosity of treacle is very great, so much indeed that there is a perceptible retention of the shape of a portion of the treacle for a short time after it is laid quickly as a heap on a table.

349. **Experimental Illustration of Viscosity.**—The following experiment illustrates the different viscosities of substances very

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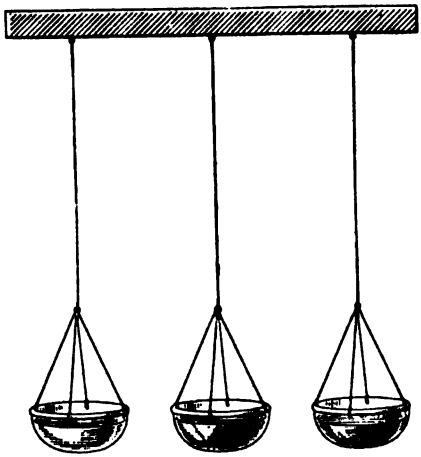
well. Three or four ordinary finger glasses are mounted, as shown in Fig. 172, in slings made of wire, each sling attached by a stiff joint, about which the sling cannot swivel, to a stiff wire, which again is rigidly attached to a hook above. The suspension wires are all made of the same material and of the same length and gauge. One of the glasses is filled rather more than half full with water, the second, third, and fourth with the same weight of sperm oil, castor oil, and treacle respectively. Small bits of paper are placed on the surfaces of the liquids in line along the radii of the glasses to indicate any relative motion of the parts of the liquid at the surface that may be set up by the motion given to the glasses.

FIG. 172.



Each wire is taken by its lower end and slowly twisted round so that each sling with the glass and contents is made to turn through approximately the same angle. Two operators can manage the four glasses, can twist them through the same angle and let them go approximately at the same moment. As soon as the wires are let go they begin to untwist and the glasses with their contents turn round vertical axes coincident with the wires. As each wire untwists the elastic couple in it producing the motion of the glass diminishes, from a maximum value when the wire is let go to nothing when the twist has just all come out; but during the period of untwisting it has had a positive value, and so the glass has been constantly receiving angular velocity. The glass then continues to turn past the position of zero twist and the wire acquires twist in the opposite direction. As that twist increases however, an increasing couple resisting the motion finally brings the glass to rest and then causes it to describe nearly the same succession of positions in the reverse order. Thus each glass performs vibrations about the vertical axis, in fact it is, with its contents to a certain extent, the bob of what we may call a *torsional pendulum*. It will be noticed on making the experiment that the amplitude or range of motion falls off in all cases, but with greater rapidity in the case of the oil than in the case of either the treacle or the water. When the glasses are left to themselves they are turned round by the slings, and if there were no disturbances such as ordinary pendulum vibration, if the glasses were quite circular and turned about their axes of symmetry, and if moreover the liquids had no viscosity and there were no drag between the

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glasses and the liquids exerted at their surfaces of separation, the glasses would turn while the liquids remained undisturbed.

This state of things is more closely approximated to by water than in the other cases, but even in water there is a tangential action between it and the glass, so that the glass drags the water in its neighbourhood round with it. This turning motion with the glass is communicated to portions of water nearer the middle by viscous action between them and the outer portions. This is shown by the displacement which takes place in the small pieces of paper. If the glass is turned round slowly, at first, the water turns round with it and the relative positions of the paper are not altered, but when the pendulum is released the to and fro motions of the pieces of paper near the outside show that the outer portions of the water are dragged with the glass through angles which are greater the greater the distance from the centre.

In the case of the oil the motion is more completely communicated from the outer to the inner portions of the liquid and the tangential drag exerted against the motion of the glass by the liquid is in that case larger and the glass is more quickly brought to rest.

In the case of the treacle however the viscosity is so great that turning motion takes place in the same phase throughout the liquid. The glass carries with it the outer stratum of the treacle, the outer stratum the stratum next it, so that the whole liquid turns approximately as a rigid body would round the vertical axis and there is but slow falling off of the amplitude of vibration.

The oscillation of the vessel generates waves in the fluid which travel in from the surface, and involve dissipation of energy in work against friction. If the ratio of the frequency of oscillation to the measure of vicosity be great the waves travel in quickly, but are very rapidly damped out, so that only a very thin stratum of the fluid is affected. When this ratio is small the fluid and vessel move together; dissipation of energy is slight, but the inertia of the oscillator is increased. In the intermediate case considerable relative motion is set up in the fluid, and the motion of the vessel is quickly damped out. (See *Viscosity* in Vol. ii. below.)

350. **Behaviour of Pitch.**—A still more extreme case would be obtained by filling the glass with pitch or shoemakers' wax. This substance is apparently hard and brittle, for when thrown down it flies in pieces with a conchoidal fracture like that of glass. A piece of it however laid on a table gradually flows out in a horizontal sheet.

There is little or no diminution of amplitude due to the viscosity of this material when it is placed in the glass of the torsional pendulum; but this is not due to the absence of viscosity, on the contrary the great viscosity of the material prevents it from undergoing in the short time during which the force acts any sensible distortion. The viscosity of the material and the fact that in ordinary circumstances it behaves as a fluid is well illustrated by the following experiment, which is due to Lord Kelvin. In a glass jar about ten inches

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in diameter and about the same depth, are placed a small quantity of water and a few ordinary corks. Then a cake of shoemakers' wax of about the diameter of the jar and about three inches thick is moulded and placed in the water above the corks so as to confine them at the bottom of the vessel, while the water rises above the wax and serves to maintain the temperature fairly constant. A few lead bullets are laid on the top of the cake and the arrangement is left to itself. It will be found that in the course of two or three weeks the corks have become lost to view by passing into the cake, and on looking at the glass from below it will be seen that the wax has closed up behind the corks so that they are now under the influence only of the action of the surrounding wax. The lead bullets also will be found to have disappeared from the surface and no trace of their passage into the wax will be discernible. A few weeks later the corks will begin to push themselves through the upper surface of the slab, pushing the wax before them, and finally breaking through the covering skin and escaping into the water above, while the bullets are finally found in the water below the slab. The wax is now found to be perfectly continuous.

The long continued action of the forces applied to the wax by the bullets has enabled them to pass slowly through against the resistance offered to the motion, a resistance approximately proportional to the velocity of progress of the bullets. In the first instance the corks were pressed against the lower face of the wax slab by the water, they gradually penetrated the slab and were then urged upwards by the force applied to them by the surrounding wax, in a manner which will be explained presently for fluids generally.

In fact the slab of wax has acted on the corks and bullets just as the treacle or oil or any other imperfect fluid would have acted. The bullets would have sunk down through the oil, and the corks would have bobbed up from the bottom very quickly as in the case of water. The only difference in the case of the wax is that the time has been prolonged to two or three months. The wax, which at ordinary temperatures breaks like a glassy solid when thrown down, behaves here as an extremely viscous fluid.

351. **Solidity of Pitch in Special Circumstances.**—There are circumstances in which pitch may be made to act as a true solid; for example, if a piece of pitch is cast into the form of a bell, and is struck with a hammer true elastic forces will be developed in it and the bell will vibrate giving out a musical note. The fact that a musical note is given out is proof that the particles of the bell when their relative positions are displaced, are urged towards their former relative positions with forces proportional to their displacement, that is, depending upon the amount of relative displacement to which they have been subjected. They seem therefore to be true elastic forces, and the body appears to have in these circumstances of rapidly alternating forces applied to it, true elasticity of shape.

If the bell however be set down on its mouth on a table and left to itself, it will become in a few days a mere sheet of pitch.

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352. **Interface in a Fluid. Fluid-Pressure. Pressure-Head.**—Now let us consider, as an approximation to an actual fluid, an isotropic fluid, free from viscosity. Since there is no elasticity of shape and no viscosity, there is no tangential action between two portions of the fluid exerted on either portion at the separating surface. Thus any action whatever exerted by one portion of fluid on another must be everywhere at right angles to the surface of separation between the two portions. We shall for brevity call the separating surface between two fluids an *interface*. An interface has of course no physical existence, it is a mere geometrical boundary between two portions of fluid. There will in general be force applied across the interface by one portion on the other, and this as we have seen is everywhere at right angles to the interface. Take any area S of the interface, and let F be the force exerted across it by one portion A on the other B. An equal force is of course exerted across the same area by the portion B on A. The ratio F/S is called the *average pressure* over the area S of the interface. If throughout any finite region of the interface the ratio of F to S is independent of the magnitude of S, uniform pressure is said to be exerted over the region of the interface. If however the pressure is not uniform we define the *pressure at a point* P in the interface in the following manner. Take any area S of the interface surrounding P and take the force F over that area, then as just stated F/S is the average pressure over the area S which includes P. Now let S be diminished indefinitely, but be always taken so as to include P, and let the ratio F/S approach as this diminution is carried on more and more nearly to a fixed value. This fixed value of the ratio is the pressure at P. It is shown in § 395 that the pressure at P is the same for all interfaces.

The height of a column of fluid required to give a pressure p at its base by its weight, is called the *head* of the fluid. If p be the pressure and ρ the density, then the weight of a column of the fluid of unit area of cross section and height h is $g\rho h$, in absolute units, or ρh in gravitation units. In the latter case if h be in feet and ρ in pounds per cubic foot, the pressure p in pounds per square foot is ρh.

353. **Hypotheses of Ordinary Hydrodynamics.**—In the definition of pressure given above we have assumed that the diminution of the area is pursued indefinitely. This statement however must not be taken as meaning more than that this fixed ratio is approached more and more nearly the farther the diminution of the area is carried, within the limits of observation or experience of areas set by the most refined instruments of physical research. No doubt if we could go beyond those limits we should come down to the grained structure of the substance, and have to take into account the want of homogeneity involved in our having molecules with interspaces between them. In that case it would be necessary to have recourse to a statistical treatment, but this is avoided by the hypothesis here adopted that the smallest portion of fluid with which we have to deal possesses all the physical properties which the fluid has in bulk. The first thing we have to do then is to calculate

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the acceleration of a portion of the fluid; for this purpose it is necessary to calculate the forces acting upon the portion. These forces fall into two sets: (1) The forces applied to the surface of the fluid by the surrounding fluid, or by the molecular actions of the molecules of the fluid itself near the surface as in capillarity. (2) Actions upon the particles of the fluid applied from without, for example, the forces of gravity on the particles of the fluid. We shall suppose the fluid at first incompressible, that is to say, that no system of forces acting upon a portion of the fluid is capable of altering its volume. Afterwards we shall consider shortly the motion of compressible fluids.

*354. **Acceleration of an Element of a Fluid.**—We have now to calculate kinematically the acceleration of a particle of a fluid. Let the velocity of a particle of the fluid at a point P be q and let the direction in which it is moving at that point be specified, let it arrive at a point Q along its path after an interval δt and let q' be the velocity it has at Q. Then the acceleration of the particle is the limit-ratio of the increase of velocity $q'-q$ to the interval of time δt in which it is effected. When the interval of time is made vanishingly small the limiting value of the ratio is $\delta q/\delta t$. It is to be observed that δq is the difference between the velocity which exists for a particle at P at time t, and the velocity which exists for a particle at Q at time $t+\delta t$. The change is therefore made up of two parts: (1) the difference between the velocity at Q at time t, and the velocity at the point P at the same time, (2) the change of velocity which the velocity at Q undergoes in the interval of time δt.

If $\partial q/\partial s$ denote the rate of variation of q along the path of the particle at time t, and ds the length of the step from P to Q be very small, the change specified in (1) is $\partial q/\partial s.ds$. Again let $\partial q'/\partial t$ be the rate of change of velocity at Q with time, then since the velocity at Q at time t is $q+\partial q/\partial s.ds$ at time $t+dt$ it has become $q+\partial(q+\partial q/\partial s.ds)\partial t.dt$, that is $q+\partial q/\partial t.dt+\{\partial(\partial q/\partial s)\partial t\}dsdt$. The last term vanishes in comparison with the others, and when dt is made very small we have for the change in q brought about by the displacement of the particle from P to Q in the interval dt the value $\partial q/\partial t.dt+\partial q/\partial s.ds$. Hence if $\dot{q}$ denote the time-rate of change of q we have since $q=ds/dt$ or $\dot{s}$,

$$\dot{q}=\frac{\partial q}{\partial t}+q\frac{\partial q}{\partial s}. \tag{1}$$

In the case of steady motion the velocity at each point of the path of the particle, and indeed at each point of the fluid, does not change with the time, and we have then

$$q=q\frac{\partial q}{\partial s}. \tag{2}$$

These equations may be put into other forms which are sometimes convenient. Thus let $\dot{u}$, $\dot{v}$, $\dot{w}$ be the components of acceleration parallel to the axes. Since $q^2=u^2+v^2+w^2$ we have $\dot{q}=\dot{u}u/q+\dot{v}v/q+\dot{w}w/q$, $\partial q/\partial t=\partial u/\partial t.u/q+\partial v/\partial t.v/q+\partial w/\partial t.w/q$, $q\partial q/\partial s$

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$= u\partial u/\partial s + v\partial v/\partial s + w\partial w/\partial s = u/q.(u\partial u/\partial x + v\partial u/\partial y + w\partial u/\partial z)$ $+ v/q.(u\partial v/\partial x + v\partial v/\partial y + w\partial v/\partial z) + w/q.(u\partial w/\partial x + v\partial w/\partial y$ $+ w\partial w/\partial z)$. Taking the sum of these expressions for $\partial q/\partial t$ and $q\partial q/\partial s$, and equating to the value of $\dot{q}$ we obtain an equation which must hold for any given case however the axes are chosen. Hence the quantities multiplied by u/q, v/q, w/q on the two sides of the equation must be equal. We obtain therefore:

$$\left.\begin{aligned} \dot{u} &= \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \\ \dot{v} &= \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} \\ \dot{w} &= \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} \end{aligned}\right\} \quad (3)$$

*355. **Equation of Continuity.**—Now we have to establish an equation which states that if there be no creation or annihilation of matter in any part of the space occupied by the fluid, whatever increase or diminution of the amount of matter within any portion of space takes place in any time must be precisely the difference between the amount of fluid that has entered the space and the amount that has passed out of the space across its boundaries in that time.

This is called the *equation of continuity.* First we shall show that the time rate of increase of volume of a rectangular element the centre of which is at the point x, y, z where the components of velocity are u, v, w, and the edges of which are lengths dx, dy, dz taken along the axes, is $(\partial u/\partial x + \partial v/\partial y + \partial w/\partial z)dxdydz$.

For consider a filament of infinitely small cross-section σ and length parallel to the axis of x. If θ_2, θ_3 be positive numerical multipliers, each less than unity, the co-ordinates of the extremities of such a filament relatively to P are $-\frac{1}{2}dx$, $\theta_2 dy$, $\theta_3 dz$ for one end, $\frac{1}{2}dx$, $\theta_2 dy$, $\theta_3 dz$ for the other. Hence the expression

$$u \mp \tfrac{1}{2}\partial u/\partial x\, dx + \theta_2 \partial u/\partial y.dy + \theta_3 \partial u/\partial z.dz$$

gives the value of u at the former or latter extremity of the filament according as the negative or the positive sign is taken. Hence the excess of the value of u at the latter extremity above that at the former is $\partial u/\partial x.dx$. This is the rate of increase of length of the filament, and produces a rate of increase of volume of the filament of amount $\sigma(\partial u/\partial x.dx)$.

The quantity $\partial u/\partial x$ is constant for all the filaments and we have therefore for the rate of increase of volume of the element due to the variation of u the value $\partial u/\partial x.dxdydz$.

Similarly for the rate of increase due to the variation of v, w through the element we have $\partial v/\partial y.dxdydz$, $\partial w/\partial y.dxdydz$ respectively. The whole rate of change of volume is, in the limit, the sum of these. The ratio of this sum to the original volume of the fluid, or what may be termed the *unital* rate of change of volume, is called the *divergence* of the flow of the fluid at P, and is denoted by

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$divq$ where q is the resultant of u, v, w at P. Hence we write

$$divq = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}. \tag{4}$$

This expression gives the divergence as far as quantities of the first order of smallness. It will be noticed for example that the rates of increase of dimensions in the directions of y and z augment the rate of increase of volume produced by the flow along x, and similarly for the other directions y, z. These effects are left out in the discussion given above, but the reader may easily verify that the terms depending on them are small compared with those retained.

The unital increase of volume in time dt sustained by a given portion of the fluid at P is thus $(\partial u/\partial x + \partial v/\partial y + \partial w/\partial z)dt$. If ρ be the density of the portion of the fluid, its density at time dt is $\rho + \dot{\rho}dt$. Hence the mass of the element after dt has elapsed is $(\rho + \dot{\rho}dt)\{1 + (\partial u/\partial x + \partial v/\partial y + \partial w/\partial z)dt\}dxdydz$. But since this must have the same value as before, namely $\rho dxdydz$, we find, neglecting the term multiplied by dt^2,

$$\dot{\rho} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0, \tag{5}$$

which is the equation of continuity. In fact, quantities of the second order being neglected, $\rho(\partial u/\partial x + \partial v/\partial y + \partial w/\partial z)dxdydz$ is the rate of increase of the mass of the element in consequence of the alteration of volume, and $\dot{\rho}dxdydz$ is the rate of increase of its mass arising from alteration of density, and these must together give a zero result.

In the same way may be reckoned the divergence of any other quantity connected with the flow of the fluid if the quantity varies continuously. Thus we have a similar expression for the divergence of the flow of inertia of the body. The rate of flow of inertia per unit of volume at P is ρq if ρ be the density at P. The components of this are ρu, ρv, ρw, so that

$$div\rho q = \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}. \tag{6}$$

Clearly since this is the rate of diminution of matter in unit volume it is the time rate of diminution of density at the fixed point P, namely $\partial\rho/\partial t$. Hence we have

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0, \tag{7}$$

which is another form of the equation of continuity. Since ρ may vary with the time for a fixed point, and also with the position of a particle of the fluid at time t, the value $\dot{\rho}$ of the rate of increase of ρ *regarded as a quantity characteristic of a portion of the fluid* is given by

$$\dot{\rho} = \frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z}. \tag{8}$$

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*356. **Velocity-Potential.**—When the distribution of velocities at any point in the fluid at any particular instant of time is obtained from a function ϕ of the co-ordinates of the point, in the manner described by the equations

$$u = -\partial\phi/\partial x, \quad v = -\partial\phi/\partial y, \quad w = -\partial\phi/\partial z, \tag{9}$$

the motion is said to be characterised by a velocity-potential ϕ. The reader will notice that this is the mode of deriving the component-forces from a force-potential as explained at § 218 above. It is an elementary theorem of what is called partial differentiation that $\partial^2\phi/\partial x\partial y = \partial^2\phi/\partial y\partial x$, so that if the velocities are so derivable from a function ϕ of the co-ordinates we must have the three relations

$$\partial w/\partial y - \partial v/\partial z = 0, \quad \partial u/\partial z - \partial w/\partial x = 0, \quad \partial v/\partial x - \partial u/\partial y = 0. \tag{9'}$$

When equations (9) hold we have

$$udx + vdy + wdz = -(\partial\phi/\partial x.dx + \partial\phi/\partial y.dy + \partial\phi/\partial z.dz) = -d\phi,$$

where $d\phi$ is the change of ϕ which takes place when x, y, z are changed to $x+dx$, $y+dy$, $z+dz$. The quantity $d\phi$ is what is called a *perfect differential* of the function ϕ of the co-ordinates. The integral of $d\phi/ds.ds$ taken along any path in the fluid is then $\phi_2 - \phi_1$ where ϕ_2, ϕ_1 are the values of ϕ at the final and initial ends of the path. This integral is not, however, free from ambiguity in certain cases, as will be explained below. (See §§ 358, 193.)

The quantities on the left of equations (9′) have a certain physical interpretation which may be explained in the following manner. Think of a small rigid sphere with its centre at the point x, y, z, and let the centre move with component velocities u, v, w. Let the angular velocities of the sphere about diameters parallel to the axes of x, y, z be θ_1, θ_2, θ_3 then the velocities of a point on the surface, the co-ordinates of which are dx, dy, dz, are by § 277,

$$u - \theta_3 dy + \theta_2 dz, \quad v - \theta_1 dz + \theta_3 dx, \quad w - \theta_2 dx + \theta_1 dy.$$

A small sphere of the fluid however does not move as a rigid body. The component velocities at a point $(x+dx,\ y+dy,\ z+dz)$ are $u + \partial u/\partial x.dx + \partial u/\partial y.dy + \partial u/\partial z.dz, \ldots$, and these can be written

$$u + \frac{\partial u}{\partial x}dx + \tfrac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)dy + \tfrac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)dz$$
$$- \tfrac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)dy + \tfrac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)dz,$$

with two similar expressions. We shall show that the last line corresponds to the terms $-\theta_3 dy + \theta_2 dz$ of the velocity of the point on the surface of the rigid sphere, that in fact the component angular velocities ξ, η, ζ (or components of *spin*) of the fluid at the point x, y, z are given by

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$$\xi=\tfrac{1}{2}\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right),\quad \eta=\tfrac{1}{2}\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right),\quad \zeta=\tfrac{1}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right). \tag{10}$$

When these quantities are all zero the motion is said to be *irrotational;* if they are not all zero the motion is said to be *rotational*, but is most frequently called *vortex-motion.*

*357. **Rotational Motion of a Fluid Element. Components of Angular Velocity.**—Consider a small sphere of the fluid the centre of which is at P (x, y, z), where the velocity components are u, v, w. If $\partial u/\partial x$, $\partial u/\partial y$, ... be the space-rates of variation of u, v, w at P, the velocity of a point very close to P, of which the co-ordinates are $x+\alpha$, $y+\beta$, $z+\gamma$ has components $u+\alpha\partial u/\partial x+\beta\partial u/\partial y+\gamma\partial u/\partial z, \ldots$, so that the component velocities of the element relatively to P are $\alpha\partial u/\partial x+\beta\partial u/\partial y+\gamma\partial u/\partial z, \ldots$. The moment of momentum h_1, of the sphere about an axis through P parallel to that of z is therefore

$$\int\rho d\varpi\left(\frac{\partial w}{\partial x}\alpha+\frac{\partial w}{\partial y}\beta+\frac{\partial w}{\partial z}\gamma\right)\beta-\int\rho d\varpi\left(\frac{\partial v}{\partial x}\alpha+\frac{\partial v}{\partial y}\beta+\frac{\partial v}{\partial z}\gamma\right)\gamma,$$

where the integrals are taken throughout the sphere. But $\partial w/\partial x, \ldots$ are constants throughout the field of integration, and hence there is for every term $\rho d\varpi.\alpha\beta\partial w/\partial x$ for which β is positive another term for which β has an equal but negative value, while everything else is the same, and so for the terms involving $\beta\gamma$.

The moment of momentum thus reduces to

$$\rho\left\{\frac{\partial w}{\partial y}\int\beta^2 d\varpi-\frac{\partial v}{\partial z}\int\gamma^2 d\varpi\right\}.$$

The moment of inertia of the sphere about the diameter is $\rho\int(\beta^2+\gamma^2)d\varpi$, and clearly by symmetry $\int\beta^2 d\varpi=\int\gamma^2 d\varpi$. Hence if μ be the moment of inertia $\rho\int\beta^2 d\varpi=\rho\int\gamma^2 d\varpi=\frac{1}{2}\mu$, we have for the moment of momentum of the sphere about the diameter parallel to x

$$h_1=\tfrac{1}{2}\mu\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right). \tag{11}$$

This proves that $\frac{1}{2}(\partial w/\partial y-\partial v/\partial z)$ is the angular velocity of the sphere about the axis of z. In the same way h_2, h_3 can be found and $\frac{1}{2}(\partial u/\partial z-\partial w/\partial x)$, $\frac{1}{2}(\partial v/\partial x-\partial u/\partial y)$ shown to be the angular velocities of the sphere about the diameters parallel to y and z.

*358. **Irrotational Motion. Single and Multiple Valued Potentials.**—The vanishing of the quantities h_1, h_2, h_3 shows that the fluid has no rotation of its elements. But the vanishing of these quantities is the criterion that u, v, w should be derivable from a function φ of the co-ordinates by the process of differentiation stated in § 354 above, that is, of the existence of a velocity-potential ϕ; and conversely, the existence of a velocity-potential proves that there is no rotation of the elements of the fluid (§ 361). It can

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be proved that if in a perfect fluid a velocity-potential exists in any part, that part retains its velocity-potential so long as the fluid continues to exist as such, provided the forces acting on it are conservative (§ 193).

The potential ϕ may be either single or multiple valued, that is for assigned values of the co-ordinates ϕ may have a single definite value or have any one of a series of possible values. In the former case the value of $\int d\phi/ds.ds$ or $\int d\phi$ taken along a path drawn in the field has the same value $\phi_2 - \phi_1$ (where ϕ_2, ϕ_1 are the values of ϕ for the final and initial points of the path) for *all* paths having the same initial and final points; in the latter case it may be impossible to vary the path indefinitely without affecting the value of the integral along it. It is generally possible, however, to draw a family of paths joining two points of the field for each of which the integral referred to has the same value. It is only necessary to assign to the initial point P of the path a value ϕ_1 of ϕ, then the value ϕ_2 at the terminal point Q will be greater than ϕ_1 by the integral $\int d\phi/ds.ds$, the value of which depends on the mode in which the path is drawn.

For example, consider an endless tube with fluid flowing along it in closed paths. The value of $\int d\phi/ds.ds$ taken along one of these paths is not zero though the initial and final points P, Q of integration coincide, for ϕ continually diminishes at positive rate $-d\phi/ds(=q)$ along the path, so that if ϕ_1 be an assigned value of ϕ for the starting-point and ϕ_2 be the corresponding value for the (coincident) final point of the integration, $\phi_2 = \phi_1 - \kappa$, where κ is what is called the *cyclic constant* for the flow along the tube.

On the other hand, when a closed path is drawn in the tube beginning at P and returning to P, and such that it can be contracted to a point without passing out of the tube, the value of $\int d\phi/ds.ds$ taken along it is zero, since whatever diminution of ϕ along one part of the path which proceeds on the whole with the flow of the fluid, is annulled in the part of the path which returns against the flow.

As an example of a single valued function, let ϕ be μ/r, where r is the distance of the point considered from a point O. The value of $\int d\phi/ds.ds(=\int d\phi)$ taken along a path from a point P to a point Q for which r has the values r_1, r_2 respectively, is $\mu/r_2 - \mu/r_1$, and is therefore zero for any closed curve not passing through the point O.

On the other hand if $\phi = \mu \tan^{-1}(y/x)$,

$$\int d\phi/ds.ds = \mu\{\tan^{-1}(y_2/x_2) - \tan^{-1}(y_1/x_1)\}$$

when taken along a path from an initial point P, (x_1, y_1) to a final point Q (x_2, y_2), Fig. 173. But this is $\mu(\theta_2 - \theta_1)$ if $\theta_2 - \theta_1$ be the angle between the radii drawn from the origin O of co-ordinates to Q and P. This angle obviously depends on the path drawn from P to Q: for if, like the path PAQ, it consists partly of a single closed loop round the origin, the radius-vector turns round as a point passes along the curve from P to Q through an angle $2\pi + \angle QOP$. If the curve makes a double circuit round the origin the value of $\theta_2 - \theta_1$ is $4\pi + \angle QOP$, and so on. The integral is thus the same for all paths

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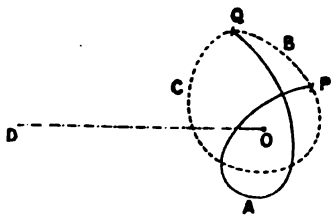
joining P and Q which make the same number of circuits round the origin.

Again in this case let curves PBQ, PCQ be drawn from P to Q, so that the two form a closed curve round O. The integral along PBQ is not equal to the integral along PCQ. For the integral taken along the closed curve from P back to P again, in the direction taken as positive (say PAQ), is $2\pi\mu$, and therefore if the integral from P to Q along PBQ be μI that along PCQ is $2\pi\mu - \mu I$. If the closed curve $PBQCP$ did not enclose O the integral round it would be zero, for as a point passed from P round the path the angle θ between a line drawn to it from O and OP would not turn through 2π but, however devious the path might be, through a zero angle on the whole.

FIG. 173.



In all cases of multiple-valued functions the integral can be deprived of ambiguity by imposing restrictions on the path. In the case just considered let a line OD be drawn in the plane of x, y from O to infinity, and let no path be drawn which crosses this line. The integral is then the same for all paths starting at P and ending at Q; in fact, the function is rendered single-valued.

Similarly in the case of the closed tube along which fluid is flowing the potential may be rendered single-valued by introducing a diaphragm across the tube which no path of integration can cross.

These considerations are of course applicable to force-potentials as well as to velocity-potentials. (See also § 486.)

*359. **Equations of Fluid-Motion.**—We may now write down the equations of motion of a fluid, and apply them to a few of the simpler cases which occur in practice. Consider again a parallelepiped of the fluid having its centre at the point x, y, z, and its edges of lengths dx, dy, dz parallel to the axes. Let X be the applied force per unit of mass on the element, then $X\rho dxdydz$ is the force acting on the element from without in the direction of the axis of X. Besides this there is force due to pressure applied to the element by the surrounding fluid. If the pressure at the centre x, y, z be P then we obtain by the process used in §§ 354, 355, for the difference of pressures on the two ends which are at right angles to the axis of x, $-\partial p/\partial x.dx$ towards the right. The force on the element due to this difference of pressure is $-\partial p/\partial x.dxdydz$. The whole force applied in the direction of x to the element is therefore

$$\{\rho X - \partial p/\partial x\}\, dxdydz.$$

This must be equal to the acceleration of the element multiplied into its mass, and similar results hold for the acceleration in the other two directions. By the values of the component accelerations given in § 355 we obtain for the equations of motion

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$$\left.\begin{aligned} X-\frac{1}{\rho}\frac{\partial p}{\partial x}&=\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\\ Y-\frac{1}{\rho}\frac{\partial p}{\partial y}&=\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\\ Z-\frac{1}{\rho}\frac{\partial p}{\partial z}&=\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}. \end{aligned}\right\} \quad (12)$$

If the forces X, Y, Z are derivable from a potential function Ω, the potential energy per unit of volume in the field of the applied forces, these equations become

$$-\left(\frac{\partial\Omega}{\partial x}+\frac{1}{\rho}\frac{\partial p}{\partial x}\right)=\dot{u}, \qquad \left(\frac{\partial\Omega}{\partial y}+\frac{1}{\rho}\frac{\partial p}{\partial y}\right)=\dot{v}, \quad -\left(\frac{\partial\Omega}{\partial z}+\frac{1}{\rho}\frac{\partial p}{\partial z}\right)=\dot{w}, \quad (13)$$

in which $\dot{u}$, $\dot{v}$, $\dot{w}$ are used for the component accelerations. Now for any instant of time p is a function of x, y, z, and the change of p corresponding to changes dx, dy, dz of x, y, z is $dp=\partial p/\partial x.dx+\partial p/\partial y.dy+\partial p/\partial z.dz$, and similarly $d\Omega=\partial\Omega/\partial x.dx+\partial\Omega/\partial y.dy+\partial\Omega/\partial z.dz$. If then we take the three equations of motion as last written, multiply the first by dx, the second by dy, and the third by dz, we obtain

$$-\left(d\Omega+\frac{1}{\rho}dp\right)=\dot{u}dx+\dot{v}dy+\dot{w}dz. \quad (14)$$

Now let ρ be a function of p so that $1/\rho$ is equal to $f'(p)$ where $f'(p)$ is the differential co-efficient with respect to p of a function $f(p)$, of p. The quantity within the brackets on the left is then, for a given instant of time, purely a function of the co-ordinates, and the whole expression on the left denotes the diminution of $\Omega+f(p)$, a function of the co-ordinates, which takes place when x, y, z are changed to $x+dx$, $y+dy$, $z+dz$, and is therefore a perfect differential (§ 356). The quantity on the right is therefore in this case proved to be also a perfect differential of a function of the co-ordinates x, y, z (which, however, may also depend upon t).

*360. **Kelvin's Theorem of Fluid Motion.**—The greater part of the theory of fluid motion may with truth be said to be contained in the following theorem due to Lord Kelvin. Before stating the theorem we lay down one or two definitions. If l, m, n be the direction cosines of an element of a line of particles moving with the fluid, the rate at which the particles are moving along the line is $lu+mv+nw$. This velocity multiplied by ds, the length of the element, is called the *flow along the element*. If dx, dy, dz be the projections of ds on he axes the values of l, m, n are, by § 24, dx/ds, dy/ds, dz/ds. Hence we may write the flow in the form $udx+vdy+wdz$.

Now consider the rate of alteration of this quantity with the time at any instant. It will depend upon two things, the rates of alteration of u, v, w, and the rates of alteration of dx, dy, dz at

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that instant. The former rates are $\dot{u}$, $\dot{v}$, $\dot{w}$; the latter are $d\dot{x}$, $d\dot{y}$, $d\dot{z}$; for if $\dot{x}$, for example, be the velocity in the direction of x at the initial (left hand) end of dx, $\dot{x}+d\dot{x}$ is the velocity at the other end. The difference between these velocities is $d\dot{x}$, and is the rate at which dx is increasing in length, and similarly $d\dot{y}$, $d\dot{z}$ are the rates at which dy, dz are increasing. For $\dot{x}$, $\dot{y}$, $\dot{z}$ we may write of course u, v, w.

Hence if at the beginning of an element of time dt the flow along the element be $udx+vdy+wdz$, at the end of that time it is

$$(u+\dot{u}dt)(dx+d\dot{x}dt)+(v+\dot{v}dt)(dy+d\dot{y}dt)+(w+\dot{w}dt)(dz+d\dot{z}dt).$$

Hence the change is $\{\dot{u}dx+\dot{v}dy+\dot{w}dz+udu+vdv+wdw\}dt$ + a quantity multiplied by dt^2. Neglecting the latter, since dt is taken very small, and dividing by dt, we get for the rate of change of flow the equation

$$\frac{d}{dt}\left(udx+vdy+wdz\right)=\dot{u}dx+\dot{v}dy+\dot{w}dz+udu+vdv+wdw$$
$$=udx+\dot{v}dy+\dot{w}dz+qdq,$$

where $q^2=u^2+v^2+w^2$, the square of the resultant velocity.

The flow along any finite curve s drawn in the fluid is the integral $\int_s(udx+vdy+wdz)$ taken along the curve from one end to the other. [That an integral is taken along a curve or over a surface we denote by the suffix s or S placed below the integral sign.] Hence the rate of alteration of the flow along the finite arc s is by the result already obtained

$$\frac{d}{dt}\int_s(udx+vdy+wdz)=\int_s(\dot{u}dx+\dot{v}dy+\dot{w}dx)+\tfrac{1}{2}(q_1^2-q_0^2), \quad (15)$$

where q_1, q_0 are the values of the velocity at the final and initial ends of the line. This is Lord Kelvin's theorem.

If the curve be closed $q_1=q_0$ and the integrated term on the right vanishes. If $\dot{u}dx+\dot{v}dy+\dot{w}dz$ be a perfect differential of a single valued function of the co-ordinates, the conditions for which have been stated in § 359, the integral on the right also vanishes, and the flow along the curve s is constant as the fluid moves.

Lord Kelvin's theorem is expressed by the equation, derived by (15) from § 359,

$$\frac{d}{dt}\int_s(udx+vdy+wdz)=-\int_s\frac{dp}{\rho}-\Omega_1+\Omega_0+\tfrac{1}{2}(q_1^2-q_0^2). \quad (16)$$

The only conditions (besides the condition that Ω shall be one valued) therefore required to prove that the quantity on the left vanishes when the integral is taken round a closed path are that $1/\rho=f'(p)$, and that $\int f'(p)dp$, or $f(p)$, and Ω should be single-valued functions of the co-ordinates.

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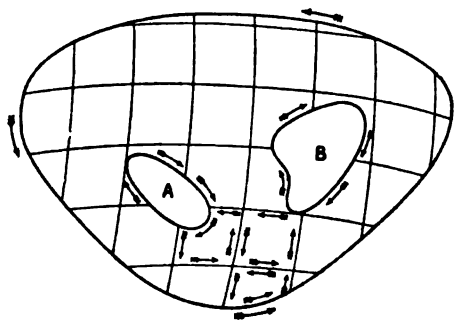
The integral $\int(udx+vdy+wdz)$ taken round any closed curve is called the *circulation* round the curve. Hence Lord Kelvin's theorem for this case asserts that under the conditions stated if the circulation is zero at one instant of time for a closed curve moving with the fluid *it is zero ever afterwards.*

361. **Curl of Velocity. Normal Spin. Circulation round Curve expressed as Surface Integral of Normal Spin.**—Now consider any closed curve moving with the fluid and any surface of which the curve is the bounding edge. Let dS be an element of the surface and ds an element of the curve, and let l, m, n be the direction cosines of the normal to dS drawn towards the positive side of the surface.

The positive side of the surface is defined in the following manner. Let the integral $\int(udx+vdy+wdz)$ be taken round the

FIG. 174.



closed curve in a direction chosen as positive, and then suppose an observer to walk round the curve in the positive direction while the surface lies on his left hand. The observer then looks down on the positive side of the surface.

There are in fact for every element of the surface two associated positive directions, one of traversing the edge, and one of the normal to the element. These are related like the direction of turning and the direction of advance of a right-handed screw.

We can divide any surface bounded by a single closed curve into elements infinitesimal in every direction, by means of lines drawn on the surface. Let this be done and let the boundary of each element in succession be traversed in the positive direction. Each linear element of boundary will in this process be traversed twice but in opposite directions if it belongs to two adjacent elements of surface, and only once if it is part of the boundary of the surface.

If the outer bounding curve enclose regions like A, B, over which the surface integral is not taken, the integral round the boundary is to be extended to the edges of all such regions. Fig. 174 shows

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the direction of the line integration for these portions of the boundary according to the observations stated above.

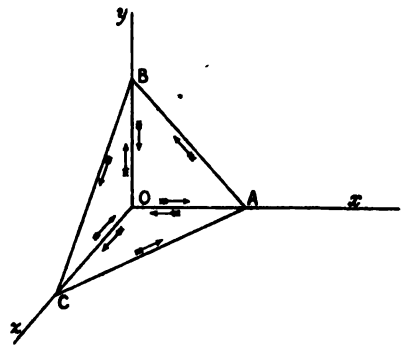
These things being understood, we can now prove the following theorem:

$$\int(udx+vdy+wdz)=2\int(l\xi+m\eta+n\zeta)dS, \tag{17}$$

where the second integral is taken over the surface and the first is taken round the whole bounding edge.

To prove this consider Fig. 175. Let $ABCA$ be a triangular

FIG. 175.



element of the surface, OA, OB, OC lines drawn from a point O to the vertices of the triangle parallel to the axis of co-ordinates. Then we have, going round in the order of the letters A, B, C,

$$\int_{ABCA}\left(u\frac{dx}{ds}+v\frac{dy}{ds}+w\frac{dz}{ds}\right)ds$$

$$=\int_{OABO}\left(u\frac{dx}{ds}+v\frac{dy}{ds}\right)ds+\int_{OBCO}\left(v\frac{dy}{ds}+w\frac{dz}{ds}\right)ds+\int_{OCAO}\left(w\frac{dz}{ds}+u\frac{dx}{ds}\right)ds.$$

If the element be very small, and the lengths of OA, OB, OC be dx, dy, dz we can use the values of u, v, w for the middle points of OA, OB, OC in calculating the three integrals on the right. Take the first of the three, it is evidently

$$\left(u+\tfrac{1}{2}\frac{\partial u}{\partial x}dx\right)dx-\left(u+\tfrac{1}{2}\frac{\partial u}{\partial x}dx+\tfrac{1}{2}\frac{\partial u}{\partial y}dy\right)dx$$
$$+\left(v+\tfrac{1}{2}\frac{\partial v}{\partial x}dx+\tfrac{1}{2}\frac{\partial v}{\partial y}dy\right)dy-\left(v+\tfrac{1}{2}\frac{\partial v}{\partial y}dy\right)dy,$$

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which reduces to

$$\tfrac{1}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)dxdy=n\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)dS,$$

if n be the cosine of the angle which the normal to the area ABC makes with OC, the normal to the area AOB, and dS denote the area of the triangle ABC.

In the same way we obtain the other two integrals and therefore we have for the element ABC.

$$\int(udx+vdy+wdz)=\left\{l\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right)\right.$$
$$\left.+m\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)+n\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)\right\}dS. \qquad (18)$$

Here the positive direction of the normal is from the surface in the direction receding from the point O.

Doing this for the whole surface divided into triangular elements (it is possible so to divide the surface) and adding the line integrals together for one sum and the surface integrals for another, the line integrals along sides which are common to two adjacent triangles cut one another out, and leave only the line integral taken once round the whole bounding edge. Hence by (10) the theorem expressed by (17) is proved.

This is a most important theorem and holds for any directed quantity of which the components are u, v, w. The components $\partial w/\partial y-\partial v/\partial z$, ..., that is 2ξ, 2η, 2ζ, may be taken as components parallel to x, y, z of another directed quantity of which the component along the normal (which is twice what may be called the *normal spin*) is given by the quantity in brackets on the right of (18). These latter components $\partial w/\partial y-\partial v/\partial z$, ..., have been called the components of the *curl* of u, v, w, and the process here illustrated of deriving one directed quantity from another is sometimes called *curling*.

Returning now to (17) we see that if ξ, η, ζ be all zero for any surface enclosed by a curve the circulation is zero round the curve. Hence if it is zero for one surface enclosed by the curve, it must be zero for all such surfaces, that is every such surface must give a zero integral of normal spin.

Now from Lord Kelvin's theorem we have the result that in the circumstances stated in § 359 the circulation, if zero at any one instant, is henceforth always zero. Hence if the surface integral of curl is zero for the surface enclosed by any curve it is henceforth always zero. Since this can be applied to all curves moving with the fluid that can be drawn, it follows that if ξ, η, ζ be once zero for any portion of the fluid they must henceforth always be zero for the same portion of the fluid.

*362. **First Integral of Equations of Motion. Steady Motion.**—Now let a velocity-potential exist and Ω be a single-valued function

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of the co-ordinates for any epoch of time that may be taken: the left-hand side of (16) is $-d(\phi_1-\phi_0)/dt$, so that the equation becomes

$$\left[-\frac{d\phi}{dt}+\int\frac{dp}{\rho}+\Omega_1-\Omega_0-\tfrac{1}{2}q^2\right]_s=0. \tag{19}$$

But $d\phi/dt=\partial\phi/\partial t+\dot{x}\partial\phi/\partial x+\dot{y}\partial\phi/\partial y+\dot{z}\partial\phi/\partial z=\partial\phi/\partial t-q^2$ so that (19) may be written

$$\left[\int\frac{dp}{\rho}+\tfrac{1}{2}q^2+\Omega\right]_s=\left[\frac{\partial\phi}{\partial t}\right]_s. \tag{20}$$

If ϕ does not vary at any particular point in the fluid the co-ordinates of which are fixed, or, as it is usually put in technical language, if ϕ is not an explicit function of the time, $\partial\phi/\partial t=0$, and the motion is then said to be *steady*. The last equation may then be written, with suffixes dropped for the integrals, in the form

$$\int\frac{dp}{\rho}+\tfrac{1}{2}q^2+\Omega=C, \tag{21}$$

where C is the value of the expression on the right of (20) for the initial end of the line. The quantity on the left is the total energy of the fluid at the point considered per unit of mass. Where p is taken in absolute units of force per unit of area $(\int dp/\rho+\frac{1}{2}q^2+\Omega)/g$ is a quantity which is sometimes called the *dynamic head* of the fluid (§ 352). Thus in the case of steady motion there is constancy of dynamic head both at a fixed point in the fluid and at each particle in its motion.

If ρ be a function of p, we have for $\int dp/\rho$ the value $f(p)$ and therefore

$$f(p_1)-f(p_0)+\tfrac{1}{2}(q_1^2-q_0^2)+(\Omega_1-\Omega_0)=0. \tag{22}$$

Thus if the pressure be the same at both ends of the line $f(p_1)=f(p_0)$, and

$$\tfrac{1}{2}(q_1^2-q_0^2)=-\Omega_1+\Omega_0. \tag{23}$$

The interpretation of (22) is that $-\Omega_1+\Omega_0$ is the amount of potential energy in the field of applied force lost by the fluid per unit of mass in passing along the curve from one end to the other: $\frac{1}{2}(q_1^2-q_0^2)$ is the kinetic energy per unit of mass gained by it. The latter is equal to the former plus the work done by fluid pressure on the fluid in the passage. But the fluid on entering the portion of curve considered has work p_0 done on it per unit of volume, and therefore work $f(p_0)$ per unit of mass, and on leaving it does work per unit of its own mass $f(p_1)$ on the surrounding fluid. Or thus: unit mass of the fluid on entering brings energy $\frac{1}{2}q_0^2+\Omega_0$, and has work done upon it $f(p_0)$; unit mass in leaving carries off energy $\frac{1}{2}q_1^2+\Omega_1$ and

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does work $f(p_1)$ and we have $\frac{1}{2}q_1^2+\Omega_1+f(p_1)=\frac{1}{2}q_0^2+\Omega_0+f(p_0)$. The equation expresses thus the conservation of energy in the fluid.

In the case of a liquid for which ρ may be taken as independent of the pressure $f(p)=p/\rho$, and equation (22) becomes

$$\frac{1}{\rho}(p_1-p_0)+\tfrac{1}{2}(q_1^2-q_0^2)+(\Omega_1-\Omega_0)=0. \tag{24}$$

It is to be carefully noticed that (20) does not give the actual value of the pressure at any point. All it does is to give when ρ and Ω are known the difference between the values of p at two points in the fluid. If ρ be uniform the equation is

$$p_1-p_0+\tfrac{1}{2}\rho(q_1^2-q_0^2)+\rho(\Omega_1-\Omega_0)=\rho\left[\frac{\partial\phi}{\partial t}\right]_s. \tag{25}$$

Of course this pressure is not necessarily unknown. It may be experimentally found in most cases.

*363. **Stream-Lines.**—An idea of great importance and utility in fluid motion is that of a *stream-line*, that is a curve drawn in the fluid so that for any given instant the direction of motion of the fluid at any element is that of the tangent to the curve. If ds be an element of length of a stream-line, dx, dy, dz its projections on the axes of x, y, z, the equations of the line are

$$\frac{dx}{u}=\frac{dy}{v}=\frac{dz}{w}=\frac{ds}{q}, \tag{26}$$

or if the velocities are derivable from a potential

$$\frac{dx}{\frac{\partial\phi}{\partial x}}=\frac{dy}{\frac{\partial\phi}{\partial y}}=\frac{dz}{\frac{\partial\phi}{\partial z}}=\frac{ds}{\frac{\partial\phi}{\partial s}}. \tag{27}$$

Hence a stream-line is perpendicular at each point to the surface of equal velocity-potential there. These stream-lines are precisely analogous to lines of force in the cases of gravitational attraction and electrical and magnetic attraction and repulsion. (See chap. x. below.)

The flow along an element of a stream-line $udx+vdy+wdz$ is qds, and hence we have

$$\frac{d}{dt}\int_s qds=\int_s \dot{q}ds+\int_s qdq=\int_s \dot{q}ds+\tfrac{1}{2}(q_1^2-q_0^2). \tag{28}$$

In the case of steady motion we have

$$\frac{d}{dt}(qds)=\frac{dq}{ds}qds+qdq=2qdq,$$

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so that Lord Kelvin's theorem when applied to a stream-line in a fluid in steady motion becomes

$$\tfrac{1}{2}(q_1^2 - q_0^2) = \int_s q ds = -\Omega_1 + \Omega_0 - \int_s \frac{dp}{\rho}. \tag{29}$$

Dropping suffixes for the values of the quantities at the final end of the line, and denoting by C the value of $\tfrac{1}{2}q_0^2 + \Omega_0 +$ *the initial value of* $\int dp/\rho$, we get

$$\tfrac{1}{2}q^2 + \int \frac{dp}{\rho} + \Omega = C. \tag{30}$$

C has thus a value peculiar to each stream-line, but in general has different values for different stream-lines. The equation is the same as that, (21), obtained for any path upon the supposition of the existence of a velocity-potential. This theorem is due to Daniel Bernoulli, and is given in his *Hydrodynamica*, 1738; it is there obtained practically from the consideration of work done on a portion of the fluid in passing along the stream-line from one point to another (see §§ 210, 362).

*364. **Two-Dimensional Motion.**—A very important particular case of motion is that which is independent of one of the co-ordinates, z say; this is commonly called motion in two dimensions. For this case the equation of continuity if the fluid is incompressible is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{31}$$

The equation of a stream-line may be written for two-dimensional motion in the form

$$udy - vdx = o, \tag{32}$$

and the equation of continuity just written is the condition that the velocities u, v are derivable from a function ψ of the co-ordinates (and possibly of t), called the *stream-function*, by the equations,

$$u = -\frac{\partial \psi}{\partial y}, \quad v = \frac{\partial \psi}{\partial x}. \tag{33}$$

The equation of the stream-line may therefore be written in the form

$$\frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy = 0. \tag{34}$$

Equation (34) integrated gives

$$\psi = F(t), \tag{35}$$

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and if the motion be steady this reduces to

$$\psi = C, \tag{36}$$

which is the integral equation of a stream-line in the case of steady motion. This equation may be regarded as giving the whole family of stream-lines if C be regarded as varying from one stream-line to another.

If there exist a velocity-potential ϕ (which of course is also a function of x, y, t in the case of two-dimensional motion) the two expressions for each component velocity give

$$\frac{\partial \psi}{\partial x} = -\frac{\partial \phi}{\partial y}, \quad \frac{\partial \psi}{\partial y} = \frac{\partial \phi}{\partial x}. \tag{37}$$

We get thus the equation of continuity for an incompressible fluid

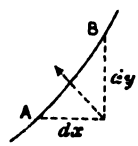
$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0,$$

and the companion equation in ψ

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0.$$

This last expresses the fact that the quantity $\partial v/\partial x - \partial u/\partial y$ (double the angular velocity of an element of the fluid about an axis parallel to the axis of z) is zero, that is that the motion is irrotational.

FIG. 176.



Consider an element ds of an equipotential line; the components of ds on the axes are dx, dy. The flow across ds is (see Fig. 176) $vdx - udy$. But

$$vdx - udy = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = d\psi,$$

that is $d\psi$ measures the rate of flow of fluid across ds (AB in Fig. 176) in the direction from right to left to an observer looking along ds in the positive direction (from A to B in Fig. 176).

The differential equation of an equipotential line is

$$\frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy = 0, \tag{38}$$

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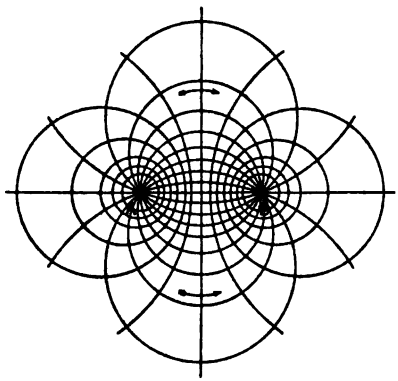
and that of a stream-line is

$$\frac{\partial\phi}{\partial y}dx - \frac{\partial\phi}{\partial x}dy = 0, \tag{39}$$

which show that equipotential lines and stream-lines are families of curves at right angles to one another.

*365. **Conjugate Functions.**—The equations (37) are the analytical conditions that $W, = \phi + i\psi, (i = \sqrt{-1})$ should be a function of the complex variable $z = x + iy$; in other words W changes with z in such a manner that the rate of variation dW/dz, of W with z, is independent of the particular change given to z, that is of the relation of dx to dy. Thus, representing, as in § 27, x by distance along the axis of x from a chosen origin and y by distance along the axis of y, and z by the step OP, where P is the point given by the Cartesian co-ordinates x, y, dW/dz is independent of the direction of the infinitesimal step PQ $(= dz)$ from P to a neighbouring point Q. The proof that equations (37) express this is left to the reader. It is an elementary theorem proved at the outset in treatises on Functions of a Complex Variable.

FIG. 177.



If then we can find a function W of the complex variable z, and if that function be expressed, as it can always be, in the form $\phi + i\psi$, where ϕ, ψ are functions of the real variables x, y, we see that the curves

$$\phi = \text{const.}, \quad \psi = \text{const.} \tag{40}$$

form a *conjugate system*, that is if *either* set of curves be taken as the equipotential lines, the other set is the corresponding system of stream-lines. Hence ϕ and ψ are what are called *conjugate functions*.

In Fig. 177 are shown two such sets of conjugate curves. They may be regarded as (1) the stream-lines and equipotential lines due to two line-sources A, B, one positive, the other negative, that is, one a line-source, the other a line-sink, at right angles to the plane of the diagram. Fluid flows out along the lines diverging from A and converging on B; the curves (circles) closed round the sources are the equipotential curves.

The curves may also be regarded as (2) the lines of flow and equipotential curves for fluid moving without rotation in closed paths round A and B, in opposite directions on the two sides of the central

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line running up and down the page. This central line is a rectilinear line of flow for both A and B. The curves radiating from A and converging to B are then the equipotential curves.

The theory of conjugate functions is dealt with in books on the *Theory of Functions of a Complex Variable* (*e.g.*, the treatise of Forsyth or that of Harkness and Morley). To these and to treatises on *Electricity and Magnetism* we must refer for a full account of the properties of these functions, and the principle of *Conformal Representation*, by which from the solution of one problem the solutions of others may be derived.

366. **Effect of Motion on Pressure. Torricelli's Theorem.**—In § 362 we have found the important result that the pressure at any point is less than it would be if there were no motion by $\frac{1}{2}\rho q^2$, where q is the velocity at the point. An excellent example of this is given by a liquid escaping under gravity from an orifice in a vessel, the free surface of the liquid in which, as well as the jet, is at atmospheric pressure, and in which either through the great cross-section of the vessel in comparison with that of the orifice, or by a continual supply of liquid, the level of the free surface is kept up. Take a path beginning in the free surface and ending in the surface of the jet, then $\Omega = -gz$, where z is measured downwards from some chosen level. Hence if h be the depth of the point on the jet chosen below the free surface we have $q_0 = 0$, $\Omega_1 - \Omega_0 = -gh$, and

$$q^2 = 2gh, \tag{41}$$

where q is the velocity in the jet at the *point on its surface*. This is known as Torricelli's theorem.

It is to be most carefully observed that this tells us nothing of the velocity inside the issuing jet. That depends on the value of p at the point considered.

367. **Velocity in Jet. Vena Contracta.**—Although the velocity at the surface of the issuing jet is given by Torricelli's theorem, this cannot be taken as the velocity for the whole jet. For some distance beyond the orifice within the vessel the stream-lines of fluid are convergent, and this convergence continues in the jet outside the orifice. The velocity inside the jet is therefore less than at the surface, and the pressure is greater. At a little distance in front of the orifice the stream-lines have become parallel, and we may take the velocity as uniform there over the cross-section, and the pressure as atmospheric. The cross-section is smaller than that of the orifice, and the velocity is that $(\sqrt{2gh})$ for the fall h from the surface to the cross-section.

This *vena contracta*, as it is called, has an area depending on circumstances, and the investigation of its amount except in particular cases is impossible. It can however be shown to be less than the area of the actual orifice. For consider the liquid in the vessel with the orifice plugged up. The pressure-force of the liquid on each element of the wall is accompanied by an equal and opposite

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force applied to the liquid by the wall. Thus the force applied to the liquid by one element of the wall may be regarded as balanced by that applied by an opposite element of the internal surface, so that no motion of the liquid takes place.

If now the orifice be opened, a force which is left unbalanced is applied to the liquid by the part of the wall opposite the orifice. If h be the depth and a the area of the orifice, the unbalanced force

FIG. 178.

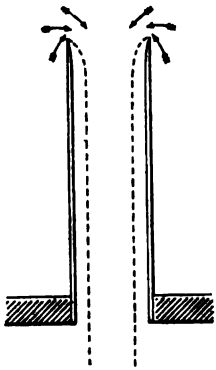

FIG. 178′.

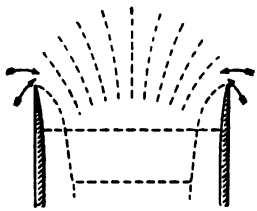

FIG. 179.

applied is $g\rho ah$, or, in gravitation units, ρah. This must be the momentum produced per unit of time in the issuing fluid, if the pressure on every other part of the wall is $g\rho ah$, which will be approximately the case if the region of motion within the vessel is remote from the wall, as when the orifice is formed by a tube with sharp edge passed some way into the liquid, Fig. 178. In this case then we have $\rho a'v^2$, for the momentum of the fluid issuing per unit of time, if a' be the effective area of the orifice, that is the area of the contracted vein, since weight $\rho a'v$ issues with velocity v. This must be $g\rho ah$, and therefore

$$v^2 = \frac{a}{a'}gh. \qquad (42)$$

This has been verified by Borda,* who obtained $a' = a/1{\cdot}924$, with an orifice formed by a re-entrant tube as described.

A concordant result was obtained by the late Mr. W. Froude, of Torquay, and published in the *Proceedings of the Glasgow Philosophical Society*, 1876, in two communications (February 23 and March 31)—one an extract from a letter of Mr. Froude to Sir William Thomson, and the other a note by Professor James Thomson. The accompanying cuts, Figs. 178, 178′, show the flow at a re-entrant orifice with thin edges: the first gives the general nature of the jet, the second the stream-lines near the edge. Fig. 179 shows the

* Mém. de l'Académie des Sciences, 1786. For the theory of the *vena contracta* see also Rayleigh, *Phil. Mag.* 1876, p. 441, or *Collected Papers*, vol. i. p. 297.

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stream-lines near an orifice with thin edge in the vertical side of a vessel.

In most cases the condition stated above is only approximately fulfilled. When, for example, the orifice is one made in the vertical side of a vessel, there is considerable motion near it in the liquid and the pressure is lowered in consequence. The flow therefore corresponds to an unbalanced pressure over an opposite area of the surface greater than that of the orifice, and the vein approaches more nearly to the orifice in area.

There is also in all cases a small diminution of velocity due to friction, so that if c_1 be the ratio a'/a, and c_2 the ratio of the actual velocity of discharge to the Torricellian value, we have for the volume discharged per second $c_1c_2a\sqrt{2gh}$ or $ca\sqrt{2gh}$, if $c = c_1c_2$.

It may be taken that for orifices in the walls or bottom of a vessel, without projecting or re-entrant mouthpiece, c is about ·62.

FIG. 180.

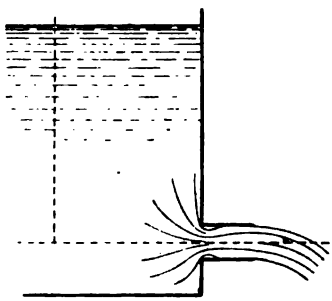
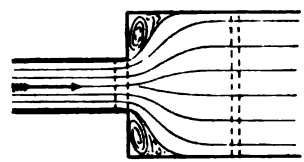
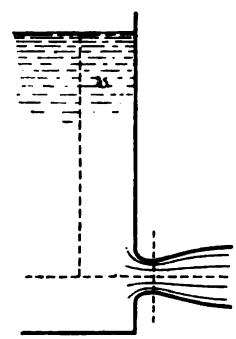


FIG. 181.



368. **Flow through a widening Spout.**—A greater flow is obtained by fitting the orifice with a projecting tube or spout, as in Fig. 180. For a mouthpiece at right angles to the side with a sharp edge of junction the value of c is about ·82. A loss of pressure takes place in consequence of the fact that the pipe just outside the inner mouth is not quite filled with the stream-lines of the liquid. Within this space eddies or whirls are formed, which involve waste of energy of head in work done against friction.

FIG. 182.



A larger rate of flow, in fact one exceeding the higher limit corresponding to the orifice in the vessel, is obtained by making the mouthpiece as nearly as may be of the form of the tube given by the stream-lines at emergence, but with a widening outer end which becomes gradually cylindrical. The loss of pressure just referred to is then avoided, and for a smooth

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well-made mouthpiece of this kind a velocity only about 3 per cent. below the limit is obtained.

The velocity is thus a little less than $\sqrt{2gh}$, but the area of the orifice is really that of the mouth of the tube. The mouthpiece must however be made to run full, that is it must be filled before the flow is allowed to begin. The velocity through the throttled part of the vein near the junction of the mouthpiece with the vessel is of course greater than that at the mouth, and the pressure is correspondingly small. If the area of the narrowest part be made very small, the pressure may fall towards zero, and approach to this value is shown by the jet leaving the surface of the mouthpiece and occupying only part of the outlet-space. For if P be the pressure where the jet has become cylindrical, p be the pressure at the neck, and V, v be the corresponding velocities, we have $P-p=\frac{1}{2}\rho(v^2-V^2)$. But if A, a be the areas, $v/V=A/a$, and therefore

$$P-p=\tfrac{1}{2}\rho V^2(A^2/a^2-1), \text{ or } P-p=g\rho h(A^2/a^2-1).$$

If then

$$P \gtreqless g\rho h\left(\frac{A^2}{a^2}-1\right),$$

$p \lesseqgtr 0$, that is if

$$\frac{A^2}{a^2} \gtreqless \frac{P+g\rho h}{g\rho h}, \tag{43}$$

the pressure at the neck is equal to or less than zero. But it is found experimentally that a liquid containing air bubbles, as water generally does, cannot sustain negative pressure, that is stretching force, and the limit is reached really before A^2 has been made so great that

$$\frac{A^2}{a^2}=\frac{P+g\rho h}{g\rho h}=\frac{r+1}{1}, \tag{44}$$

where r is the ratio of the pressure p to that due to the head h. Thus, if $g\rho h=P$, we have $A^2/a^2=2$, or $A=a\sqrt{2}$.

Thus it is not sufficient in determining the supply of water from a reservoir to consider only the area of the orifice in the wall of the reservoir. The form of the tube beyond must also be considered. The Romans were well aware of this method of increasing the flow through an orifice, and a law had to be passed to prevent its use by persons supplied with water from the public reservoirs.

369. **Experimental Illustrations of Torricelli's Theorem.**—That the square of the velocity of efflux is proportional to the height of the free surface above the orifice in the case here considered is frequently illustrated by the apparatus indicated in Figs. 183, 184. The range of the jet on a horizontal plane below the orifice is measured for various depths of the orifice below the surface and heights above the plane. Let h be the depth of the orifice below the surface, x its

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height above the horizontal plane on which the jet impinges. Let the jet emerge horizontally. Then the horizontal velocity remains v throughout; the vertical velocity of a portion of the jet at any time t after it left the orifice is gt. Hence in time t the horizontal distance travelled is vt, and the vertical distance fallen is $\frac{1}{2}gt^2$. But this is x, and therefore we have $t = \sqrt{2x/g}$. If y be the range on the horizontal plane,

$$y^2 = v^2 2x/g = 4hx,$$

by Torricelli's theorem.

The point at which the jet meets the plane is therefore a point on a parabola of which the orifice is the vertex and of which the latus-rectum is $4h$. Also, if the vertical distance x between the plane and the orifice be fixed, the ranges for different depths h of

FIG. 183.

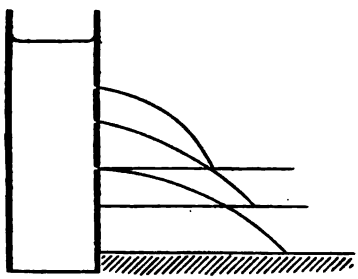
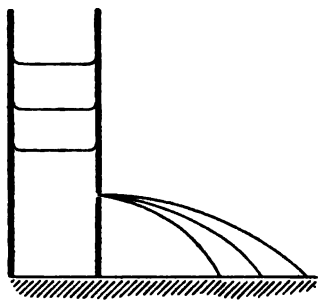


FIG. 184.



the orifice below the free surface are the ordinates of a parabola of which the values of h are the abscissæ and the latus-rectum is $4x$.

The square of the range is thus for given positions of the orifice and the free surface proportional to the distance x of the plane below the orifice, and for a given position of this plane is proportional to the depth h of the orifice below the surface. A vessel is therefore constructed with orifices at different points in the same vertical, and for a certain position of the free surface (which can be kept constant by having a sufficient supply and an overflow or a siphon to prevent the level from rising above the required position), the ranges on the horizontal plane of the jets from these orifices are calculated and marked on the plane, and the result is verified by causing the jets to flow. Fig 183 gives the range for different values of h in the proportion 9, 16, 25, on a plane at the same distance below the orifice in each case. Fig. 184 shows the ranges on the same plane for depths of the orifice in the proportion stated. The figures are taken from Winkelmann's *Handbuch der Physik*.

It will be noticed that if any two values of x and h be taken these may be interchanged, that is the range for a distance x of the plane below the orifice and a depth h of the orifice below the surface is the same as that for a distance h of the plane below the orifice and

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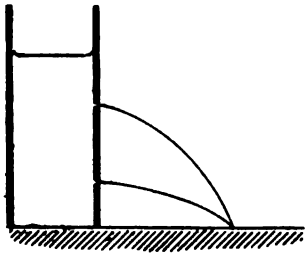
a depth x of the orifice below the surface. This is illustrated by Fig. 185.

*370. **Flow of a Compressible Fluid. Adiabatic Flow of a Gas.**—So far we have considered mainly the case in which ρ is constant. In a gas however the density varies with the pressure and in different ways according to the condition imposed on the fluid. If the temperature is kept constant the density is proportional to the pressure, that is $p/\rho = p_0/\rho_0$. If the heat contained by each portion of the fluid remains constant, that is for *adiabatic* compression or rarefaction, the condition is $p/\rho^\gamma = p_0/\rho_0^\gamma$, where γ is the ratio of the specific heat of the fluid pressure constant to the specific heat volume constant.

FIG. 185.



In the former case we have $\int_s dp/\rho = (p_0/\rho_0)\log p_1/p_0$ where p_0, p_1 refer to the initial and final ends of the line of integration s. Hence (22) is in this case

$$\frac{p_0}{\rho_0}\log\frac{p_1}{p_0} + \tfrac{1}{2}(q_1^2 - q_0^2) + \Omega_1 - \Omega_0 = 0. \tag{45}$$

In the other case

$$\int_s dp/\rho = \gamma(p_1/\rho_1 - p_0/\rho_0)/(\gamma - 1),$$

so that for (22) we have

$$\frac{p_0^{\frac{1}{\gamma}}}{\rho_0}\frac{\gamma}{\gamma-1}\left(p_1^{1-\frac{1}{\gamma}} - p_0^{1-\frac{1}{\gamma}}\right) + \tfrac{1}{2}(q_1^2 - q_0^2) + \Omega_1 - \Omega_0 = 0. \tag{46}$$

If at the initial end of the line s the velocity is zero or negligible and $\Omega_1 - \Omega_0$ be negligible we have

$$q_1^2 = \frac{2\gamma}{\gamma-1}\frac{p_0}{\rho_0}\left\{1 - \left(\frac{p_1}{p_0}\right)^{\frac{\gamma-1}{\gamma}}\right\} = \frac{2\gamma}{\gamma-1}\left(\frac{p_0}{\rho_0} - \frac{p_1}{\rho_1}\right). \tag{47}$$

This equation may be applied to integration along a stream-line in all circumstances, or to integration along *any* line moving with the fluid if the motion is irrotational.

*371. **Steady Flow of a Gas into a Vacuum.**—It will be shown later that the square of the velocity of propagation of waves of voluminal compression and compression through an elastic fluid is equal to the value of $\partial p/\partial\rho$, where $\partial p/\partial\rho$ is the rate of variation of pressure with density taken under the condition of constancy of temperature, or of zero transmission of heat from one part to another, or whatever the condition may be which obtains when the pulse is transmitted. It is known, from the small amount of stifling of

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sound with distance traversed in ordinary cases, that in air and other gases waves of the kind here referred to are transmitted adiabatically. Hence $\partial p/\partial\rho$ must be calculated from the equation $p/\rho^\gamma = C$, so that we have $\partial p/\partial\rho = \gamma p/\rho$. Hence if the velocity of sound at pressure p_1 and density ρ_1 be V_1, and at pressure p_0 and density ρ_0 be V_0,

$$q_1^2 = \frac{2}{\gamma - 1}\left(V_0^2 - V_1^2\right). \tag{48}$$

From the first expression for q_1^2 in (47) it will be seen that $V_1 = 0$ when $p_1 = 0$, and therefore when the gas escapes adiabatically into a vacuum the velocity of efflux is $V_0\sqrt{2}/\sqrt{\gamma - 1}$. The value of γ is for air, oxygen, nitrogen, and hydrogen 1·41 nearly, so that the limiting velocity is $2{\cdot}209\,V_0$.

Here an interesting point arises. The mass of gas flowing from an orifice into a vacuum is $q_1\rho_1 a'$ where a' is (as at § 367) the effective area of the orifice. Hence we have by the value of q_1 just obtained

$$q_1\rho_1 a' = a'\frac{\sqrt{2}}{\sqrt{\gamma - 1}}V_0\rho_1. \tag{49}$$

But since ρ_1 is zero, this would give zero rate of flow of gaseous matter, a result which requires careful consideration. The case of $\rho_1 = 0$ is a limiting one which has no existence in practice. The gas cannot at once expand into the vacuous space so as to fulfil just beyond the orifice the condition $\rho_1 = 0$, but follows for some way stream-lines which finally become lost by viscosity and diffusion of the gas.

Let us suppose that at the orifice the pressure is p and the density ρ. If the flow is steady the amount of matter crossing a section of a tube of flow must be the same at all parts of the tube. The amount per unit area is therefore $q\rho$, and this must be the same at every point in a stream-line. If then a be the area of a tube and M be the rate of flow of fluid across a section, we have $a = M/q\rho$.

Since M is constant the value of a will be least where $q\rho$ is greatest. Now we have for a maximum or minimum of $q\rho$

$$\frac{d}{dp}(q\rho) = \rho\frac{dq}{dp} + q\frac{d\rho}{dp} = 0,$$

or

$$\frac{dq}{dp} = -\frac{q}{\rho}\frac{d\rho}{dp}.$$

But also by (22), since $\Omega = 0$ and $q_0 = 0$

$$q^2 = -2\int_{p_0}^{p}\frac{dp}{\rho},$$

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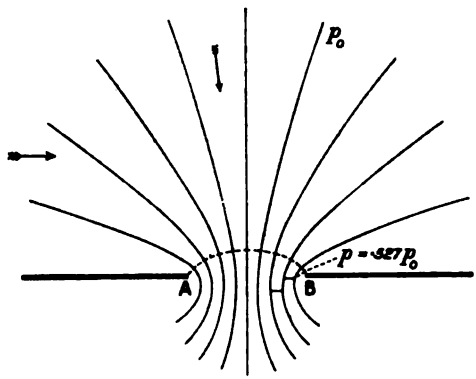
and therefore $qdq/dp = -1/\rho$. Substituting in the value of dq/dp just found we obtain $dq/dp = -1/q\rho = -q/\rho . d\rho/dp$, or $d\rho/dp = 1/q^2$ that is

$$q^2 = \frac{dp}{d\rho}, \tag{50}$$

or the velocity is that of the propagation of sound. When this is the case the cross-section of the stream is a minimum, for clearly this must give a *maximum* value of $q\rho$.

Now in the case of adiabatic flow by (48) $q^2 = 2(V_0^2 - V^2)/(\gamma - 1)$;

FIG. 186.



hence, since for maximum flow along a stream-line we have $q^2 = V^2$, $V^2 = 2V_0^2/(\gamma+1)$ or $p/\rho = p_0/\rho_0 . 2/(\gamma+1)$. This last result, with $p/\rho^\gamma = p_0/\rho_0^\gamma$ gives

$$\frac{\rho}{\rho_0} = \left(\frac{2}{\gamma+1}\right)^{\frac{1}{\gamma-1}}, \quad \frac{p}{p_0} = \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}},$$

or if $\gamma = 1{\cdot}41$, $\rho = {\cdot}635\rho_0$, and $p = {\cdot}527p_0$.

The conclusion is that if p_1, the pressure in the receiving chamber, be less than this, the stream, after coming to a minimum width, widens again somewhat, as shown in Fig. 186. The curve drawn normally across the streams from A to B is evidently concave outwards, and therefore the narrowest parts of the tubes must be on the outward side of that curve, as indicated by the cross-lines.

The pressure in the narrowest part of each stream-tube cannot be less than ${\cdot}527p_0$, nor ρ less than ${\cdot}635\rho_0$. Outside this the streams must widen and the velocity increase since ρ diminishes.

If, however, the pressure p_1 outside the orifice be greater than this the flow will be in parallel tubes at the narrowest part and the pressure will be p_1.

When the velocity of flow is greater than V no disturbance of pressure can be propagated back along the outflowing gas, since this

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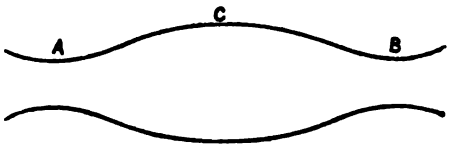
would take place at velocity V, so that diminution of pressure below $\cdot 527p_0$ cannot affect the pressure behind the contracted vein.

The discussion of this case is due to Professor Osborne Reynolds, *Phil. Mag.* March 1886. The results agree with experiments made by Mr. R. D. Napier, *Engineer*, 1867, January 4 and 25.

For adiabatic flow of a gas there are thus two values of the pressure for a value of the cross-section of the tube greater than its minimum value, one greater than $\cdot 527p_0$ and one less. Hence there are two possible values of the pressure at C (Fig. 187) in a tube of

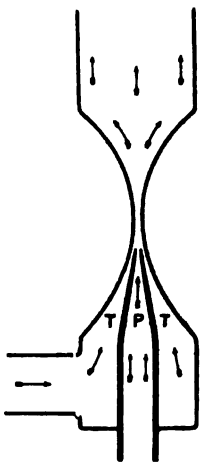
FIG. 187.



varying section for which the pressure at the two equal necks A, B is $\cdot 527p_0$. For a liquid the pressure at C is *greater* than at A or B.

*372. **Jet-Pump.**—Fig. 188 shows the action of Professor James Thomson's jet-pump. A jet of water is driven at high velocity through the converging mouthpiece P, which opens at a point just a little short of the most constricted part of the surrounding tube T. This tube is connected by a side branch with the space from which the water is to be drawn. The flow of the water along the constricted neck of T gives by Bernoulli's theorem a lower pressure at the constriction than at the part beyond, and there is consequently a flow of fluid from the space surrounding and behind the nozzle. The increasing pressure from the constricted part of T forwards is employed in retarding the fluid flowing into it.

FIG. 188.



The pump may be employed of course to transfer one fluid from the space by means of a jet of another fluid applied by P. In this case a mixture of the two fluids is delivered by T to the delivery pipe. We shall suppose that the average pressure of the fluid in the nozzle is p_1, in the surrounding pipe p_2, and in the delivery pipe p, while the densities of the fluids and their velocities are ρ_1, ρ_2, ρ, v_1, v_2, v respectively, while the areas of delivery in the three cases are a_1, a_2, a. The momentum carried in by the jet in unit of time is $\rho_1 v_1^2 a_1$, and the rate at which momentum is generated by the pressure is $p_1 a_1$; similarly in the other cases. The rate at which momentum is delivered by the jet and the surrounding pipe is therefore $(p_1 + \rho_1 v_1^2)a_1 + (p_2 + \rho_2 v_2^2)a_2$.

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The momentum delivered per second across a cross-section of the delivery tube is $(p+\rho v^2)a$, and we have

$$(p_1+\rho_1 v_1^2)a_1+(p_2+\rho_2 v_2^2)a_2=(p+\rho v^2)a. \tag{51}$$

Besides this we have the equation of continuity, namely, the condition that as much matter is delivered as flows from the two jets. This condition is

$$a_1\rho_1 v_1+a_2\rho_2 v_2=a\rho v. \tag{52}$$

Of course the engineering student will understand that here and elsewhere, if pressure is measured in pounds per square foot or square inch, the p in the formula as written above is g (that is 32·2) times the pressure thus reckoned, and equation (51) will have the form

$$\left(p_1+\frac{\rho_1 v_1^2}{g}\right)a+\left(p_2+\frac{\rho_2 v_2^2}{g}\right)a_2=\left(p+\frac{\rho v^2}{g}\right)a. \tag{53}$$

Thus if p_1, p_2, p be taken in pounds per square foot, v_1, v_2, v in feet per second, ρ_1, ρ_2, ρ will be taken in pounds per cubic foot, and a_1, a_2, a in square feet. The quantities p_1/ρ_1, p_2/ρ_2, p/ρ will be the heads in feet corresponding to the three pressures.

*373. **Numerical Example on Action of Jet-Pump.**—We take an example from Professor Perry's treatise on "The Steam Engine" (to which the reader may very profitably refer for further practical examples on this and similar subjects). Let the rate of flow in the jet be 1 cubic foot per second under a head of 60 feet, and let it mix with 6 cubic feet drawn in the same time through the surrounding tube T from a tank, the surface of the water in which is at the same level as the jet: it is required to find how high the mixed jet will rise with atmospheric pressure above it. All frictional loss in addition to that involved in the mixing is to be neglected, and the pressure at the nozzle is to be taken as 9/34 of an atmosphere.

Taking the weight of a cubic foot of water as unity, and as the unit of pressure the gravity of a cubic foot of water (that is 62·5 pounds) per square foot we obtain $v_1^2/2g=94-9=85$, $v_2^2/2g=25$, so that (if $g=32{\cdot}2$) $v_1=74$, $v_2=40$ (in feet per second). But $v_1a_1=1$ and $v_2a_2=6$, so that $a_1=1/74$, $a_2=3/20$, and $a=1/74+3/20=121/740$ (in square feet). Hence $v121/740=7$ and $v=5180/121$, in feet per second.

If then p be the average pressure where the streams have become completely mixed we have

$$\left(p+\frac{v^2}{g}\right)\frac{121}{740}=(9+170)\frac{1}{74}+(9+50)\frac{3}{20},$$

$$p+\frac{v^2}{2g}=\frac{740}{121}\left(\frac{179}{74}+\frac{177}{20}\right)-\frac{5180^2}{121^2\times 64{\cdot}4}$$

$$=69-28{\cdot}458=40{\cdot}542.$$

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Since the water is driven out against atmospheric pressure, taken as equivalent to a head of 34 feet of water, the height to which the pump can raise the water above the jet in the circumstances stated is $40{\cdot}542 - 34 = 6{\cdot}542$ in feet.

*374. **Variation of Pressure in Horizontal Tapering Pipe.**—It is important that the physical reader should consider what happens in as many practical cases as possible. We therefore give here a few cases of variation of fluid pressure,* which may help to correct preconceived and erroneous ideas. Fig. 189 shows discharge of a liquid

FIG. 189.

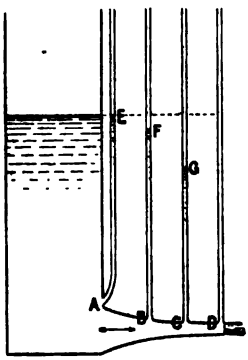
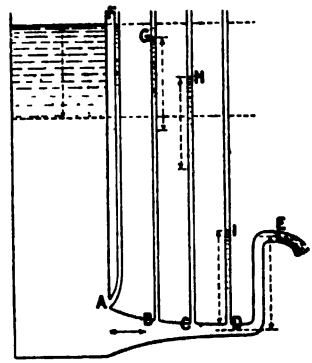


FIG. 190.



from a vessel by a very gradually tapering tube, which at the extremity of the nozzle is nearly straight. The variation of pressure is shown by glass tubes kept filled by the fluid to heights which indicate the pressures at the points at which the tubes are attached. Let us suppose that at D the orifice has half the area that the cross-section of the mouthpiece has at C, that the latter area is half the cross-section that the mouthpiece has at B, and that at A the area of cross-section is so great that there is no sensible motion.

At D the pressure is atmospheric, at C the velocity is half that at D, and therefore the pressure p at C is given by the relation $p + \frac{1}{2}(q/2)^2 = P + \frac{1}{2}q^2$; and since $q^2 = 2gh$ where h is the height of the level of the liquid above the orifice, we have $p = P + \frac{3}{4}g\rho h$. The gauge-glass at G shows the difference of pressures $\frac{3}{4}g\rho h$. Thus the motion has brought down the pressure from the static value that would exist at C by $\frac{1}{4}g\rho h$. Similarly the pressure at B is greater than that at C by $\frac{3}{4}$ of the amount $\frac{1}{4}g\rho h$ which that at C is below the static value, and so on, until the great width at A is reached, where the diminution of pressure is not sensible.

Another case is that shown in Fig. 190. There the pipe is carried up and bent horizontally. The pressure at the nozzle is again atmospheric, and the velocities are the same, so that the

* Mostly taken from the Address of Mr. W. Froude, Brit. Assoc. Rep. 1875.

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pressure p at the lower horizontal part of the mouthpiece at D is by Bernoulli's theorem connected with that at E by the relation $p - g\rho(H+h) = P - g\rho H$, where H is the vertical distance of the orifice E below the level of the liquid, and h is the further distance ID. Hence $p = P + g\rho h$. Then, just as before, the vertical heights HI, GH, ..., correspond each to $\frac{3}{4}$ of the static pressure corresponding to the depth of the orifice below the free level of the liquid.

The flow is the same, in fact, as it would be if the mouthpiece were opened at D and the height of the free surface were brought down by the distance ID.

FIG. 191.

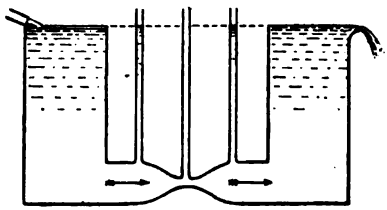
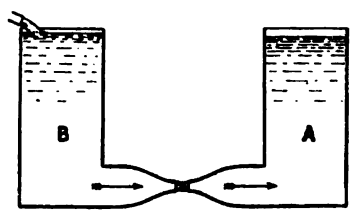


Fig. 191 shows two vessels connected by a channel at the bottom of a double conoidal form so that it gradually narrows at the middle to a cylindrical neck, and then widens again in like manner. The flow in the cylindrical neck is that due to the loss of head, and the excess of pressure above atmospheric shown by the gauge-glass there is zero. At the very wide parts of the channel where the velocity is insensible, the excess pressure shown is that due to the head.

Liquid is supplied to one vessel, and flows out of the other at the same rate; and the level (save for a little loss due to friction) is the same in both vessels.

FIG. 192.



The following experiment, made by Mr. Froude, is also interesting. Two vessels are made like those which would be formed by cutting the cylindrical neck of the connecting pipe across the middle, so that they have two conoidal mouthpieces. The two nozzles are placed opposite one another at the same level and a little distance apart, as shown in Fig. 192. When both vessels are filled up to the same level, and the orifices are then opened, both vessels discharge, and the two jets meeting form a plane sheet, at right angles to the horizontal line joining the centres of the nozzles. As one vessel, A, empties rather more rapidly than the other, B, the jet from B gains the preponderance and the disk formed by the meeting fluids moves towards A. The flow from A is diminished, while that of B is increased; and the level of B falls more rapidly, and the opposite state of motion of the disk takes place, and so on. There is thus an oscillation of the disk of meeting fluid between the two orifices.

If the level in B, say, be adjusted by an overflow to be less than that at A by a sufficient amount, the jet from A will enter B, and the level in B will be maintained if a supply to A makes good the

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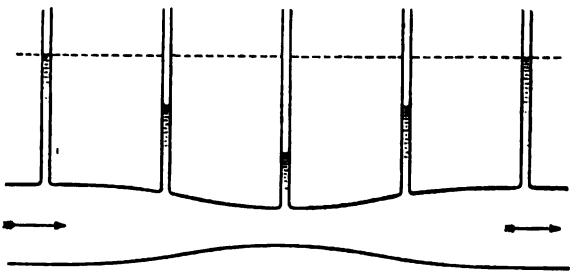
loss of level, so that the flow between the mouthpieces is kept at a constant value.

In Mr. Froude's experiment the recipient vessel B had a head of 18 inches above the orifices, while A had a head of $20\frac{1}{2}$ inches. The difference, $2\frac{1}{2}$ inches, represents the head employed in overcoming friction.

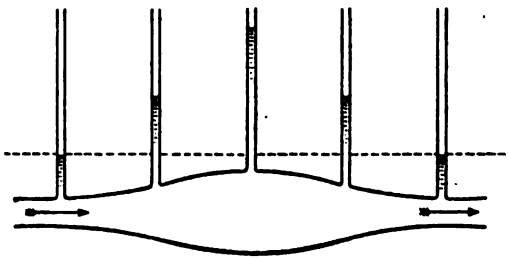
375. **Flow along Pipes of Variable Section.**—The variation of pressure in tubes of non-uniform cross-section is illustrated by

FIG. 193.



Figs. 193, 194, which show respectively a widening and a constriction of a horizontal pipe running full of liquid in steady motion. The gauge-tubes show, by the height of the liquid standing in them, the variation of pressure. Thus the flow is faster where the tube is narrower, and slower where the tube is wider than the average;

FIG. 194.



and consequently the pressure is greater in the wider parts and smaller in the constricted parts than the average.

That the contrary is the case is a very common opinion, but its falsity is easily seen by the most elementary considerations. When a portion of the liquid is passing from a wider to a narrower part of the pipe, Fig. 193, running full, it must undergo acceleration, since in steady motion the same quantity passes all sections in the same time. The pressure must therefore on any part be greater behind than in front. Thus the pressure must continually diminish along a part of the tube.

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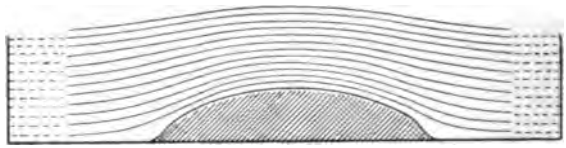
Similarly when the fluid is moving from a constricted path to a wider there must be retardation; that is, the pressure must be greater in front than behind on any part of the liquid.

It is also not an uncommon idea that fluid flowing along a pipe, like that in Fig. 194, will tend to push the pipe before it. This is, of course, an entire mistake, as will be understood from the fact that the pressure on the tube is at right angles to its sides, and does not depend on the direction of motion.

376. **Stream-Line Flow Past an Obstacle. Steady Motion of Solid in Fluid.**—Another common notion, which however has more foundation in experience, is that any obstacle whatever in a stream is pushed in the direction of flow. This is not the case in a perfect fluid if the body be of such a shape (see Fig. 195) as will allow of continuous stream-line motion of the fluid past it. If the fluid is

FIG. 195.



viscous there is always, whatever the shape of the body and whether or not it is completely submerged, a drag on the body due to what is called skin-friction, which tends to carry the body with the stream; and if there are discontinuities of curvature eddies or whirls of the fluid are produced.

The stream-lines of the fluid past one side of a submerged obstacle are shown in Fig. 195. It will be seen that they are crowded together midway between the ends, and are wider apart beyond the obstacle both in front and behind, and that at each of these places they approach to coincidence with the undisturbed stream-lines, shown by the dotted straight lines. They are also wider apart than the average just after they have begun to change from the straight course, and just before they resume it.

There is therefore a diminution of pressure below the average opposite the middle of the body on both sides, and an increase of pressure in front of and behind the body. As a result of this, when a ship moves through the water the water rises above the undisturbed level at the bow and stern and falls below it amidships.

Since the stream-lines at points at a considerable distance from the body, up stream and down stream, are undisturbed by the presence of the body, whatever forcive (system of forces) is applied to the body on the up-stream side, in consequence of the deviation of the stream-lines, is balanced by an equal and opposite forcive due to the resumption by the fluid of the undisturbed course.

From this we can pass to a submerged body of fair shape moving

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uniformly through a perfect fluid. It is only necessary to impose on the fluid and body a motion equal and opposite to that of the undisturbed fluid in the case just considered. The fluid will be brought to rest at some distance in front and behind; the body will move with a speed equal to that of the fluid in the former case, and the actions between the body and the fluid will not be altered. Hence the body will experience no resistance to its motion.

Sir George Stokes has shown (*loc. cit.* below) that a viscous liquid flowing between parallel plane walls in a thin film round an obstacle has stream-lines identical with those of the two-dimensional steady motion of a perfect liquid round an infinite cylinder of cross-section represented by the obstacle. In this way Professor Hele Shaw has shown the stream-lines of a perfect liquid (*Brit. Assn. Rep.* 1898).

In water or other actual fluids the body experiences a drag exerted by the fluid on its surface. This is called skin-friction, and is the chief resistance experienced by a body of good lines in passing through the fluid. The friction causes a deviation of the stream-lines from the forms they would have in a perfect fluid, and sets up eddies in a thin stratum of the fluid at the surface of the body. Also if the body is of a bad shape there is a formation of eddies or whirls of the fluid where the stream-lines fail to close in round the body. There is in this case great differences of velocity between different parts of the fluid, and consequently great loss from friction.

377. **Wave-Resistance to Motion.**—A completely submerged boat is free from another serious cause of loss of energy which affects many ordinary vessels. In the case of the latter there is, as has been noted above, a rise of the level of the water at the bow and stern, and a depression amidships according to the excess and defect of pressure already described. These changes of level form the origin of waves which spread out from the position of the ship in all directions along the surface, and are gradually wiped out by fluid friction or dissipated by breaking on the shore. Hence the ship in its motion is a source from which energy is radiated in wave motion, and this energy is supplied by the motive power of the ship.

There is therefore a great deal to be said, on the ground of economy of power, in favour of submarine vessels, or for vessels which disturb the free surface as little as possible: of course the objections on other grounds are obvious.

*378. **Resistance to Flat Disk Moving through Fluid.**—Though we never have to deal with perfect fluids, there is in an ordinary fluid a considerable force exerted on a body of unfavourable shape (*e.g.*, the disk shown in Fig. 197) which tends to carry it down stream, and which cannot be put down to the action of friction. According to a theory worked out by Kirchhoff and Lord Rayleigh, the motion is really discontinuous, so that (eddies neglected) a region of dead water is formed behind the obstacle in which the pressure is the same as that in the moving fluid at a considerable distance behind. This region is separated from the rest of the fluid by a surface tangential to the edges of the plate, and is

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determined by the condition that the pressure is the same at every point of it as in the stream at a distance behind. Fig. 197 shows the form of this separating surface near a long blade with parallel edges immersed as described below.

The fluid is thus brought to rest, or nearly so, just in front of the centre of the disk, and there is an excess of average pressure on the front of the disk above that behind, which pushes the disk down stream. The action is thus very similar to that of the impact of a jet on a disk,

FIG. 196.

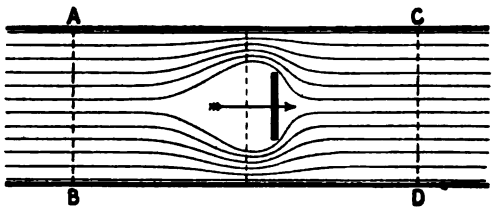
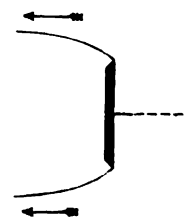


FIG. 197.



which is the view frequently taken in the attempts made to reckon the effect quantitatively.

If the motion were perfectly continuous there would be no such force tending to carry the body down stream, but the continuity of the motion would require infinite velocity at a sharp edge of the obstacle, and the pressure there would be of infinite negative value, that is, the fluid would be subject at the edge to infinite tension. Of course a perfectly sharp edge cannot be obtained, but a considerable falling short of perfect sharpness would result in tension far greater than the fluid in ordinary circumstances (for example, water containing small air-bubbles) could bear.

*379. **Rayleigh's Theory of Resistance to Motion of Plate in Fluid.**—According to Lord Rayleigh's theory (see *Phil. Mag.* Dec. 1876, to which the reader must refer for the analytical proof), if a long plane lamina with parallel edges be held in a stream so that the angle between its plane and the direction of flow is α, and the edges are perpendicular to the direction of flow, the mean unbalanced pressure on the body is $\rho V^2 \pi \sin\alpha/(4+\pi\sin\alpha)$ where V is the undisturbed velocity of the fluid in the stream. Thus if $\alpha = 90°$, this becomes $\pi\rho V^2/(4+\pi)$.* This multiplied by the area of the lamina is the force down stream.

According to the ordinary theory of a jet in which the velocity of the fluid is supposed to be destroyed by impact on the anterior face of the plate the force would be $\frac{1}{2}\rho V^2 \times$ area. Now $\pi/(4+\pi) = \cdot 44$ nearly, so that the force is $\cdot 44\rho V^2 \times$ area, which shows that the ordinary view gives in this case a fair approximation.

* This result is also given by Kirchhoff, *Vorles. ü. Math. Physik., Mechanik*, p. 308. It ought to be stated here that the application of this theory to actual fluids has been forcibly objected to by Lord Kelvin (*Nature*, 50, 1894), who quotes experiments by Dines (*Proc. R.S.* June 1890) which give much greater resistance.

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It will be observed that if the plate is oblique to the direction of flow the formula gives a transverse as well as a down-stream component of force. The average force per unit area of the plate

$$\rho V^2 \pi \sin\alpha / (4 + \pi \sin\alpha)$$

is perpendicular to the plate, and has therefore components in the directions specified of amounts

$$\rho V^2 \pi \sin\alpha \cos\alpha / (4 + \pi \sin\alpha), \ \rho V^2 \pi \sin^2\alpha / (4 + \pi \sin\alpha).$$

It will be observed that the transverse component has a maximum value for $\alpha = 39^\circ$ nearly.

These results are of course equally applicable to a plate moving steadily through a fluid at rest everywhere at a distance from the plate. It is only necessary to impress on the fluid and plate a velocity equal and opposite to the fluid. The force on the plate will not be altered, and we have the case of a plate moving through fluid otherwise at rest.

A very remarkable conclusion follows from these results, and may be very easily verified qualitatively by the reader, by the following simple experiment suggested by Mr. Froude. Standing in a boat moving through the water, immerse the blade of an oar vertically in the water, so that the blade is broadside on to the direction in which the boat is going, and make a mental estimate of the muscular effort required to overcome the resistance to the motion. Now without changing the position in which the blade is held, move it to and fro in the horizontal transverse direction. It will be found that the resistance offered to the forward motion has been sensibly increased by the transverse motion.

The explanation is furnished by Lord Rayleigh's theory of the force on the plate held obliquely in the stream. Let the plate supposed held at right angles to the stream have a velocity v in its own plane. If an equal and opposite velocity be supposed imposed on both the fluid and plate, the plate will be brought to rest and the fluid will have a resultant velocity $\sqrt{V^2 + v^2}$ in a direction inclined at the angle $\alpha = \sin^{-1} V / \sqrt{V^2 + v^2}$ to the plate. Hence the average force at right angles to the plate is

$$\frac{\pi \sin\alpha}{4 + \pi \sin\alpha} \rho (V^2 + v^2) = \frac{\pi V}{\pi V + 4\sqrt{V^2 + v^2}} \rho (V^2 + v^2). \qquad (54)$$

The force at right angles to the blade is therefore altered in the ratio $(4 + \pi)(V^2 + v^2) / V(\pi V + 4\sqrt{V^2 + v^2})$.

This increase of broadside-on resistance due to motion of an elongated body edgeways through water is, as Mr. Froude has also pointed out, illustrated by the diminution of leeway made by a vessel produced by increase of her speed to windward. Thus he says (*loc. cit.*):

"When a vessel was working to windward, immediately after she

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had tacked, and before she had gathered headway, it was plainly visible, and it was known to every sailor, that her leeway was much more rapid than after she had begun to gather headway. The more rapid her headway became the slower became the lee-drift, not merely relatively slower, but absolutely slower."*

The place of maximum pressure is where the stream divides in front of the plate, and therefore when the plate is perpendicular to the stream lies midway between the edges. Not so, however, when the plate is inclined to the stream: it is easy to see that the place of division is then nearer to the upstream edge than to the other. The analysis shows, in fact, that the resultant of the parallel forces on the plate acts at a distance $3l\cos\alpha/4(4+\pi\sin\alpha)$ from the middle of the plate, where l is the breadth of the plate.

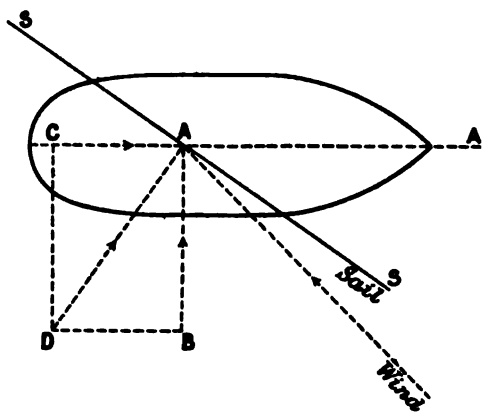
Hence if α be zero, that is, if the plate be edge-on to the stream, this distance is $3l/16$. If then the plate be pivoted about a vertical axis in its plane at this distance from the middle, the plate will be in stable equilibrium in the position in coincidence with the stream with the edge nearer to the axis up-stream. If the vertical axis be at a greater distance from the middle the plate will be in stable equilibrium in the same position. If the vertical axis be at a less distance, x, from the middle, the position parallel to the stream is unstable, and there are two positions of stable equilibrium equally inclined to the stream given by

$$x = 3l\cos\alpha/4(4+\pi\sin\alpha). \tag{55}$$

Lord Rayleigh points out that the force per unit area may be

* This will be understood from the diagram, Fig. 198. *SS* shows the direction of the sail for a ship sailing in the direction *AA*. The wind gives a component of force *DA* perpendicular to the sail which resolves into two

FIG. 198.



components, one *CA* in the direction of motion, the other *BA* at right angles to that direction The motion produced by the latter is called *leeway* or *lee-drift*.

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made accurately $\frac{1}{2}\rho q^2$, if the obstacle be in the form of the bottom of a box, at right angles to the stream, with two vertical sides projecting up the stream so far that the fluid is very nearly at rest at the surface of the obstacle.

*380. **Stable Position of Moving Plate.**—If $x = 0$ the plate is stable only when broadside on to the stream. To verify this, Lord Rayleigh has suggested a simple experimental illustration, which any one can carry out. A slip of brass is pointed at the ends and mounted with its points in two opposite indentations formed by a centre punch in the inner surface of the limbs of a U-shaped strip of thicker material.

The strip when moved through water will set itself perpendicular to the direction of motion.

It is an objection to the kind of motion here assumed, namely, that of a region of non-moving fluid separated from the moving fluid by a surface of discontinuity, that such a mode of motion would be unstable. For let any swelling of the surface into the region of moving fluid take place. Into that space is thus introduced an increase of pressure which would tend to spread the swelling further, and the deviation of the surface from the equilibrium form determined by the condition of constant pressure would be augmented. Lord Rayleigh however doubts whether the value of the resistance would be materially affected by this, since the instability would begin only at some distance behind the plate, and the pressures in front would be only slightly affected.

The results are confirmed by some old experiments of Vince, published in the *Phil. Trans.* for 1798. The common theory gives the force for an inclined lamina as proportional to $\sin^2\alpha$ instead of to $\sin\alpha/(4+\pi\sin\alpha)$ as in Lord Rayleigh's theory. The following table taken from Lord Rayleigh's paper gives a comparison of numbers proportional to the resistances obtained by Vince with those obtained by calculation from the formula here given;

α	$\sin^2\alpha$	Vince's Numbers.	$\frac{\sin\alpha(4+\pi)}{4+\pi\sin\alpha}$
90°	1·0000	1·000	1·0000
70°	·8830	·974	·9652
50°	·5868	·873	·8537
30°	·2500	·663	·6411
20°	·1170	·458	·4814
10°	·0302	·278	·2728

The agreement of the fourth column with the third is very good; there is really no agreement between the second and the third.

The investigation of the resistance in this case of motion is of the highest importance for the theory of aëroplanes and the flight of birds.

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*381. **Liquid Revolving with Uniform Angular Velocity. Forced Vortex.**—A case in which stream-lines exist is that of a liquid revolving in steady motion about a vertical axis under gravity. This is known as a *forced vortex*, and like the "free vortex," to be considered presently, is of great importance in the theory of turbines and water motors.

A line of particles moving with the fluid is one drawn on the free surface at right angles to the stream-lines at all points. The flow along that line is zero, and consequently the left-hand side of (16) is zero. Hence we have by (16)

$$-\int_s \frac{dp}{\rho} - \Omega_1 + \Omega_0 + \tfrac{1}{2}(q_1^2 - q_0^2) = 0. \qquad (56)$$

Let the line begin at the intersection of the axis with the free surface and end at a stream-line of radius r. The velocity at the axis is zero, at the final end $q_1^2 = \omega^2 r^2$. Also since $dp = 0, \int dp/\rho = 0$, and we have

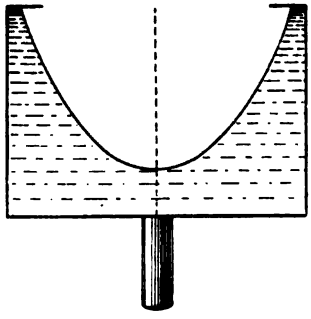
$$\Omega_1 - \Omega_0 = \tfrac{1}{2}\omega^2 r^2.$$

Now Ω, the potential energy of the liquid per unit volume, may, as we have seen, be taken as $-gz$ where z is the distance of the element considered below a fixed horizontal plane. Let h be the distance of the intersection of the axis with the surface below the plane chosen, and z the distance below the same plane of the element considered. We have $\Omega_1 - \Omega_0 = g\,(h - z)$. Thus we have

$$\omega^2 r^2 = 2g(h - z). \qquad (57)$$

This relation between r and z represents a paraboloid of revolution, which is therefore the form of the free surface. If we write x' for $h - z$ its equation is

FIG. 199.



$$r^2 = \frac{2g}{\omega^2}x'. \qquad (58)$$

The latus-rectum of the paraboloid is thus $2g/\omega^2$. A section of a paraboloid is shown in Fig. 199; and gives the form of the free surface at a distance from the side of the revolving vessel.

This case may be treated in the following more elementary manner, as one of equilibrium of a rotating body. Let A, B, Fig. 200, be sections by the paper of two surfaces of equal pressure. Consider a small rectangular element with faces in the two surfaces, and therefore the other two pairs of sides perpendicular to them. For equilibrium, the difference of pressures dp must supply the vertical

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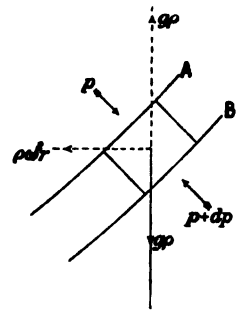
force necessary to balance that due to gravity, and the horizontal force required to give the acceleration $\omega^2 r$ towards the axis; or, as it is usually put (when the case is reduced to a purely statical one by the introduction of a centrifugal force $m\omega^2 r$ from the axis), the difference of pressures must be equal and opposite to the resultant of gravity and centrifugal force.

The acceleration at right angles to the surfaces A, B, is the equilibrant of $\omega^2 r$ outwards and g downwards, and therefore the tangent of the inclination of the section of A or B to the horizontal is $\omega^2 r/g$, which, since z is taken positively downwards, is $-dz/dr$. Hence integrating we obtain $-gz = \frac{1}{2}\omega^2 r^2 + C$. But at the axis $r = 0$ and $z = H$, say. Hence $C = -gH$. Therefore the integral equation of a surface of equal pressure is

FIG. 200.



$$2g(H - z) = \omega^2 r^2,$$

which agrees with the result already obtained.

The rate of variation of pressure along r is evidently given by $dp/dr = \rho\omega^2 r$, and therefore

$$p = \tfrac{1}{2}\rho\omega^2 r^2 + C, \tag{59}$$

where C is a constant. This shows how pressure increases with distance from the axis, and is, of course, merely the result already obtained in equation (56), since for variation in a horizontal direction Ω is a constant.

If p be measured in gravitation units (as in pounds per square foot), the above equation is

$$p = \tfrac{1}{2}\frac{\rho}{g}\omega^2 r^2 + C, \tag{60}$$

where ρ is the weight of liquid in a unit of volume (*e.g.*, pounds per cubic foot), and g is the acceleration due to gravity (*e.g.*, 32·2 foot-second units).

For the direction of the normal to the surface of constant pressure, that is of the resultant of gravity and centrifugal force, we have $dz/dr = g/\omega^2 r$, or

$$z = \frac{g}{\omega^2}\log r + C', \tag{61}$$

where C' is another constant.

To find the pressure for a point within the liquid, draw a path radially into the liquid from any surface stream-line, and let r now denote the distance from the axis of the stream-line arrived at. If p be the pressure at the final point, we have as in (56)

$$p - P + g\rho(H - z) = \tfrac{1}{2}\rho\omega^2 r^2. \tag{62}$$

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This result is really that expressed in (59), which was obtained in another way.

The surface of equal pressure is still a paraboloid of revolution about the axis of rotation, and it is easy to see that it differs from the former only by a change of position of the vertex, which is at a distance $(p - P)/g\rho$ below that of the surface paraboloid.

382. **Plummet Immersed in Forced Vortex.**—It is interesting to consider the position of equilibrium of a plummet hung from a point on the axis of such a vortex, and having its bob immersed in the liquid and turning with it. By the discussion in last Article the plummet will be in equilibrium if the thread be perpendicular to the surface of equal pressure on which it lies. Clearly it will not be in equilibrium unless this is the case, as may be seen by the fact that the centrifugal and gravity forces are in the ratio $\omega^2 r/g$, and their resultant makes an angle with the axis the tangent of which has this value.

The pull applied by the cord must balance this resultant, less what is balanced by the immersion of the bob in the liquid.

If the plummet-cord be vertical, and the length of the cord be greater than the radius of curvature g/ω^2, at the vertex of the paraboloid on which the bob lies, the plummet if deflected will find itself in a position in which the cord makes a smaller angle with the vertical than does the normal to the paraboloid on which the bob now lies. The string will not supply the necessary horizontal force $\omega^2 r$, and the cord will move farther off from the vertical. The plummet is in this case in unstable equilibrium. If the length be less than g/ω^2 the equilibrium will be stable.

If the bob be specifically lighter than the liquid the cord will have to be moored to a point of the revolving vessel below the surface, and the plummet will be turned in the opposite direction. In this case it will be stable when in the axis.

It will be shown in connection with the Principle of Archimedes that if w be the weight of the plummet bob, s its specific gravity, its weight in water is $w(1 - 1/s)$. The downward vertical force, and the centrifugal force (relative to the fluid exerted by the bob on the fluid in which it is immersed) are $wg(1 - 1/s)$ and $w(1 - 1/s)\omega^2 r$. The latter force is negative when $s < 1$, and so the bob tends to move towards the lowest point. Thus, as every one has observed, a body floating in a whirlpool moves towards the lowest part as a place of equilibrium.

383. **Forced Vortex of several Non-Mixing Liquids.**—If the fluid consists of different non-miscible parts of different densities, it will arrange itself in a forced vortex in a series of strata increasing in density from stratum to stratum from the free surface into the fluid. That this will be the case is evident from § 381. The pressure-force, which applies at a given radius and level the required centre-ward acceleration to an element of fluid of density ρ, will be insufficient to give that acceleration to an element of equal volume, but of greater density, at the same distance and level.

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This is taken advantage of for various purposes in the arts. For example, a bowl kept filled with milk is rotated rapidly about a vertical spindle the axis of which coincides with the axis of figure of the bowl. The lighter part, the cream, is thus brought to the central part of the bowl, and the heavier more watery part passes to the outside in consequence of centrifugal force. The cream and skimmed milk thus separated are drawn off by pipes communicating with the inner and outer regions of the bowl and are received in separate vessels.

*384. **Free Vortex.**—If the angular velocity be not constant but a function of the distance from the axis, we must proceed in a different manner, inasmuch as a line of constant (zero) flow cannot now be obtained by taking a radial line of particles. Let a velocity-potential exist; then for any line of particles in the fluid we have

$$\frac{p}{\rho} + \Omega + \tfrac{1}{2}q^2 = C,$$

where C is a constant throughout the fluid.

Applying this to a line drawn in a radial plane from a point at which $p = P$, $\Omega = -gh$, and $q^2 = Q^2$, to any chosen terminal point, we have

$$\frac{p}{\rho} - gz + \tfrac{1}{2}q^2 = \frac{P}{\rho} - gh + \tfrac{1}{2}Q^2,$$

or

$$\frac{p-P}{\rho} = -g(h-z) - \tfrac{1}{2}q^2 + \tfrac{1}{2}Q^2. \tag{63}$$

In the case of rotation, the component velocities are given by $u = -\omega y$, $v = \omega x$, where x, y are the horizontal rectangular co-ordinates of a point considered, reckoned from an origin on the axis. Hence

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 2\omega + x\frac{\partial \omega}{\partial x} + y\frac{\partial \omega}{\partial y} = 2\omega + r\frac{\partial \omega}{\partial r}.$$

Therefore if $2\omega + r\partial\omega/\partial r = 0$, a velocity-potential will exist. This relation is equivalent to $\omega r^2 = c$, where c is a constant, that is, the angular velocity must be inversely proportional to r^2. The actual linear velocity q is thus c/r. For a surface of equal pressure we get then

$$\frac{p-P}{\rho} - \tfrac{1}{2}Q^2 + g(h-z) = -\tfrac{1}{2}\frac{c^2}{r^2}. \tag{64}$$

If for the quantity on the left we write $-gz'$, we have for the equation of a surface of equal pressure

$$r^2 z' = \tfrac{1}{2}\frac{c^2}{g}.$$

A surface of equal pressure is, therefore, that generated by the revolution, about the axis, of the quasi-hyperbolic curve, the equation

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of which is $x^2z' = c^2/2g$. The value of x^2 is smaller the greater z', and therefore as the depth of a section below a fixed plane increases, the section of the free surface narrows. Also the rate of diminution of x diminishes with increase of z' (for d^2x/dz'^2 is positive), and the curve is therefore convex to the axis.

If P be atmospheric pressure, the free surface is got from (64) by putting $p = P$. This is the so-called *free vortex*. If the level at which the velocity is Q be the free undisturbed surface far from the vortex, $Q = O$; and we see that the velocity at any point on the free surface is that given by $v^2 = 2gh'$ where h' is the depth of the point below the undisturbed surface. The velocity at any point in the interior of the fluid is the same as that of the point vertically above on the free surface.

Since every particle has velocity c/r, its moment of momentum about the axis is the same whatever its distance from the axis; and consequently if an element of the fluid alters its distance from the axis, there is no turning motive required to give it the moment of momentum proper to its new position.

It is to be observed, however, that in reality it is not a case of *vortex-motion* in the proper technical sense. There is no rotation of the elements of the fluid, since the conditions for a velocity-potential are fulfilled. At every instant each elementary part of the fluid is being carried bodily in a stream-line about the axis, while every particle of the element is moving at the same instant with velocity identical in magnitude and direction.

*385. **Steady Radial Flow.**—Now let us consider the case of steady radial flow outwards from or inwards towards an axial right circular cylinder, which in the first case is a surface from which the liquid is delivered, in the second case is a surface at which the fluid is received and withdrawn from the space considered. In the first case the surface may be called a *surface-source*, in the second a *surface-sink*.

We suppose the velocity of the fluid to be everywhere the same over cylindrical surfaces co-axial with the surface source or sink. Since the same quantity of fluid crosses in the same interval of time all cylindrical surfaces, we have, if, as we suppose, ρ is constant and v, v_0 are the velocities at cylinders of radii r, r_0, $v/v_0 = r_0/r$.

This is clearly a case in which a velocity-potential exists. In fact, the velocity-potential is $\phi = -v_0r_0 \log r$, which gives

$$-d\phi/dr = v_0r_0/r.$$

We can apply exactly the same analysis to this as to the former case, and arrive at exactly the same result. For Bernoulli's theorem gives precisely equation (64) with $c^2 = r_0^2$, and the surfaces of equal pressure are the same as in the former case.

*386. **Spiral Vortex.**—If, then, we have in any case of a free vortex portions of the liquid moving say inwards at any part with a motion compounded of a circular and radial motion, the velocity in each being inversely as the distance from the axis, the surfaces

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of equal pressure will be exactly the same as if the motion were purely circular or purely radial. Thus the two motions may co-exist, and we have the so-called spiral vortex. There is no cylindrical surface, as supposed above, at which the fluid is received, but there is a free surface of the form given by (64). The fluid has, combined with the motions just discussed, a downward motion, by which it is withdrawn from the vortex at the lowest part.

In a spiral vortex in which the radial and circular components of velocity are inversely proportional to the distance from the axis the stream-lines are everywhere equally inclined to the radius vector drawn to the point considered. For if a/r, b/r be the circular and radial velocities, and ϕ be the angle the stream-line makes with the radius, we have $\tan\phi = a/b$. The stream-lines are, therefore, equiangular spirals, with their poles on the axis.

This is the kind of motion in the whirlpool chamber of a centrifugal pump constructed according to the principles laid down by the late Professor James Thomson. The water is delivered at the circumference of the wheel, and flows spirally outwards in a surrounding chamber of about twice the diameter of the wheel, until at the outside circumference it reaches the delivery-pipe, and leaves the chamber tangentially, or nearly so. The velocity is thus diminished and the pressure correspondingly increased, and therefore also the efficiency of delivery of the pump when used to deliver water at a higher level.

The water in the wheel is revolving with the wheel, and thus there is a combination of the forced and free vortices.

Cyclones are, no doubt, vortices of the kind here described, with, however, a central part, in which the fluid is rotating with nearly the same angular velocity at every point, so that the velocity at the centre is very small. This is confirmed by the fact that there is comparative calm at the centre of a cyclone, together with very low pressure, which is what we should expect by § 384.

Also the fluid near the base of the vortex is retarded by friction with the water or the ground, and therefore as the pressure in the outer parts of the vortex is higher than in the inner parts, the air near the bottom flows inward as well as revolves. The air thus arriving spirally at the centre is there carried upward together with sometimes dust or water in a divided state, as spray or spindrift.*

Water revolving in a washhand basin as it escapes by a hole in the bottom is a case of a free spiral vortex, interfered with no doubt by friction, especially where the water is in contact with the surface of the basin. The velocity is great in the middle near the axis, and is small near the sides, where also the pressure is greater. In consequence of the retardation of the sides of the vessel, there is a downward flow of the liquid along the sides.

387. **J. Thomson's Theory of Formation of River Bends.**—Another case of a free spiral vortex is to be found in a river bend,

* Professor J. Thomson on *Whirlwinds and Waterspouts*. *B.A. Rep.* 1884.

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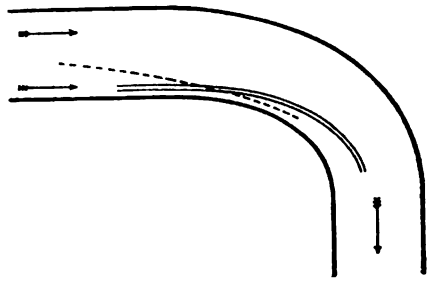
and Professor James Thomson has given by its means an explanation of the wearing away of the outer bank of the river at the bend, and the formation of deltas. As the water flows round the bend (Fig. 201), the pressure is greater towards the outer bank by the theory

FIG. 201.



just given, and consequently the level is higher there, otherwise the flowing water would not be in equilibrium. But in consequence of the retardation due to the friction of the bottom, the water there requires less acceleration towards the inner bank, and consequently the excess of pressure sets up an obliquely inward flow along the bottom, which begins above the bend as shown by the dotted line in Fig. 201. Also at the outer bank the water which is there moving rapidly is retarded to some extent by the friction of the bank, and a component flow downward along the bank is produced as shown in

FIG. 202.

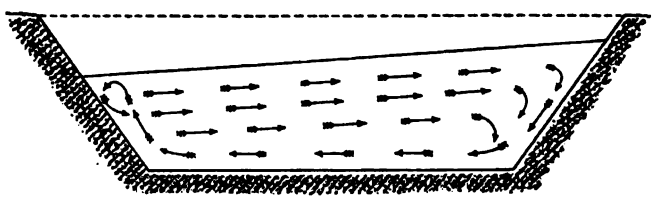


Fig. 202. The smoothly moving liquid at the outer bank scours the surface, and the downward current continued by the bottom current carries mud and gravel, and deposits it to some extent on the inner. The inner bank is moreover protected by being covered by upward slowly-moving fluid. Thus the bend becomes sharper and sharper, until the river at some favourable place and time cuts across the narrowed piece of land between the two limbs of the bend, and insulates a portion between two channels and forms a delta.

Fig. 202 shows the general nature of the flow in a section of the stream. It is to be remembered, of course, that the diagram shows only the transverse component of flow.

The currents are easily illustrated on a small scale, as was done

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by Thomson, by causing water to flow in an artificial river channel and watching the flow in different places by introducing small crystals of permanganate of potash or aniline. The crystals dissolve, and show by the filamentary streams from them the direction of flow.

*388. **Kinetic Energy of Liquid in Simply Connected Space. Theorem of Minimum Energy. Physical Analogies.**—The kinetic energy T in any space filled with fluid is $\frac{1}{2}\int\rho q^2 d\varpi$ where $d\varpi$ is an element of volume, and the integral is taken throughout the space. If there be a velocity-potential throughout the space we have

$$T=\tfrac{1}{2}\int\rho q^2 d\varpi=\tfrac{1}{2}\iiint\rho\left\{\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2+\left(\frac{\partial\phi}{\partial z}\right)^2\right\}dxdydz. \qquad (65)$$

If the space be simply connected or rendered so by diaphragms we can integrate this by parts and obtain

$$T=-\tfrac{1}{2}\int_S\phi\left(l\frac{\partial\phi}{\partial x}+m\frac{\partial\phi}{\partial y}+n\frac{\partial\phi}{\partial z}\right)dS-\tfrac{1}{2}\iiint\rho\nabla^2\phi\, dxdydz, \qquad (66)$$

where $\int_S$ denotes integration taken over the bounding surface of the space, and the triple integration is taken throughout the space enclosed by the surface. But if ρ be invariable, $\nabla^2\phi=0$ by the equation of continuity. Hence for irrotational motion in a fluid of invariable density the kinetic energy within the surface is given by

$$T=-\tfrac{1}{2}\rho\int_S\phi\frac{d\phi}{dn}dS, \qquad (67)$$

where $d\phi/dn$ is written for $l\partial\phi/\partial x+m\partial\phi/\partial y+n\partial\phi/\partial z$, the rate of variation of ϕ along the normal inwards from the surface towards the space.

It is to be remarked that if $\int_S d\phi/dn.dS=0$ (and this condition is always fulfilled when the whole boundary of a space in which $\nabla^2\phi=0$ is considered) the addition of an indeterminate constant to ϕ in no way affects the value of the kinetic energy.

It follows from (68) that if (*a*) $\phi=0$ over the surface, or (*b*) $d\phi/dn=0$ over the surface, or (*c*) $\phi=0$ over part of the surface, and $d\phi/dn=0$ over the remainder, $T=0$. Moreover, since the element of integration $q^2 d\varpi$ is everywhere positive, the fluid is at rest everywhere if (*a*), (*b*), or (*c*) is fulfilled.

If then at no point of the surface there is motion of the fluid normal to the surface, the fluid is everywhere at rest. But if the surface is at rest, that is if the space considered is bounded by perfectly rigid walls, $d\phi/dn=0$, and there can be no irrotational motion of the liquid anywhere within the space. This is only true, of course, if ρ be invariable.

Consider another motion of component velocities $u+u_0$, $v+v_0$, $w+w_0$, where $(u, v, w)=-(\partial\phi/dx, \partial\phi/dy, \partial\phi/dz)$, and lu_0+mv_0

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$+nw_0=0$, so that the velocity normal to the surface is the same at every element as in the motion just considered. Also let $\partial u_0/\partial x + \partial v_0/\partial y + \partial w_0/\partial z = 0$.

From the conditions imposed on u_0, v_0, w_0, it follows that

$$\iiint\left(\frac{\partial\phi}{\partial x}u_0+\frac{\partial\phi}{\partial y}v_0+\frac{\partial\phi}{\partial z}w_0\right)dx\,dy\,dz=$$

$$-\int\phi(lu_0+mv_0+nw_0)dS-\iiint\phi\left(\frac{\partial u_0}{\partial x}+\frac{\partial v_0}{\partial y}+\frac{\partial w_0}{\partial z}\right)dx\,dy\,dz=0.$$

But for the kinetic energy of the new motion we have

$$T=\tfrac{1}{2}\rho\{\int(u^2+v^2+w^2)d\varpi+\int(u_0^2+v_0^2+w_0^2)d\varpi+2\int(uu_0+vv_0+ww_0)d\varpi\}.$$

The third term on the right vanishes by the result just obtained: hence

$$T=T_1+T_0, \tag{68}$$

where T_1, T_0 are the kinetic energies of the irrotational motion (u, v, w) and the motion (u_0, v_0, w_0) which is unrestricted except by the conditions stated above.

Hence the kinetic energy of the irrotational motion (u, v, w) of the liquid is less than the kinetic energy of any other motion, fulfilling the same surface conditions and the equation of continuity, by the kinetic energy of the motion which must be compounded with the irrotational motion to produce the other.

This theorem, which is due to Lord Kelvin, is a particular case of the more general theorem, also due to Lord Kelvin, that a material system, if started from rest by impulses applied to certain points of the system and adjusted to communicate certain specified velocities to those points, has smaller kinetic energy than that of any other possible motion of the system fulfilling the same velocity conditions (see § 251 above).

In a fluid of invariable density, and filling infinite space outside a closed region S_1, the kinetic energy is given by

$$T=-\tfrac{1}{2}\rho\int_{S_1}\phi\frac{d\phi}{dn}dS.$$

It is not legitimate to assume this from (68) on the ground that S_1 is now the whole boundary of the fluid. A surface S_2, including S_1, however, may be regarded as drawn in the fluid, and if the motion is irrotational in the space between S_1 and S_2 we have

$$\int_{S_1}\frac{d\phi}{dn}dS_1+\int_{S_2}\frac{d\phi}{dn}dS_2=0.$$

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It will be observed that wherever the outer surface S_2 may be taken the integral $\int_{S_2} d\phi/dn.dS_2$ has always the same value. Now, by taking S_2 far enough from S_1 at every point, we can make the velocity-potential have as nearly as we please at all points of S_2 a constant value. Let this be C, then we have for the kinetic energy within the space

$$T = -\tfrac{1}{2}\rho\left\{\int_{S_1}\phi\frac{d\phi}{dn}dS_1 + \int_{S_2}C\frac{d\phi}{dn}dS_2\right\}. \tag{69}$$

If there is no flux of the fluid on the whole across the outer boundary, then $\int_{S_2} d\phi/dn.dS_2 = -\int_{S_1} d\phi/dn.dS_1 = 0$, and we have

$$T = -\tfrac{1}{2}\rho\int_{S_1}\phi\frac{d\phi}{dn}dS_1. \tag{70}$$

This is the kinetic energy of the motion outside any closed surface enclosing sources and sinks of fluid, the sources on the whole giving out at each instant fluid at the same rate as fluid is being received by the sinks.

A source from which fluid is produced at the rate q, and a sink at which fluid is produced at rate $-q$, that is disappears at rate q, are perfectly analogous to positive and negative (§ 474) charges of gravitationally attracting matter, and solutions of problems of fluid motion in which these are concerned are at once translateable into solutions of problems of attraction due to corresponding distributions of attracting matter. For the velocity-potential ϕ, and the gravitational potential V fulfil the same space differential equation, and the same surface conditions.

It will be seen from (70) that if $d\phi/dn$ be everywhere zero over S_1, T is zero. Hence there can be no irrotational motion in infinite space external to the surface S_1, for the surface integral is zero, and therefore u, v, w must be zero at every point of the space considered. Also if S_1 be infinitely small there is no motion.

Irrotational motion is thus impossible in the space within and the space without S_1, if S_1 be everywhere at rest; for then the liquid in contact with the surface at any instant cannot have any component of velocity $d\phi/dn$ at right angles to the surface. Also it is impossible in a liquid filling infinite space and at rest at infinity.

There is a similar analogy between the velocity of a fluid surrounding a system of vortices and the magnetic force due to a system of currents, but the subject is too complicated to be dealt with here. (See *Magnetism and Electricity*.)

*389. **Steady Motion of Sphere and Cylinder in Infinite Liquid.**—As an example we may find the kinetic energy of a perfect fluid of invariable density in which a solid sphere of radius a moves with uniform velocity V. The radial velocity of the surface of the

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sphere at any point the radius to which makes an angle θ with the direction of motion is $V\cos\theta$.

This value of $-d\phi/dn$ or $-d\phi/dr$ is obtained from

$$\phi = \tfrac{1}{2}Va^3\frac{\cos\theta}{r^2},$$

which satisfies the equation of continuity $\nabla^2\phi = 0$, and gives, when $r = a$, $-d\phi/dn = V\cos\theta$. Hence when $r = a$ we have $-\phi d\phi/dn = \frac{1}{2}aV^2\cos^2\theta$, and $dS = 2\pi a^2\sin\theta\, d\theta$, so that

$$T = \pi\rho a^3 V^2 \int_0^\pi \cos^2\theta d(-\cos\theta)$$

$$= \frac{1}{3}\pi\rho a^3 V^2.$$

The kinetic energy of the fluid is therefore half the kinetic energy of a sphere of the same density as the fluid moving with velocity V, without rotation. If M be the mass of the sphere, and m the mass of a sphere of the fluid of radius a, we have for the total kinetic energy

$$T' = \tfrac{1}{2}(M + \tfrac{1}{2}m)V^2.$$

This is the work that must be done to set the sphere in motion with velocity V in the liquid, and shows that the presence of the liquid gives the sphere an apparent increase of inertia of amount $\frac{1}{2}m$.

This result is of importance, as was first pointed out by Green, for the correction of the period of a pendulum-bob vibrating with small maximum velocity (and therefore slight acceleration) in a fluid of constant density, and shows that in strictness a correction for the inertia of the fluid is necessary, as well as one for the virtual diminution of gravity due to the weight of the liquid displaced. It was shown by Green that when the bob is of the form of a very oblate spheroid of equatorial radius a', and polar radius b', and vibrates in the equatorial plane, the density of the solid should be increased from ρ to $\rho + \pi b'\rho'/4a'$ where ρ' is the density of the fluid. This correction is small in the flat bobs parallel to the plane of vibration with which pendulums are frequently furnished.

If a velocity $-V$ be imposed on both sphere and fluid, the former will be brought to rest, and we shall have the case of a steady stream past a spherical obstacle. Neither in this case, however, nor in the former, is there any force exerted on the sphere. Once started in an infinite perfect fluid, the sphere will continue to move for ever with uniform velocity. The impulse required to start it is, however, greater than if there were no fluid by the amount $\frac{1}{2}mV$. When the sphere is reduced to rest in a steady stream there is no force upon it, for the distribution of velocity with respect to the sphere is the same up-stream as it is down-stream.

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For the velocity-potential we have in this case

$$\phi = V\left(r + \tfrac{1}{2}\frac{a^3}{r^2}\right)\cos\theta.$$

In the case of a cylinder moving with steady velocity V perpendicular to its axis in an infinite liquid the equation of continuity becomes

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0, \quad \text{or } \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0,$$

if r be the distance of the point considered from the axis of the cylinder, and θ the angle which the plane through the axis and the point makes with the plane of the axis and the direction of motion.

The velocity of a point on the surface of the cylinder is $V\cos\theta$. We write, therefore, for the velocity-potential ϕ, and the stream function ψ, which are found to be:

$$\phi = Va^2\frac{\cos\theta}{r}, \quad \psi = -Va^2\frac{\sin\theta}{r}.$$

This value of ϕ fulfils the equation of continuity, and gives $-d\phi/dn = V\cos\theta$ for $r = a$. Hence when $r = a$, $-\phi d\phi/dn = V^2 a\cos^2\theta$, and the kinetic energy of the motion of the liquid per unit of length of the cylinder is

$$T = \tfrac{1}{2}\rho V^2 a^2 \int_0^{2\pi} \cos^2\theta\, d\theta = \tfrac{1}{2}\pi\rho a^2 V^2,$$

that is the kinetic energy of the motion of the fluid is equal to that of an infinite cylinder of the fluid of the same radius and moving with the same velocity as the given cylinder. Thus if M be the mass of unit length of the latter, and m denote $\pi\rho a^2$, we have

$$T = \tfrac{1}{2}(M + m)V^2.$$

By imposing on the whole material system a velocity $-V$, we obtain the case of a frictionless fluid flowing past a fixed cylinder in a direction perpendicular to its axis. The potential and stream functions are now

$$\phi = V\left(r + a^2\frac{\cos\theta}{r}\right), \quad \psi = V\left(r - \frac{a^2}{r}\right)\sin\theta,$$

as the reader may verify. As before, the stream exerts no force on the solid.

*390. **Equations of Equilibrium of a Fluid.**—The equations of equilibrium of a fluid are given at once by (12). It is only necessary to put in these the right-hand sides, that is the component accelerations of the element of the fluid, equal to zero. Thus we get

$$\frac{\partial p}{\partial x} - \rho X = 0, \quad \frac{\partial p}{\partial y} - \rho Y = 0, \quad \frac{\partial p}{\partial z} - \rho Z = 0, \tag{71}$$

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which simply express the fact that the force per unit of area applied to a short straight filament having its ends along either of the three axes must be balanced by the difference of pressures between its two ends.

It is important to notice that if X, Y, Z be zero, there is no variation of pressure from point to point in the fluid, that is the pressure is the same at every point.

If the forces X, Y, Z are derived from a potential Ω we have the corresponding equations

$$\frac{\partial p}{\partial x}+\rho\frac{\partial \Omega}{\partial x}=0,\quad \frac{\partial p}{\partial y}+\rho\frac{\partial \Omega}{\partial y}=0,\quad \frac{\partial p}{\partial z}+\rho\frac{\partial \Omega}{\partial z}=0. \tag{72}$$

From (71) since $dp=\partial p/\partial x.dx+\partial p/\partial y.dy+\partial p/\partial z.dz$ we have

$$dp=\rho(Xdx+Ydy+Zdz), \tag{73}$$

which serves to determine the pressure.

This equation is easily obtained from first principles as follows:—Consider a prismatic filament of the fluid of which the length is ds, and the area of cross-section σ. Let dx, dy, dz be the projections of ds on the axes of co-ordinates, and let P be the applied force per unit volume in the direction taken as positive for the length of the filament. The force tending to move it in the direction of P is $P\rho\sigma ds$. But if p be the pressure at the negative end of the filament, the pressure at the other end is $p+dp/ds.ds$. The pressure-force resisting motion in the positive direction is therefore $\sigma dp/ds.ds$. Thus we obtain

$$\frac{dp}{ds}=P\rho,\quad \text{or } dp=P\rho ds. \tag{74}$$

But $Pds=Xdx+Ydy+Zdz$, and therefore

$$dp=\rho(Xdx+Ydy+Zdz),$$

as already obtained.

*391. **Case of Pressure a Function of the Co-ordinates.**—Now we suppose that p is a function of the co-ordinates taken with respect to a system of axes fixed relatively to the fluid. Therefore, the left-hand side is a perfect differential (see § 193) of a function of x, y, z: it follows that the right-hand side is also a perfect differential. The criterion of this is given by the three equations:

$$\frac{\partial(\rho Z)}{\partial y}-\frac{\partial(\rho Y)}{\partial z}=0,\quad \frac{\partial(\rho X)}{\partial z}-\frac{\partial(\rho Z)}{\partial x}=0,\quad \frac{\partial(\rho Y)}{\partial x}-\frac{\partial(\rho X)}{\partial y}=0.$$

If the first of these be multiplied by X, the second by Y, and the third by Z, and the equations be added, we obtain

$$X\left(\frac{\partial Y}{\partial z}-\frac{\partial Z}{\partial y}\right)+Y\left(\frac{\partial Z}{\partial x}-\frac{\partial X}{\partial z}\right)+Z\left(\frac{\partial X}{\partial y}-\frac{\partial Y}{\partial x}\right)=0. \tag{75}$$

If we draw a line of force through the point x, y, z considered, its

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direction-cosines are proportional to X, Y, Z. We have then in (75) three multipliers, a, b, c, which give $aX + bY + cZ = 0$, and these (§ 24) must be proportional to the direction-cosines of a line passing through x, y, z at right angles to the line of force. The values of a, b, c show that the lines of force are cut orthogonally by a system of surfaces.

For let $F(x, y, z) = c$ be the equation of a family of surfaces, the individual members of which are obtained by giving different values to c. Then the direction-cosines of the normal to the surface of this family passing through the point considered must be proportional to X, Y, Z if the surface cuts the line of force at right angles. Hence we have

$$\frac{\partial F}{\partial x} = hX, \quad \frac{\partial F}{\partial y} = hY, \quad \frac{\partial F}{\partial z} = hZ,$$

where h is a multiplier, the same for X, Y, Z, but in the general case a function of the co-ordinates. This gives $\partial(hZ)/\partial y = \partial(hY)/\partial z$, &c., from which we obtain again (75). The latter equation is therefore the condition stated above.

392. **Surfaces of Equal Pressure are Surfaces of Equal Potential.**—The density ρ is thus an integrating factor of the quantity $Xdx + Ydy + Zdz$, that is, converts the expression into a perfect differential. If ρ be invariable, that is if the fluid be incompressible, $Xdx + Ydy + Zdz$ is itself a perfect differential, that is the component forces are derivable from a potential function. Hence the fluid is in this case in equilibrium only if the system of forces is derivable from a potential function.

If a force-potential exist, we have, whether ρ is constant or not,

$$dp = -\rho d\Omega. \tag{76}$$

The surfaces of equal pressure are, therefore, also surfaces of equal potential of the field of applied force, for if dp vanishes so also does $d\Omega$. It follows also that ρ is constant over a surface of equal pressure in this case: for if dp be the constant step of pressure from one equipressure surface to another adjacent, $d\Omega$ is the constant step of potential for the same two surfaces, and ρ must therefore be constant over the surface.

If the applied forces be those due to gravity, surfaces of equal pressure must be horizontal surfaces, that is, surfaces perpendicular to the vertical, in other words, to the plumb-line.

393. **Surfaces of Equal Pressure Cut Lines of Force at Right Angles.**—The equations of equilibrium express the fact that in all cases the surfaces of equal pressure cut the lines of force orthogonally. For p is a function of x, y, z, say $f(x, y, z)$. If then we take any constant value of p the equation $f(x, y, z) = p$ is the equation of an equipressure surface. The direction-cosines of this surface are proportional to $\partial f/\partial x$, $\partial f/\partial y$, $\partial f/\partial z$, that is to $\partial p/\partial x$, $\partial p/\partial y$, $\partial p/\partial z$, and equations (71) state that these are proportional to X, Y, Z. The resultant of the applied forces is therefore at each point perpendicular to the surface of constant pressure at that point.

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We have seen already that if the applied forces are derivable from a potential, the surfaces of equal pressure and the surfaces of equal density coincide. If however the forces be not so derivable, this coincidence does not necessarily exist. As in all the practical cases we shall have to consider the forces have a potential, we need not occupy space with the theory of the other case. We may merely remark that a surface of equal pressure has the equation

$$Xdx + Ydy + Zdz = 0, \tag{77}$$

while one of equal density has the equation

$$\frac{\partial \rho}{\partial x}dx + \frac{\partial \rho}{\partial y}dy + \frac{\partial \rho}{\partial z}dz = 0. \tag{78}$$

The intersection of two surfaces, one of each kind, gives a line of equal pressure and density.

394. **Compressible Fluid in Field of Force.**—In the case of a compressible fluid the density is a function of the pressure p and the absolute temperature θ (see Part II.) of the form $p/\rho\theta = p_0/\rho_0\theta_0$ or $p/\rho = R\theta$ where R is a constant. Thus the equations of equilibrium become

$$\frac{1}{p}\frac{\partial p}{\partial x} = \frac{1}{R\theta}X, \quad \frac{1}{p}\frac{\partial p}{\partial y} = \frac{1}{R\theta}Y, \quad \frac{1}{p}\frac{\partial p}{\partial z} = \frac{1}{R\theta}Z, \tag{79}$$

and the equation for the determination of the pressure is

$$\frac{dp}{p} = \frac{1}{R\theta}\left(Xdx + Ydy + Zdz\right). \tag{80}$$

If a force-potential exists, we have

$$\frac{dp}{p} = -\frac{1}{R\theta}d\Omega, \tag{81}$$

which cannot be integrated unless we know the distribution of θ in the fluid. If, for example, θ be constant, we have

$$\log p = -\frac{1}{R\theta}\Omega + C. \tag{82}$$

This integral is taken along a line drawn in the fluid, and C is the value of $\log p + \Omega/R\theta$ for the initial end of the line. The equation may be written

$$p = ce^{-\Omega/R\theta} = ce^{-\Omega/k}, \tag{83}$$

if when θ is constant we write instead of $p/\rho = R\theta$, $p = k\rho$, where $k = R\theta$, and put c for the value of e^C.

The pressure therefore falls off in geometric progression, as the value of Ω increases in arithmetical progression. In the atmosphere, when the positive direction is taken upwards, Ω is positive and varies as the height above a chosen zero level.

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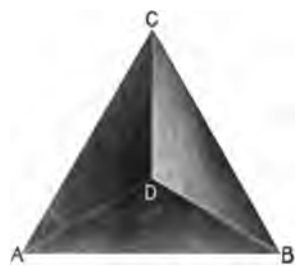
CHAPTER IX.

HYDROSTATICS.

395. **Elementary Theory of Equilibrium of a Fluid. Pressure at a Point is the same in all Directions.**—We may deal with the equilibrium of a fluid, and especially of a fluid under gravity, in the following more elementary manner. First of all we can show that at any point in the fluid the pressure is the same in all directions, a proposition which is true for all actual fluids, however viscous, if they be in equilibrium.

FIG. 203.



Consider a small regular tetrahedron of the fluid. Fig. 203 shows such an element of the fluid, as it would appear to an observer viewing it in a direction at right angles to one face from beyond the opposite vertex. All the four faces are equilateral triangles. This portion of fluid is kept in equilibrium under two systems of forces applied to the matter contained in it: (1) the external applied forces of which the components are X, Y, Z; (2) the pressure-forces applied over the triangular faces. If no momentum of the tetrahedron, or any part of it, be generated, the equilibrium is the same as that of a portion of rigid matter of the same shape, size, and density, replacing the tetrahedron of fluid under the same system of forces.

If s be the length of edge AB of the tetrahedron, the volume of it is $\sqrt{2}s^3/12$, and the area of each face is $\sqrt{3}s^2/4$. The external force in the direction of x is therefore $\sqrt{2}s^3\rho X/12$, and the pressure-force over a face is $\sqrt{3}s^2p/4$. The ratio of the former to the latter is $\sqrt{2}s\rho X/3\sqrt{3}p$. Now let s be indefinitely diminished. The ratio just found diminishes towards zero, and can be made as small as may be desired by making s small enough. Thus the applied force in any direction tends to vanish in comparison with the pressure-force on a face. For a very small element, therefore, the external applied forces may be neglected in comparison with the pressure-

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forces, with approximation approaching perfect exactness as the element is made more and more nearly of zero dimensions.

Considering therefore only the pressure-forces on the tetrahedron, it is obvious that for equilibrium these must be equal forces. Thus the average pressure p is the same over each face, and in the limit we have the result that the pressure at the centre of the tetrahedron is the same in all the four directions perpendicular to the faces.

Now keeping the plane of one of the faces fixed in position take a new tetrahedron in the position which the former would have if slewed round through any angle about an axis perpendicular to the plane of that face, say through its centroid. We get the same result for the direction that has remained fixed and the three directions normal to the positions of the three other faces. By repeating this process we can prove the proposition for successive groups of four directions perpendicular to the faces, and thus show that the pressure is the same in all directions at the centre of the element of the fluid.

This proved, it is seen at once, by considering the equilibrium of a prism-shaped element of the fluid, that the pressure is the same at all points in the fluid if there be no applied forces.

396. **Fluid under Gravity. Surfaces of Equal Pressure are Horizontal.**—Now consider a fluid of uniform density under the action of gravity.

First, it can be proved that the pressure is the same at all points in a horizontal surface. So far as effects of fluid pressure have to be considered in ordinary cases, such a surface at any place may be taken as a plane. Take then a thin prism of the fluid the length of which is horizontal and the ends of which are vertical. If the fluid is in equilibrium the prism is in equilibrium in all respects, and has therefore no acceleration in any direction. Now the forces acting on it are: (1) the external applied forces due to gravity which act vertically downwards on the particles of the fluid; (2) the pressure-forces applied to its bounding surface by the surrounding fluid. The latter fall into two sets, the forces applied to the sides of the prism and the forces applied to its ends.

Consider acceleration of the prism in the direction of its length. Neither the applied forces, which are vertical, nor the pressure-forces on the sides can have any influence in producing or preventing such acceleration, since their directions are at right angles to the length of the prism. There remain therefore only the pressure-forces on the ends. One of these urges the prism in one direction, the other in the opposite direction. They must therefore be equal, otherwise there would be acceleration. Hence the pressures at the two ends are also equal; and since the pressure at any point is the same in all directions, the pressure in any direction at one point in a horizontal plane is equal to the pressure in any direction at another point in the horizontal plane.

397. **Variation of Pressure with Level.**—Again consider a thin prism with its length in the vertical direction, and its ends in two

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horizontal planes at a distance h apart. As regards acceleration in the vertical direction the pressure-forces on the sides have no influence, and there remain only the pressure-forces on the ends, and the gravity-forces. If p be the pressure in the lower plane, p' that in the upper, and a be the area of cross-section of the prism, there is applied to the prism an upward pressure-force pa in the lower interface, and a downward pressure-force in the upper interface. Also a downward force is applied by gravity of amount in absolute units $g\rho ah$. The upward force must balance the total downward force and therefore:

$$pa = p'a + g\rho ah$$

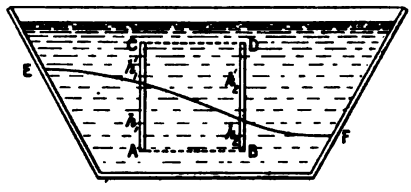
or

$$p = p' + g\rho h; \qquad (1)$$

the pressure in the lower horizontal interface therefore exceeds that in the upper by $g\rho h$, or in gravitation units of pressure by ρh.

For example, consider a pond 10 feet deep. The pressure at the surface is atmospheric, P say. The weight of a column of the water of one square foot cross-section is in pounds $10 \times 62{\cdot}5$, since 1 cubic foot of water weighs roughly 62·5 pounds. The pressure at the bottom is therefore in pounds per square foot 625. Atmospheric pressure is equal to $34 \times 62{\cdot}5$ lbs. per square foot, and the whole pressure is 2750 lbs. per square foot, or about 19 lbs. per square inch.*

FIG. 204.



398. **Vessel containing Different Fluids which do not Mix.**—It may be remarked that since a system of bodies is in stable equilibrium only when the potential energy of the system is a minimum, a number of fluids, which are of different densities and which do not mix, will when placed in the same vessel arrange themselves in order of density with the fluid of greatest density lowest. Only with this arrangement can the potential energy of the system of attracting earth and the fluids in the vessel (the position of which is supposed given) be a minimum.

399. **Surface of Separation of Two Fluids is Horizontal. Communicating Vessels.**—We can now show that the surface of separation of two adjacent fluids is horizontal. For draw two horizontal surfaces AB, CD (Fig. 204), one in each fluid. The pressure is the same at all points in each. Take two narrow vertical prisms with their

* A cubic foot of pure water at 53° F. weighs 62·4 lbs. and at the temperature of maximum density, that is 4° C. or 39·2° F., it weighs 62·425 lbs. Sea water weighs about 64 lbs. per cubic foot. Muddy water has generally a markedly higher density.

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lower ends in the lower surface, and their upper ends in the upper surface. Their heights are the same, but if the surface of separation EF be not horizontal they can be taken in positions such that the lengths of them in the lower and upper fluids are not the same for both.

Let h_1, h_2 be the lengths of the prisms in the lower fluid, h'_1, h'_2 their heights in the upper, ρ, ρ' the densities of the fluids, and p, p' the pressures in the lower and upper horizontal surfaces. By (1) we have

$$p = p' + g(\rho h_1 + \rho' h'_1) = p' + g(\rho h_2 + \rho' h'_2).$$

Hence $\rho h_1 + \rho' h'_1 = \rho h_2 + \rho' h'_2$. Also $\rho(h_1 + h'_1) = \rho(h_2 + h'_2)$.

We get from these by subtraction

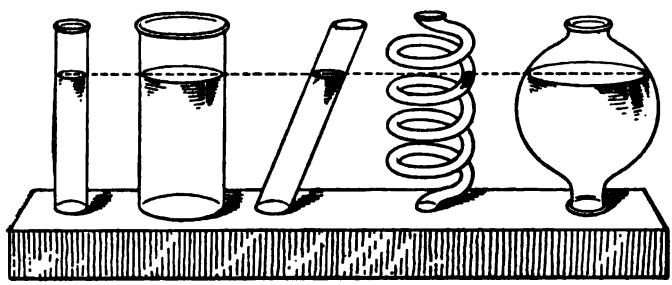
$$h'_1(\rho - \rho') = h'_2(\rho - \rho'),$$

that is

$$h'_1 = h'_2;$$

or the lengths of the prism contained in the upper fluid are the same.

FIG. 205.



Hence the surface of separation is horizontal, and so for any other separating surface.

Let us suppose that we have given two liquids contained in a vessel consisting of an upper and lower chamber connected by cross pipes of different sizes and shapes. If one liquid be contained in the lower chamber and part of the connecting pipes, and the other liquid occupy the upper chamber and the remaining space in the pipes, it will be found that the level of the surface of separation is the same in all the connecting channels.

A particular case is that shown in Fig. 205. There the lower chamber and the under parts of the tubes of different shapes communicating with it are filled with water, the upper parts of the tubes and the whole atmospheric space into which their upper ends open are filled with air, and the level is the same in every tube or connecting channel.

This result is described often in popular language by saying that "water finds its level." The proof by hydrostatic principles that

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different parts of the fluid which are in communication by channels filled with the same fluid stand with their surfaces at the same level is perfectly simple. Take (Fig. 206) the horizontal prism in the lower part the extremities of which are AB. Let B be on a vertical drawn from the free surface, but let A be situated beyond any such vertical. The pressure at A is the same as the pressure at B, and the pressure at B is greater than that at E by $g\rho h$. Again the pressure at D is the same as the pressure at C, which is greater than the pressure at E by $g\rho h'$. Hence the height of the free surface above D must be h'.

Let two liquids, for example water and mercury, be in contact in

FIG. 206.

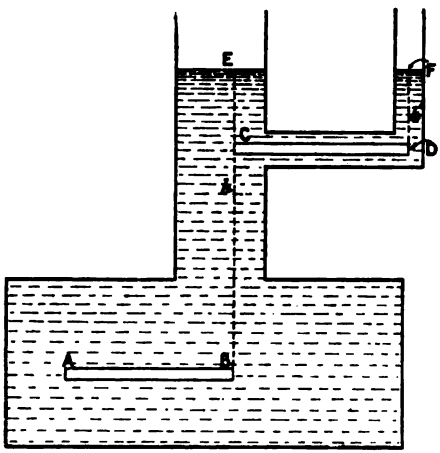
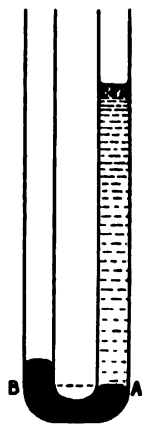


FIG. 207.



a U-shaped tube as shown in Fig. 207. If they are immersed in the atmosphere let P be atmospheric pressure at the top of the right-hand column, h, h' the heights of the right and left columns above AB, the level of the surface of separation, ρ, ρ' the densities of the liquids, and ρ_a the density of the air. The pressure at AB on the right is $P+g\rho h$, and on the left it is $P+g\rho_a(h-h')+g\rho' h'$. These pressurse must be equal, and therefore we have

$$(\rho-\rho_a)h=(\rho'-\rho_a)h'. \tag{2}$$

The densities $\rho-\rho_a$, $\rho_1-\rho_a$ are therefore inversely proportional to the heights h, h' above the common surface of separation. If ρ_a be neglected or the arrangement be in a vacuum

$$\frac{\rho}{\rho'}=\frac{h'}{h}. \tag{2'}$$

It will be noticed that the cross-sections of the tubes do not enter here. The two limbs may in fact have any relative cross-sections provided they are large enough to obviate capillarity effects.

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400. **Interference of Capillarity with Horizontality of Surface.**—It is necessary to notice here, what we shall consider in more detail later, that the horizontality of the surface of separation of two fluids does not hold except at a distance from the walls of the containing vessel given by experience. The surface of water standing in a capillary tube of bore a millimetre or less in diameter is nowhere horizontal, except at one point. If the water wets the vessel the surface is concave upwards, and this concavity leads to an elevation of the surface above the level at which it would stand if it were plane.

On the other hand, mercury in a narrow glass tube is convex, and the surface stands at a lower level than it would if the surface were plane.

It may be taken that there is no sensible rise or depression of liquids in tubes $\frac{3}{4}$ inch in diameter or more. At the walls, however, there will be found a curvature of the liquid surface, either concave or convex.

Any one can see the upward curvature of tea round the wall of the cup, or notice the meniscus formed by wine in a glass by holding the glass between the eye and a light. The convexity of the surface of mercury in a barometer is well known, and is used to determine whether the surface of the column is rising or falling. When the surface is rising in the tube the sides of the column retarded by the tube lag behind, and the central part of the column rises relatively, increasing the convexity above the average value which it has when the column is stationary. The reverse takes place when the surface is falling; the convexity is then less than the average.

401. **Transmission of Pressure. Hydrostatic Paradox.**—It follows from what has been stated that if a liquid be contained in a space bounded by unyielding walls, any pressure applied to it at any part of the wall will be communicated throughout the liquid. For the differences of pressure due to differences of level must remain the same as before, so that the pressure everywhere has been increased by the same amount.

This gives a very remarkable result which has been called the *hydrostatic paradox*, and is illustrated in Fig. 208.

The containing vessel consists of two cylinders communicating as shown. One cylinder is of one or two inches diameter, the other of diameter comparatively large. We shall suppose their areas to be a, A. The cylinders are fitted with pistons, supposed water-tight and frictionless, and for convenience of description are supposed to be vertical.

If, now, a weight of P pounds be applied to press down the small piston, the large piston will be pressed up, and a weight $W = PA/a$ will be required to keep it in the same position as before. For if the space occupied by the liquid is not altered the pressure of the fluid against the under surface of the smaller piston must have been increased by P/a. Hence the pressure against the under surface of the larger piston has been increased by the same amount, and the

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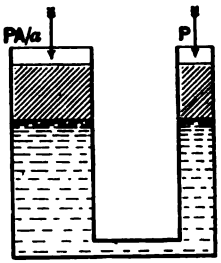
extra pressure-force applied to it is PA/a. That the force P applied by the small piston to the fluid should give a force PA/a applied to the large piston by the fluid has been supposed to be paradoxical.

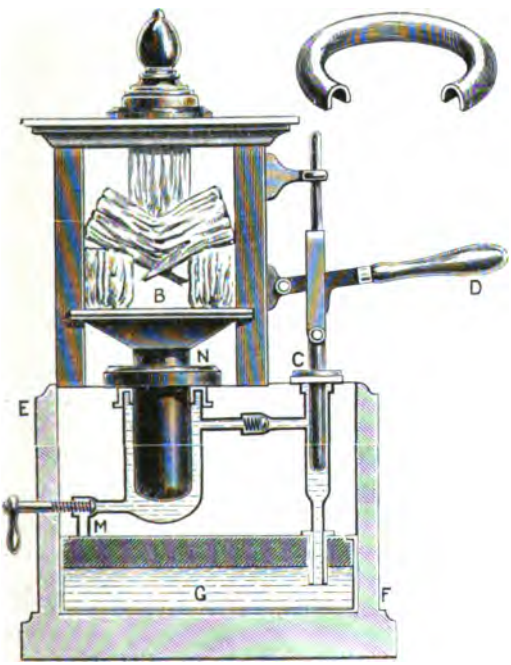
There is no paradox, however, when the matter is viewed in the light of the principle of work. If the small piston be driven down a distance h the work done by the force P is Ph. But if the volume of the fluid remains the same as before the large piston must rise, so that the space has been increased by just as much as it was diminished by the descent of the small piston. This volume is ah. Hence the large piston rises a distance $H = ah/A$. The work done in raising the weight W is then Wah/A. But $W = PA/a$, so that the work Wah/A is Ph; that is, the work spent in depressing the small piston is precisely equal to that spent in raising the large one.

FIG. 208.



In practice, as the result of friction, the work done in raising the large piston is never so great as that spent in depressing the other.

FIG. 209.



402. **Hydraulic Press.**—This arrangement illustrates the principle of the hydraulic press invented by the celebrated Blaise Pascal. The pressure is applied to the water in the small cylinder by forcing down the plunger by the lever D (Fig. 209). The upward motion of the large piston is resisted by a body placed between the "ram" carried by the large piston and a beam overhead. It was made a practical machine by Bramah, who overcame the serious difficulty of making the pistons water-tight by fitting them with the leather collar of U section shown apart in Fig. 209, and also in cross-section at the upper end of the

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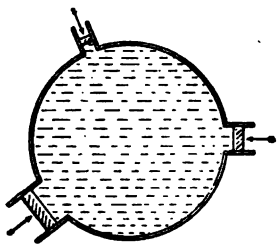
ram-cylinder. The water enters the annular space between the two edges of the leather, and forces the inner wall against the cylinder. Other details not shown in the representation of the press in Fig. 209, which is taken from an old cut, are found in modern apparatus. Water is kept under sufficient head in a reservoir below, and flows through a valve opening upwards at the bottom of the small cylinder when the plunger is lifted. A strong tube of copper of fine bore now generally connects the plunger cylinder with the ram-cylinder. The channel is here shown cut in a block of metal and fitted with a spring-valve which opens towards the ram-cylinder.

If we have a closed vessel and a number of pistons working in attached cylinders as in Fig. 210; then, if a high pressure be maintained in the vessel, work may be done on these different pistons by the pressure which they all receive from the liquid. This is realised in practice by high pressure in mains which supply power for working lifts and other hydraulic engines. For example, the Hydraulic Power Company supply water in London for such purposes at a pressure of 750 lbs. per square inch. It is easy to calculate the work done in any given expenditure of such water. Since p is constant, the work done in an expenditure of a volume v is simply pv. If p be in lbs. per square foot, and v be in cubic feet, the work done will be given by pv in foot-pounds. A consumption of 4,000,000 gallons, or 640,000 cubic feet per twenty-four hours, gives a rate of working of about 1500 horse-power.

FIG. 210.



Hydraulic power is now used for many purposes. For example, for opening and shutting dock-gates, for lifts, for riveting iron and steel plates, for working guns on board ship, in "jacks" for lifting locomotives, &c. The hydraulic power is frequently applied by means of an *accumulator* as it is called. This consists of a heavy weight, which forces down a plunger and gives the necessary pressure. This arrangement is very convenient when the power is not continually required, as in opening dock-gates, &c., for the accumulator may be gradually pumped up again by an engine of comparatively small power working during the intervals

The energy given out by the engine is stored up in the lifted weight, or rather in the system of earth and weight separated against their mutual attraction, and can be used up in a comparatively short time. The storage battery performs the same part in electricity. Energy stored at a slow rate in chemical change of the substances in a battery can be given out again as quickly as may be required.

403. **Pressure-Forces. Thrust on a Plane Surface.**—Let dS be an element of a surface so small that the pressure (which is supposed

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to vary continuously from point to point) may be taken as uniform over it. The product pdS is sometimes called the "whole pressure" over the element, and the sum $\Sigma(pdS)$ of such products taken for the whole surface or any part of it is called the "whole pressure" over the surface or over that part. This designation seems very undesirable; pdS has the dimensions of a force, and is the thrust exerted by the fluid on the element dS, and in no circumstances ought to be called a *pressure* either whole or partial. Again, $\Sigma(pdS)$ has the dimensions of force, but it is a mere addition of the numerical values of forces which have different directions in the general case, in which the surface is not plane.

We shall call pdS a thrust, or sometimes a pressure-force. When the surface is a plane $\Sigma(pdS)$ is the total thrust excited on the surface by the fluid. When the surface is not plane $\Sigma(pdS)/\Sigma dS$, the sum of the thrusts on the elements divided by the area of the surface is the *average pressure* on the surface. If θ be the inclination of the normal at dS to a chosen direction, $\Sigma p\cos\theta dS$ is the thrust on the surface in that direction.

The calculation of the thrust exerted on a plane surface is important for many practical purposes; for example, the total thrusts exerted by the water inside and outside on a dock-gate.

If we denote the thrust exerted on a plane surface by T we have

$$T=\int_S pdS, \tag{3}$$

where p is the pressure at the element dS, and $\int_S$ denotes integration taken over the whole plane.

If the fluid be of uniform density, and the pressure be due to gravity, we have

$$T=\int_S (P+g\rho h)dS, \tag{3'}$$

where P is the pressure at the free surface, and h is the depth of dS below it. To take account of P it is only necessary to multiply the area by P; hence we may disregard P and calculate only $\int_S g\rho h dS$.

As a first example consider a triangular plate immersed so that the vertex is in the surface and the base is parallel to the surface. Denote the inclination of the plane to the vertical by θ, the distance of the vertex from the base by b, and the length of the base by a, The area of a strip at distance h from the vertex, and of breadth dh is $ahdh/b$, and the pressure is $g\rho h\cos\theta$, since the vertical depth of the strip is $h\cos\theta$. Hence

$$T=g\rho\cos\theta\frac{a}{b}\int_0^b h^2dh=\frac{1}{3}g\rho ab^2\cos\theta. \tag{4}$$

This is the magnitude of the resultant of the thrusts on the elements of the plate. It is the product of $g\rho$ and the volume $\frac{1}{3}ab_2\cos\theta$ of the prism formed by drawing from BC lines BD, CE

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(Fig. 211), perpendicular to the plane of the triangle ABC, and of length equal to the depth of BC below the surface.

404. **Thrust on the Bottom of a Vessel. Experiment on Pascal's Vases.**—It was shown by Pascal that the total pressure-force on the bottoms of vases of different shapes, as in Fig. 212, and filled with liquid, had the same value. This follows, of course, from the action of hydrostatic pressure, but it can be verified by supporting the vessel independently, and counterpoising a disk closing the bottom of the vessel. It is found that in all cases the counterpoise is equal

FIG. 211.

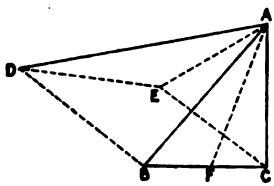
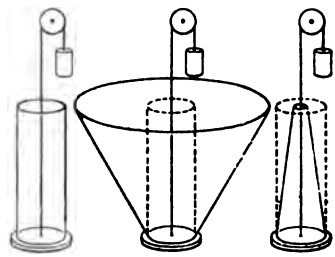


FIG. 212.



to the weight of the disk together with that of a cylindrical column of liquid of section equal in area to the circular opening of the bottom of the vase.

This has been supposed to be paradoxical, but the explanation is evident. The additional weight in the case of the wide-mouthed vase is borne by the sides of the vessel; in the narrow-mouthed vase the whole weight is less than the thrust on the bottom, but the total downward force exerted by the liquid is the downward thrust on the bottom *minus* the total upward thrust exerted on the inward sloping sides of the vessel, and this downward force is precisely the weight of the liquid. For the upward thrust is exactly the weight of the liquid required outside in the latter case to make up the cylinder of liquid, as shown by the dotted lines in the third diagram of Fig. 212.

405. **Centre of Pressure.**—It is clear that a pull of amount $g\rho\int hdS$, applied perpendicularly to an immersed plane plate at some point, will equilibrate the resultant of the thrusts on the elements of one face of the plane. To find this point we take any two non-parallel lines of reference in the plane, and put x, y for the distances of dS from them.

For the sums $M_1 M_2$ of the moments of the thrusts about these lines of reference we have

$$M_1 = g\rho\int hxdS,\ M_2 = g\rho\int hydS,$$

and if ξ, η, denote the distances of the line of action of the resultant from the same lines

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$$\xi = \frac{\int hxdS}{\int hdS}, \quad \eta = \frac{\int hydS}{\int hdS}, \tag{5}$$

where the integrals are taken over the plate.

The point found is the centroid of a distribution of matter over the plate, of surface-density proportional to h, the depth of the point below the surface. It is called the *Centre of Pressure* of the plate for the distribution of pressure here supposed.

For the triangular plate situated as described above we take one reference line in the plane and one in the free-surface. A second line is unnecessary, as clearly the line of action of the resultant must in this case pass through a point in the line joining the vertex with the middle of the base. We have therefore

$$\xi = \frac{\int_0^b h^3 dh}{\int_0^b hdh} = \tfrac{2}{3}b. \tag{6}$$

If the base of the triangle were in the surface we should have for the length of a strip parallel to the base and of distance h from the base the value $a(b-h)/b$, and therefore

$$T = g\rho\frac{a}{b}\cos\theta\int_0^b (b-h)hdh = \tfrac{1}{6}g\rho ab^2\cos\theta, \tag{7}$$

or half the former result (4).

The centre of pressure is easily found to be in this case at a distance $\frac{1}{2}b$ from the base. This can be seen without calculation, as the thrusts on strips at equal distances on the two sides of the middle of AF have evidently equal moments about the base.

406. **Centre of Pressure for any Plane Area.**—For any plane area immersed in a uniform liquid under gravity the pressure at any point is proportional, as we have seen, to the depth of the point below the free surface if we take the surface pressure as zero. It follows that the pressure is proportional to the distance of the point from the line of intersection AB of the plane of the area with the free surface. If x, y be the co-ordinates of any point P in the plane area, with reference to axes in the plane, and α be the angle which a perpendicular let fall from the origin on a line through P parallel to AB makes with the axis of x, the perpendicular distance of the origin from the line is evidently $x\cos\alpha + y\sin\alpha$. If then ϖ be the distance of the origin from AB we have for the distance of P from AB, $\varpi - x\cos\alpha - y\sin\alpha$. Hence for the co-ordinates of the centre of pressure we have, integrating over the immersed area,

$$\xi = \frac{\int x(\varpi - x\cos\alpha - y\sin\alpha)dS}{\int(\varpi - x\cos\alpha - y\sin\alpha)dS}, \quad \eta = \frac{\int y(\varpi - x\cos\alpha - y\sin\alpha)dS}{\int(\varpi - x\cos\alpha - y\sin\alpha)dS}. \tag{8}$$

If now the origin of co-ordinates be taken at the centroid, and

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the axes be the principal axes of inertia there, we have, §§ 147, 168, $\int xdS = 0$, $\int ydS = 0$, $\int xydS = 0$, so that

$$\xi = -\frac{\cos\alpha}{\varpi S}\int x^2 dS, \quad \eta = -\frac{\sin\alpha}{\varpi S}\int y^2 dS. \tag{9}$$

But $\int x^2 dS$, $\int y^2 dS$ are the moments of inertia of the area about the axes of y and x respectively. Denoting these by Sa^2, Sb^2 respectively, where a, b are proportional to the reciprocals of the semi-axes of the momental ellipse for the plane, we have

$$\xi = -\frac{a^2}{\varpi}\cos\alpha, \quad \eta = -\frac{b^2}{\varpi}\sin\alpha. \tag{10}$$

These enable us to write the equation of the line AB, which is $x\cos\alpha + y\sin\alpha - \varpi = 0$, in the form

$$\frac{x\xi}{a^2} + \frac{y\eta}{b^2} + 1 = 0, \tag{11}$$

so that the line AB is the polar of $-\xi$, $-\eta$ with respect to the ellipse $x^2/a^2 + y^2/b^2 = 1$, that is ξ, η is what is called the antipole of the line AB with respect to this ellipse.

If the plane be so placed that a principal axis of the ellipse, say that of x, is horizontal, $\cos\alpha = 0$, and $\sin\alpha = 1$, so that $\xi = 0$, $\eta = -b^2/\varpi$. The centre of pressure is then on the perpendicular drawn through the centroid to the line AB.

If the surface pressure P be not zero, it is only necessary to increase ϖ by the amount corresponding to the depth of a stratum of the liquid which would produce P.

407. **Centre of Pressure of a Triangle with Vertices at any Depths.**—As a final example we may find the centre of pressure of a triangle, the vertices of which, A, B, C, are at depths f, g, h. We shall take the axis of x horizontal. The plane of the triangle may be taken as vertical without alteration of the position of the centre of pressure relatively to the triangle. We have $\xi = 0$, $\eta = -b^2/\varpi$.

Now the moment of inertia of the triangle about a horizontal axis in its own plane through the vertex A is

$$\tfrac{1}{6}S\{(g-f)^2 + (g-f)(h-f) + (h-f)^2\}.$$

The depth of the centroid below the vertex A is $\frac{1}{3}(g+h-2f)$, and consequently the moment of inertia of the triangle about a horizontal axis in its own plane through the centroid is

$$\begin{aligned}\tfrac{1}{6}S\{(g-f)^2 + (g-f)(h-f) + (h-f)^2 - \tfrac{2}{3}(g+h-2f)^2\}\\ = \tfrac{1}{18}S(f^2 + g^2 + h^2 - fg - gh - hf).\end{aligned}$$

Hence, since $\varpi = \frac{1}{3}(f+g+h)$, the distance of the centroid below the surface,

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$$\frac{b^2}{\varpi}=\frac{f^2+g^2+h^2-fg-gh-hf}{6(f+g+h)}$$
$$=\frac{1}{12}\frac{(f-g)^2+(g-h)^2+(h-f)^2}{f+g+h}.$$

The depth of the centre of pressure below the surface is thus

$$\varpi-\eta=\tfrac{1}{4}\{(f+g)^2+(g+h)^2+(h+f)^2\}/(f+g+h). \qquad (12)$$

408. **Thrust in Given Direction on Curved Surface.**—The whole thrust in any direction over a curved surface is to be found by taking the pressure over each element dS of the surface and multiplying that by the cosine of the angle θ between the normal to dS and the proposed direction. Thus summing for the whole surface we have

$$T=\int p\cos\theta\, dS. \qquad (13)$$

If p be constant over the surface

$$T=p\int\cos\theta.dS. \qquad (14)$$

But $\int\cos\theta.dS$ is the area of the projection of the bounding edge of the curved surface on a plane perpendicular to the proposed direction. If we call that area S_θ we have

$$T=pS_\theta;$$

If p be not constant we can find a quantity p_a which fulfils the equation

$$T=\int p\cos\theta.dS=p_aS. \qquad (15)$$

p_a is called the average component of pressure in the proposed direction.

The average pressure over the surface is not this but $(\int pdS)/S$, where S is the total area.

409. **Pull required to separate Magdeburg Hemispheres.**—As an example consider a sphere made up of two hemispheres of external radius r united by flanges round a great circle of the sphere and exhausted of air. It is required to find the force required to separate the hemispheres against the external air pressure. This arrangement is known as the *Magdeburg hemispheres*, from its having been invented by von Guericke of Magdeburg to illustrate air pressure.

If the external breadth of the flanges be b, and the external pressure P, the force required is $\pi P(r+b)^2$; since the projection of the surface acted on, taken perpendicular to the direction of pull, is a circle of radius $r+b$. Thus, for Magdeburg hemispheres of radius 3 inches and breadth of flange 1 inch, the force required would be about $14{\cdot}7\times16\pi$, or nearly 740 pounds. that is nearly a third of a ton. In an old cut illustrating experiments with the air-pump two teams of horses are shown pulling apart two Magdeburg hemispheres of no very great size.

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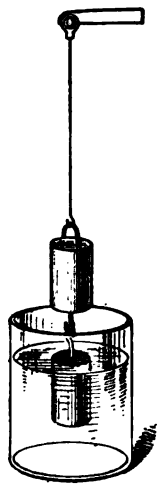
410. **The Principle of Archimedes.**—Let a body be immersed in a fluid under gravity, and suppose the thrust exerted by the fluid on the surface of the body to be the same at each element as it would be on the corresponding element of a mass of the fluid in equilibrium, substituted for the body without disturbing the surrounding fluid in any way. On this supposition the vertical components of pressure-force on the different parts of the surface of the fluid replacing the body are precisely the same as those on the body. But the resultant upward thrust exerted on the fluid is equal to the gravity of the fluid replacing the body, hence there is a resultant upward thrust exerted on the body of exactly the same amount. The body appears therefore to lose gravity by immersion to an amount equal to that of the fluid displaced by it.

FIG. 213.



411. **Experimental Verification of Principle of Archimedes.**—This result can easily be verified by the arrangement shown in Fig. 213. A balance is arranged with a solid cylinder of copper or brass which has above it a hollow cylinder the internal volume of which is equal to the volume of the solid cylinder. When the cylinders are equipoised by weights in the other scale of the balance, a vessel of water is introduced below the solid cylinder so as to immerse the latter. The equilibrium of the balance is disturbed by the upward thrust exerted on the immersed cylinder, and is restored by filling the upper cylinder with water.

It may be remarked here that if the vessel of water be placed on one scale of another balance the immersion of the solid cylinder will cause an apparent increase of the weight of water and vessel. This is due to the increased depth of water in the vessel, which gives a thrust on the bottom greater than before by the weight of water displaced by the cylinder. Thus the weight of the cylinder apparently lost from the first balance is added to that of the vessel of water. Archimedes' principle may be verified also by weighing a body of regular shape the dimensions of which have been accurately measured and its volume V in cubic centimetres calculated. The body is then weighed and the apparent loss of weight in water found. This should be V grammes since a cubic centimetre of water is very approximately one gramme.

412. **Application of Principle to Detection of Adulteration of Gold.**—The principle stated above was obtained by Archimedes, who had been commissioned by Hiero, King of Syracuse, to ascertain whether a crown which had been made for him, and which was supposed to be of pure gold, had been adulterated by any baser metal. The larger the body immersed the greater is the quantity of fluid required to replace the body, and therefore the greater is the upward thrust exerted on the body. Now a certain weight of an alloy

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of gold and silver, or of gold and copper, would have a greater volume than an equal weight of pure gold. Thus there would be a greater apparent loss of weight of the alloy than of the gold produced by immersion in water; and the question as to the purity of the crown could be tested. It was only necessary to take a weight of gold of the required standard of purity exactly equal or in a known ratio to the weight of the crown, and note the loss of weight by immersion in water in the two cases.

It is related that this mode of deciding the question occurred to Archimedes on observing the supporting effect of the water on his body in the bath.

413. **Correction in Weighing for Air Displaced.**—The principle applies to all fluids on which gravity has any sensible action. Thus a piece of wood entirely immersed in water experiences an upward thrust measured by the weight of water displaced, a force which is greater indeed than the weight of the wood, and causes ascent of the body when no other force is applied; a balloon is acted on by an upward force equal to the weight of air displaced, greater it may be than the weight of the balloon and its appendages, and therefore causing ascent of the balloon.

It follows that in the exact weighing of bodies it is necessary to take account of the weights of air displaced by the body and by the weights put in the other scale-pan. Hence we have a practical method of determining the specific gravities, or in other words, comparing their densities (§ 135). We shall return to this presently.

414. **Work done on Liquid in Immersing a Body. Example.**—The principle of Archimedes may be deduced in the following manner from the principle of energy. Let a body of volume V be totally immersed in a liquid of density ρ. If the depth of the centroid of the volume V be increased by an amount x, there is no alteration of the position of the free surface, and the alteration of distribution of the liquid is exactly that which would be effected by raising a volume V of the liquid through a distance x. Hence the work done on the liquid is in gravitation units of work ρVx, that is, the upward force exerted by the liquid on the body is ρV in gravitation units of force.

Thus ρVx is the increase of energy of the liquid caused by the change of position of the body in this case.

If the body be only partially immersed, and the volume in the liquid be increased from V to $V+v$, the work done on the liquid is that involved in raising the centroid of the liquid displaced through the distance through which that of the former volume V has descended, together with the work done in raising a volume v of the liquid above the former free surface in the space surrounding the body. This work is $\rho vx'$ if x' be the distance through which the centroid of this portion of the liquid has been raised.

Let a, a' be the cross-sections at the free surface of the body and vessel respectively, and let the body be lowered vertically without other change of position through a small distance x. If only the volume V formerly below the free surface were further immersed, the work

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done on the liquid would be ρVx as just found. The liquid would then have the same free surface as before, and this would stand a height x above the highest point of the volume V.

The addition of a length x to the part of volume V will raise the fluid which thus stands above into a layer above the former free surface of thickness $ax/(a'-a)$. The "centre of gravity" of this portion of liquid is thus raised through a height $\frac{1}{2}x+\frac{1}{2}ax/(a'-a)$ or $\frac{1}{2}a'x/(a'-a)$. The work done is the product of this by ρax, that is, $\frac{1}{2}\rho aa'x^2/(a'-a)$. The whole work w done on the liquid is therefore given by

$$w=\rho Vx+\tfrac{1}{2}\rho x^2 a\frac{a'}{a'-a}. \tag{16}$$

For the average force F exerted on the fluid by the solid, and on the solid by the fluid, we have, therefore, in gravitation units,

$$F=\rho V+\tfrac{1}{2}\rho xa\frac{a'}{a'-a}. \tag{17}$$

The second term is negligible if x be infinitesimal, and therefore the force is ρV as before.

As an example, we may show that when a sphere of radius r is held just immersed in a cylindrical vessel of radius R containing water, and is caused to rise gently just out of the water, the loss of potential energy of the water is $W'r(1-\frac{2}{3}r^2/R^2)$ if W' be the weight of the water displaced by the sphere.

When the sphere is immersed to a distance h measured from its lowest point to the free surface, and is then caused to rise in actual level a distance dx, the depth h diminishes by an amount

$$dh=a'dx/(a'-a).$$

But $(a'-a)/a'=(R^2-2rh+h^2)/R^2$, so that $dx=(R^2-2rh+h^2)dh/R^2$. Putting dx for x in (16), and for V its value $\frac{1}{3}\pi h^2(3r-h)$, we get for the whole potential energy lost by the water in falling a distance dh along the sphere the equation

$$dw=\frac{1}{3}\pi\rho\frac{3r-h}{R^2}(R^2-2rh+h^2)h^2dh.$$

The last term in (16) is neglected since it involves dx^2, and we make here dx infinitesimal. The whole potential energy lost by the water is therefore

$$\frac{\pi\rho}{3R^2}\int_0^{2r}(R^2-2rh+h^2)(3r-h)h^2dh=\frac{4}{3}\pi\rho r^4\left(1-\frac{2}{3}\frac{r^2}{R^2}\right)$$
$$=W'r\left(1-\frac{2}{3}\frac{r^2}{R^2}\right),$$

which was to be proved.

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The whole change of level of the sphere x is given by

$$x = \frac{1}{R^2}\int_0^{2r}(R^2 - 2rh + h^2)dh = \frac{2r}{R^2}\left(R^2 - \frac{2}{3}r^2\right).$$

Hence the potential energy gained by the solid sphere in rising is, if W be the weight of the sphere,

$$2Wr\left(1 - \frac{2}{3}\frac{r^2}{R^2}\right).$$

These results are given in gravitation units; *e.g.*, if ρ be in lbs. per cubic foot, and R, r be measured in feet, the energies are given in foot-pounds by the expressions found. This example is taken from the unsolved Exercises in Greenhill's *Hydrostatics*, to which the reader may refer for instructive examples in all parts of the subject.

415. **Centre of Buoyancy. Buoyancy of a Body.**—The vertical components of pressure-force have a resultant which acts in a definite vertical in the body. For let the body be removed and equilibrium maintained by filling up the space vacated by it with fluid (the same as that in which the body is immersed) without disturbing the arrangement of the surrounding fluid in any way whatever. The forces of gravity on the fluid thus replacing the body have a resultant which acts downwards through the centroid of the replacing portion of fluid, and we assume that the pressure-forces exerted by the surrounding fluid on the body were the same as those now exerted on the surface of the portion of fluid replacing it. The thrust exerted on this fluid, and therefore that on the body, must act upward through the same point. The point of the portion of the space occupied by the body in the fluid which would thus coincide with the centroid of the replacing fluid is called the *centre of buoyancy* of the body in the state of immersion, whole or partial as the case may be, which it has in the fluid.

The upward force equal to the weight of the displaced fluid is called the *buoyancy*, and is therefore for a floating body equal to the weight of the body. A floating body is said to have a reserve of buoyancy depending on the amount of the body above the surface of the liquid. The *freeboard* or height of the side of a ship above the water-line is an indication of the vessel's reserve of buoyancy.

On every British ship of over eighty tons displacement must be painted a load water-line (the upper edge of an inch broad horizontal line across a circle), beyond which the ship should not be sunk in salt water. This ensures that the ship shall have a reserve of buoyancy of one-fourth of her total buoyancy. The buoyancy with any load is strictly the weight of water she displaces with that load: but the term is very frequently used, especially in connection with buoys or life-belts, to denote the reserve of buoyancy which the body when floating possesses.

By building vessels in watertight compartments, a reserve of buoyancy may be retained after collision has caused one or more compartments of the vessel to be filled. The buoyancy lost is of course

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only that due to the filling up with water of space formerly unoccupied by cargo.

If a body of weight W which floats displace when completely immersed a weight W' of water, the reserve of buoyancy is $W' - W$ or $W(W'/W - 1)$. The ratio W/W' is the apparent specific gravity of the body. It is frequently denoted by s. Hence the reserve of buoyancy is $W(1/s - 1)$.

416. **Interchange of Buoyancy and Reserve of Buoyancy.**—Suppose a homogeneous body of specific gravity s to float in water with a certain water-line. The buoyancy is W, and the reserve of buoyancy $W(1/s - 1)$. Let now the specific gravity be changed to $1 - s$, the buoyancy will become $W(1 - s)/s$, and the reserve of buoyancy $W(1 - s)/s.\{1/(1 - s) - 1\}$ or W. By the change of specific gravity the buoyancy and reserve of buoyancy have been interchanged. Therefore the body after the change of specific gravity will float if inverted with the same water-line as before.

Moreover, in the inverted position the body will have the same righting moment as before for the same angle of heel. For conside the portions of the body on the two sides of the plane of the water-line. Since the body is homogeneous the centroids of these, which are the centres of buoyancy for the two positions of the body, lie on a line through G, and are at distances from G which are inversely as the volumes of the portions. But the buoyancies in the two positions are directly as these volumes; hence the proposition.

Hence, as pointed out by Professor Elgar, in a letter to the *Times*, September 1, 1883, the stability of a vessel with deep draft and low freeboard is similar to that of a vessel with light draft and high freeboard.

417. **Righting Moment.**—The notion of centre of buoyancy is necessary for the proper understanding of the conditions of equilibrium of a body immersed in a fluid. The resultant downward force due to gravity on the body acts in a vertical through the centroid of the body, the upward resultant of the pressure-forces on the surface of the body acts in the vertical through the centroid of the buoyancy. For equilibrium these two forces must be equal and act in the same vertical. If the body be displaced through a small angle from the position in which this condition is fulfilled, the two forces are no longer in the same vertical, and form a couple which turns the body either towards or from the position of equilibrium. In the former case the equilibrium is stable, in the latter unstable. The moment of the couple brought into play for any position of the body inclined to that of equilibrium is called the *righting* or the *capsizing moment*, according as it tends to cause the body to move towards or from the equilibrium position.

418. **Metacentre. Metacentric Height. Curves of Stability.**—Now let a body floating in equilibrium be inclined over through any angle so as to maintain the same displacement. The position of the centre of buoyancy and the centroid of the vessel will, in general, be no longer in the same vertical. The vertical through the new position

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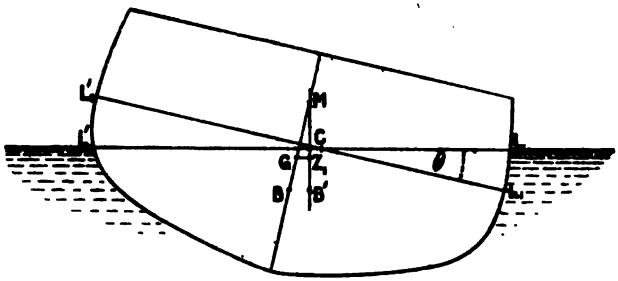
of the centre of buoyancy will meet the line through the centroid of the vessel which is vertical in the equilibrium position in a point M, which is called the *metacentre*. The position of the metacentre relatively to the centroid of the vessel determines whether the equilibrium is stable or unstable.

Thus, let Fig. 214 represent a cross-section of a ship heeled over through the angle θ indicated, between the two water-lines, LL', L_1L_1',

FIG. 214.



and let the centre of buoyancy be B', while B is the point which would coincide with the centre of buoyancy if the vessel were on even keel. Also let G be the centroid of the vessel. If the displacement has not been altered by the heel, the section has been turned about an axis through C, the centroid of the *plane of flotation* or *water-line area* of the ship, as the reader may prove.

If the weight or "displacement" of the ship be W, the righting moment N is $W \times$ *distance between G and the line B'M*, that is

$$N = W.\overline{MG}\sin\theta = WH_m\sin\theta,$$

if H_m denote $\overline{MG}$. Thus the righting moment is greater for a given angle of heel the greater the height H_m of the metacentre above the centroid of the vessel. Clearly for stability M must be above G; the equilibrium is unstable if M is below G. Thus the equilibrium is stable according as the metacentric height H_m is positive or negative.

The position of the metacentre varies with the value of θ, and not infrequently becomes very small or even negative in ships for very large values of θ. In such a case the greatest care must be taken by the navigating officers to prevent the ship from getting broadside on to waves of period nearly equal to that of the free rolling of the ship, lest so great an amplitude θ of heeling oscillation should be produced as may go beyond the point at which H_m vanishes. If this amplitude of rolling is reached the ship will capsize unless prompt means be taken, as causing a body of men to move across the deck, to bring into play a moment aiding the recovery of the ship.

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The position of M for an infinitely small angle of heel is sometimes defined as the metacentre, and the corresponding value of H_m as the *metacentric height*. When used in this sense the metacentric height is a correct measure of the stability of the ship when upright, or, as it is called, is "on even keel."

As the ship heels over about a horizontal line at right angles to the cross-section represented in Fig. 214, and passing through C the centre of buoyancy moves into successive positions B_1, B_2, ..., which lie on a curve called the *curve of buoyancy*.

Generally the stability of a ship for small angles of heel is slight, in consequence of $\overline{MG}$ for such displacements being small. In such cases $\overline{MG}$ rapidly increases with θ, so that for large inclinations the vessel is thoroughly stable. For moderately small angles, therefore, the ship has a long period of oscillation, and possesses great steadiness. A ship in which the metacentric height is great for small inclinations is said to be *stiff*. The degree of stiffness for different inclinations may be exhibited in a curve of which the abscissæ are angles of heel, and the ordinates are the corresponding metacentric heights.

A curve in which the metacentric heights are replaced by the moments of the righting couples is called the curve of *statical stability*. Another curve is also drawn called the curve of *dynamical stability*. In the latter the ordinates are proportional to the amounts of work which must be spent in heeling the vessel slowly over from the vertical to the inclined positions. It is clear that each ordinate of the curve is proportional to the area of the curve of statical stability from the initial point up to the corresponding ordinate of the latter curve.

419. **Longitudinal Metacentre. Surface of Buoyancy. Metacentric Heights.**—A ship may, however, turn about a horizontal axis perpendicular to a longitudinal section through G and the keel, as when she "pitches" when kept with her head to the waves, or in any way undergoes change of *trim*, as it is called, that is of the difference of draft of water at bow and stern. For this inclination there also exists a metacentre which is called the *longitudinal metacentre*. For this the metacentre height is evidently much greater than the ordinary or transverse metacentric height. The righting moment has an expression corresponding to that already obtained.

Hence when the ship is inclined in a combination of heeling and pitching the centre of buoyancy moves to a succession of positions which lie on a surface, which is called the *surface of buoyancy*.

The metacentric heights, both transverse and longitudinal, are determined experimentally for ships by moving a weight along the deck and observing the inclination produced by it in different positions, or by filling the boats on the two sides of the vessel alternately with a known weight of water. This is done for different displacements of the vessel, so that the stability in different circumstances may be known.

Let the weight moved be denoted by P, and let it be moved

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across the deck through a distance b. Then if W be the displacement of the ship and θ the inclination, we have for equilibrium $Pb\cos\theta = W.GZ = WH_m \sin\theta$, or

$$Pb = WH_m \tan\theta.$$

But $H_m \tan\theta$ is the distance through which the centroid of the ship has been displaced in the same direction relatively to the ship as has P, for if this distance be denoted by GG_1 we have $b/GG_1 = W/P$. If then the displacement W of the ship be known, and θ be observed by means of a clinometer (or pendulum hung in the ship with a graduated circular arc to measure its angular deflection from a line fixed in the ship through the point of suspension), the value of Pb/W or GG_1 is calculated, and from this H_m is obtained as $GG_1/\tan\theta$.

To take an example (see Greenhill, *Hydrostatics*, p. 150), H.M.S. *Achilles*, of displacement 9000 tons, was inclined over by moving 20 tons across the deck through a distance of 42 feet, so that the bob of a pendulum 20 feet long moved through a distance of 10 inches, and the angle of heel was changed from $-\theta$ to $+\theta$.

Here $\tan 2\theta = 10/(20 \times 12) = 1/24$, nearly, and $2GG_1 = 20 \times 42/9000$, in feet. Thus $MG = 20 \times 24 \times 42/9000 = 2{\cdot}24$, in feet. The value of θ in degrees is $57{\cdot}3/48$, very nearly, or $\theta = 1^\circ\ 12'$.

420. **Wedges of Emersion and Immersion.**—When a ship is heeled over as in Fig. 214, a wedge of the vessel CLL_1 is brought under water, while another wedge of equal volume $CL'L'_1$ emerges. The former is called the *wedge of immersion*, the latter the *wedge of emersion*. The righting couple, L, due to these wedges, is the sum of the moments about the longitudinal line through C of the buoyancy of the wedge of immersion, and the negative buoyancy of the wedge of emersion. These give two moments in the same direction. Let $d\varpi$ be any volume of the wedge of immersion, x its horizontal distance from the line through C, and let $\rho d\varpi$ be the weight of water filling $d\varpi$. Also let the same symbols accented apply to the wedge of emersion. We have

$$L = \int \rho x d\varpi + \int \rho x' d\varpi', \tag{18}$$

where the integrals are taken throughout the wedges.

Take two parallel cross-sections at distance dy apart, and let θ be small. Then we have for an element of the space between these $d\varpi = x\theta\, dx\, dy$.

But $dxdy$ is an area dA parallel to the water level, is, in fact, an element of what is called the *plane of flotation*, the section of the ship by the water level. Hence, summing for all slices of the wedges thus taken, we may write (18),

$$L = \rho\theta \{\int x^2 dA + \int x'^2 dA'\}. \tag{19}$$

The quantity on the right is the moment of inertia about the longitudinal line through C of a plane of matter, of amount $\rho\theta$ per unit of area, coinciding with the plane of flotation. If A be

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the whole area of the plane of flotation, and k be its radius of gyration about the longitudinal through C, we have

$$L = \rho\theta A k^2. \tag{20}$$

If the centroids of the two wedges be found, and be denoted by g, g', the line gg' must evidently be parallel to the line BB' (Fig. 214), joining the centres of buoyancy for the two positions of the ship. Also if U be the volume of a wedge, V the total immersed volume, q the length of the projection of gg' on the water line in the displaced position, and p the length of the perpendicular let fall from B on $B'M$, clearly $q = pV/U$. Now when θ is infinitesimal gg' is infinitely nearly horizontal, and therefore BB' is then horizontal. Hence the tangent to the curve of buoyancy at B is parallel to the water line.

By the relation between p and q just stated we have for the righting moment

$$\begin{aligned} W.GZ &= W(p - GB\sin\theta) \\ &= W\left(\frac{U}{V}q - GB\sin\theta\right) = \rho(Uq - V.GB\sin\theta). \end{aligned}$$

The last relation is known as Atwood's theorem.

The term ρUq is plainly the couple $L(=\rho\theta A k^2)$ due to the wedges of emersion and immersion. The total righting moment is therefore to be determined, in any given case of a floating body, by calculating $\rho\theta A k^2$ for the plane of flotation and subtracting from the result the value of $W.GB\sin\theta$. Thus for a homogeneous right circular cylinder of radius r, and height $2h$, floating with its axis vertical and immersed to a depth d, we have θ being small,

$$\rho\theta A k^2 = \tfrac{1}{4}\rho\theta\pi r^4, \quad W.GB\sin\theta = \rho\pi r^2 d(h - \tfrac{1}{2}d)\theta,$$

so that

$$\text{righting moment} = \rho\theta\pi r^2\{\tfrac{1}{4}r^2 - (h - \tfrac{1}{2}d)d\}.$$

The equilibrium is unstable only if $Ak^2 < V.GB$. Hence in the case just considered the equilibrium is unstable only if $d^2 - 2hd + \frac{1}{2}r^2 < 0$, and is thoroughly stable if $\frac{1}{2}r^2 > h^2$, that is if $h < r/\sqrt{2}$, since this ensures that $d^2 - 2hd + \frac{1}{2}r^2$, or $(d-h)^2 + \frac{1}{2}r^2 - h^2$ shall be positive whatever d may be.

If the body be completely immersed the centre of buoyancy is fixed in the body, since the shape of the space occupied by it in the fluid in no way alters. It will be clear by merely drawing a figure, that for stability of equilibrium the centre of buoyancy B must be above the centre of gravity G. The righting moment for an inclination θ is evidently $W.GB.\sin\theta$.

421. **Oscillations of a Floating Body.**—For small oscillations of a floating body, such as a ship, we can obtain an approximate expression by § 53 if we neglect the motion of the water. Calling

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WK^2 the moment of inertia of the ship about the horizontal line through O, and T the period, we have

$$T = 2\pi\sqrt{\frac{K^2}{H_m}}. \tag{21}$$

Thus the greater the righting moment the smaller the period of rolling. The quantity WH_m may be taken as a measure of what is called in § 418 the *stiffness* of the ship.

The theory of the oscillations of a ship is a very important part of applied mechanics, and cannot be dealt with here. The reader must refer to treatises on the stability of ships.

422. **Heel Produced in Screw Steamer by Propeller.**—A screw steamer is heeled over by the reaction of the water on the propeller, and a sailing ship by the action of the sails, which also tends to send down the bow and raise the stern. In the former case the *torque*, or total turning motive applied by the engines to the propeller, measures the torque N of reaction exerted by the water. This torque is balanced by the heeling over of the vessel through an angle θ in the reverse direction to that in which the engines are turning the screw. If the horse-power given out by the engines at the propeller be U, and n be the number of revolutions made by the propeller per minute, the angular velocity of the screw in radians per minute is $2\pi n$. We have then

$$33000U = 2\pi n \times N \text{ or } N = 33000U/2\pi n$$

in pound-foot units of torque. If H_m be as before, the metacentric height in feet, and θ the angle of heel produced, we have

$$\frac{33000U}{2\pi n} = WH_m \sin\theta,$$

or if θ be small

$$\theta = \frac{33000U}{2 \times 2240\pi n WH_m}, \tag{22}$$

where W is to be taken in tons.

For a warship of 12,000 tons and, say 2 feet metacentric height, the engines of which develop, say 12,000 horse-power, with the screw running at 100 revolutions per minute, the inclination in degrees is $33 \times 57{\cdot}3/(4\pi \times 224) = 1{\cdot}5$.

The effect of the reaction of the water on the wheels of a paddle steamer is in a similar way to alter the trim of the vessel—throwing up the bow and depressing the stern. The opposite effect of the action of the wind on the sails of a sailing vessel is, to some extent, counteracted by the "rake" of the masts toward the stern.

We have not space to pursue further here the subject of the equilibrium of floating bodies, important as it is from a practical point of view. The reader will find a detailed and interesting prac-

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tical treatment in Professor Greenhill's treatise on *Hydrostatics*. He may consult also treatises on Naval Architecture.

423. **Specific Gravity. Determination of Specific Gravity of Solid. Relation between Specific Gravity and Density.**—As remarked above, an important application of Archimedes' principle is to the determination of specific gravities. The specific gravity of a body is the ratio of the true weight of the body (the weight in vacuo) to the true weight of an equal volume of water at the temperature of maximum density. [Water taken at the temperature of melting ice and warmed first contracts until the temperature 4° centigrade is reached, then expands as the temperature is raised further.]

These weights can be determined approximately by weighing the body in air, and then ascertaining the loss of weight when the body is weighed in water. The former weighing is performed with the body hung by a fine thread from a "short pan" attached to one end of the beam, an equal length of thread being placed with the weights in the other pan. Then a beaker of water is placed in the balance-case, and raised so as to completely immerse the body. But several corrections are necessary to obtain an accurate result. First, the true weight of the body must be ascertained by correcting for the air displaced by the body and the air displaced by the weights, the weight of the body in water must be similarly corrected, and the temperature of the water must be observed, and a correction applied to take account of the fact that the water in which the body was weighed was not at maximum density.

The following precautions are necessary in carrying out the experiments. The water must have been freed from air by boiling. Bubbles of air carried in by the bodies immersed and clinging to them should be removed by touching them with a clean wire from the outside. The body when weighed in water should not be allowed to touch the side of the vessel, or to come very near it anywhere.

The relation between specific gravity and density is evident. Let W be the weight of a body, w the weight of an equal volume of water at its temperature of maximum density.* Then if G be the specific gravity, we have

$$W = Gw. \tag{23}$$

But $W = V\rho$ where V is the volume of the body and ρ its density. Also $w = V\rho_w$ if ρ_w be the density of the water (pounds per cubic foot, or grammes per cubic centimetre, &c., as the case may be). Hence we have

$$\rho = G\rho_w, \tag{24}$$

that is the density of the body is equal to its specific gravity multiplied by the density of water: in other words, the specific gravity of the body is the ratio of the density of the body to the

* In future, in this connection, this temperature will be understood unless the contrary is stated.

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density of water, as is evident from the meaning of density and the definition of specific gravity given above.

It is to be observed that the specific gravity of a body does not depend on the units employed, while the density does. The dimensional formula of density is evidently ML^{-3}.

If the system of units be such that the density of water is unity the specific gravity is numerically the same as the density. This is approximately the case in the metric system, where the weight of a cubic centimetre of water is very nearly 1 gramme. It is, however, to be observed very carefully that in this system a gramme is not defined as the mass of a cubic centimetre of water, but as 1/1000 of the mass of Borda's standard kilogramme (see §134).

Particulars regarding methods of weighing and correcting for the buoyancy of air are given in chap. xvi. on *Measurements and Instruments*. We shall merely give here some account of the determination of the specific gravities of solids and liquids, leaving the determination of the densities of gases for discussion in Part II.

424. **Determination of Specific Gravity of a Solid which floats in Water.**—When a solid is of smaller specific gravity than that of water, it is necessary to attach it to a sinker, preferably made of copper or brass wire, in order that the weight of an equal volume of water may be ascertained. It is not necessary that the true weight of the sinker should be known, but merely its weight in water. For let the true weight of the solid be W, that of the solid and sinker together in water be W_1, and that of the sinker in water W_2. Put for a moment W' for the true weight of the sinker. Then weight of water equal in volume to body and sinker together $= W + W' - W_1$. The weight of water displaced by the sinker $= W' - W_2$. Hence weight of water displaced by the solid $= W + W' - W_1 - (W' - W_2) = W - W_1 + W_2$. Therefore

$$G = \frac{W}{W - W_1 + W_2}. \tag{25}$$

In the determination of the specific gravities of a number of substances specifically lighter than water the same sinker of copper wire may be used and left in the water. The body can easily be attached when required.

In the determination of the specific gravities of bodies which are soluble in or are chemically attacked by water, it is necessary to weigh the solid in some other liquid by which it is not so affected, then determine the specific gravity of the liquid by some suitable method. If W be the weight of the body and W_2 be the weight in the liquid, what we may call the specific gravity of the body relatively to the liquid is given by

$$G_2 = \frac{W}{W - W_2}. \tag{26}$$

If now the weight of any volume of the liquid be found to be w_2 and that of the same volume of water be w, a volume of water equal to

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that of the solid will weigh $w(W - W_2)/w_2$, or if s be the specific gravity w_2/w of the liquid this volume of water will weigh $(W - W_2)/s$. Hence the true specific gravity is

$$G = \frac{Ws}{W - W_2} = G_2 s. \tag{27}$$

425. **The Specific Gravity Bottle.**—The densities of both solids and liquids can be determined by means of the pyknometer or specific

FIG. 215.

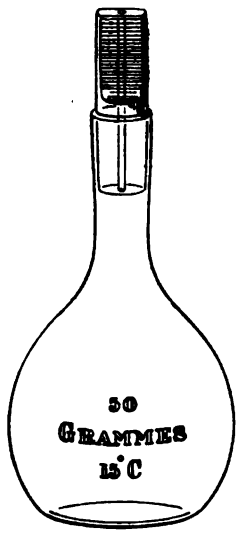
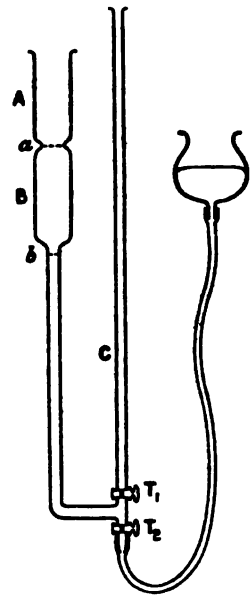


FIG. 216.



gravity bottle. This is a small glass flask provided with a ground neck and a glass stopper accurately fitting it so that the stopper can always be put in to exactly the same distance. Through the stopper from top to bottom runs a narrow perforation to allow air and liquid to escape through the stopper when it is inserted, so that the flask may be exactly filled with liquid up to the stopper. (See Fig. 215.)

The true weight of the flask, and the weight of water which it contains having been ascertained, it is then well dried * and filled exactly with liquid and weighed, and the weight of the contained liquid

* A good method of drying a flask is to force air from a bellows through a narrow glass tube a good length of which is kept hot in the flame of a Bunsen lamp. If the end of the tube be inserted in the neck of the bottle a stream of dry warm air is made to circulate in the interior and quickly evaporates any moisture remaining. It is of little use, besides being dangerous to the flask, to attempt to drive off the moisture by holding the flask above the gas flame.

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found by applying the necessary corrections to the result. If W_2, W be the weights of liquid and water respectively contained by the bottle we have

$$G = \frac{W_2}{W}. \tag{28}$$

To determine the specific gravity of a solid, for example, a few fragments of a rare metal, or an insoluble powder, the weight W_s of the solid is ascertained, then the solid is placed in the bottle, the bottle filled up with water, and the weight W' of the contents found. The weight of water in the bottle along with the substance is now $W' - W_s$. The weight of a volume of water equal to that of the body is therefore $W - (W' - W_s) = W + W_s - W'$. Hence we have

$$G = \frac{W_s}{W + W_s - W'}. \tag{29}$$

426. **Volumenometer.**—The volume of a body and therefore its specific gravity can be determined by means of a volumenometer, or stereometer as it is sometimes called. This was invented by a French officer named Say for the determination of the specific gravity of gunpowder.

One form of the instrument is represented in Fig. 216. AB are two parts of a vessel connected by a short neck. The lower part B is continued by one limb of a U-tube, the other limb of which is furnished with two taps one above and one below the junction. The lower end of the second limb C can be left free or have attached to it a flexible tube connected with a cistern containing mercury, by which mercury can be raised to any required level in the U-tube, The upper lip of the vessel A is ground so as to fit air-tightly a greased plate of glass slipped over it.

The instrument is first calibrated as follows. The lower end of the limb C is left free and the tap T_2 is closed, the other left open. Mercury is then poured in at the mouth of C until it has risen to the top of A, which is then closed by the glass plate, care being taken to include no air. The tap T_1 is then closed, and T_2 opened, and the mercury run out from A to the mark a, and weighed. This gives the volume of A, V_1 say. The mercury is then run out to b, and weighed, and gives the volume of B, V_2 say.

In a determination with the instrument the movable cistern is attached for convenience, though it is not absolutely essential, and the mercury is made to stand at the level a while both limbs are open to the atmosphere. The plate is then slipped over A, and the cistern is lowered until the mercury falls to b. The air contained in the space AB thus falls below atmospheric pressure, by an amount indicated by the vertical distance of the top of the column in C below b. Let this height be h, then if pressure be reckoned in terms of the height of a column of mercury, and H be the height of the barometric column, we have for the pressure in AB, $H - h$. The mercury is again brought to a with the vessel A open. The body of

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which the volume is to be formed is then placed in A, the plate replaced, and the cistern lowered until the mercury falls to b in the left-hand limb. The vertical depth h' of the column in C below b is now read off.

The calculation of the volume v of the body is then carried out as follows. By Boyle's law of the pressure of air (§ 431) we have from the first determination of pressure $(H-h)/H = V_1/(V_1+V_2)$, and from the second $(H-h')/H = (V_1-v)/(V_1+V_2-v)$. From these two equations we obtain

$$v = \frac{h'-h}{h'}(V_1+V_2) = \frac{h'-h}{h'}\frac{H}{h}V_2. \tag{30}$$

Thus v having been obtained, and the weight W of the body having been determined we get at once the density

$$\rho = \frac{W}{v}. \tag{31}$$

It is necessary of course that h, h' should be large enough to be accurately determined.

The instrument is liable to inaccuracy through moisture contained in the air, leakage of the plate, and variation of temperature.

If the atmospheric pressure is varying it is necessary to read the barometer when the mercury stands at a, and when h and h' are observed, and use the observed values in a formula obtained without supposing H constant, to avoid any error from this cause.

A description of another form of volumenometer invented by Prof. W. Stroud, for which great accuracy is claimed, will be found in the *Phil. Mag.* August 1893.

427. **Watt's Hydrometer.**—The specific gravities of liquids can be compared by an apparatus due to James Watt and indicated in Fig. 217. In two beakers containing the liquids to be compared stand two glass tubes, A, B, which are open at the bottom, and are united by a bend at the top, provided with a tube T, by which air can be sucked out of both tubes. The tube can be closed when required by a pinch-cock on a piece of rubber tubing terminating T.

When air is sucked out at T the liquids rise in the tubes above the levels in the beakers. Let the two heights above these levels in AB be h, h', let P denote the pressure of the atmosphere, p the pressure in the bend above the columns (neglecting the pressure-difference in the air at the tops of the columns due to difference of levels), and ρ, ρ' the densities of the liquids. We have $P-p = g\rho h = g\rho' h'$, so that

$$\frac{\rho}{\rho'} = \frac{h'}{h}. \tag{32}$$

428. **Specific Gravity of Fluid. Hydrometer of Variable Immersion. Graduation.**—The specific gravity of a liquid can be determined by ascertaining the apparent loss of weight sustained in

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the liquid by a body of known volume. If this loss be w grammes, and the volume of the body be V cubic centimetres, the specific gravity sought is w/V.

The specific gravities of fluids are, however, when not very accurately required, determined by *hydrometers*. These are of two

FIG. 217.

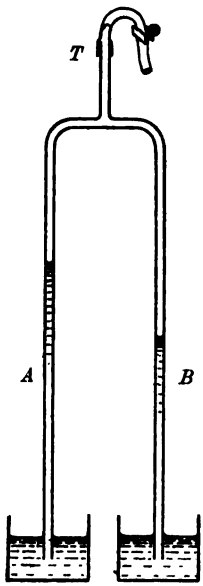
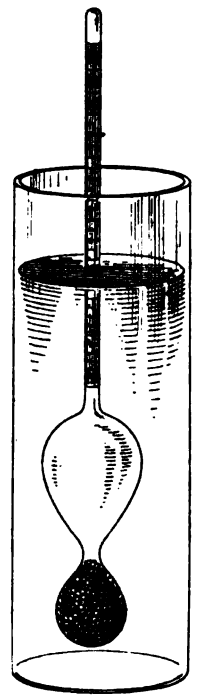


FIG. 218.



kinds—hydrometers of variable immersion, and hydrometers of constant immersion. A hydrometer of variable immersion consists of a hollow body furnished with a uniform graduated stem as shown in Fig. 218. It is weighted with shot or mercury in the lower part of the body so as to float in a fluid with the stem vertical. The free surface of the liquid stands at some point of the stem, and, as the stem is of small cross-section, the instrument is approximately a body completely immersed, and is subject to very nearly the same conditions of stability.

It is clear that the greater the specific gravity of the liquid the lower will be that cross-section on the stem which coincides with the free surface, so that the position of this cross-section determines the specific gravity of any liquid in which the hydrometer may be

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placed. The specific gravity may be obtained from a knowledge of the volume of the body of the instrument up to some cross-section of the stem, the weight of the instrument, and the cross-section of the stem. These may be determined in the following manner: First, the instrument is weighed. It is then immersed in a liquid the specific gravity of which has been determined accurately by some other means. The cross-section which coincides with the free surface is observed, and is marked on the stem. The volume of the instrument up to this mark is equal to the volume of the liquid which has the same weight as the hydrometer, and this volume can be ascertained from the known weight of the instrument and the specific gravity of the liquid.

The cross-section of the stem is found either by measuring the diameter at a number of places by means of a screw gauge or vernier callipers (chap. xvi.), or by placing the instrument in a second liquid also of known specific gravity. In the latter case the instrument will sink to another cross-section of the stem, and if V be the volume immersed in the first liquid, $V+v$ the volume immersed in the second liquid, s, s' the specific gravities of the two liquids, the difference of volume is plainly equal to $V(s-s')/s'$. The length of stem between the two cross-sections being measured, the cross-section is v/l, thus the cross-section of the stem being known the volume of any length of the stem is of course also known.

To graduate the instrument we proceed as follows. First, the specific gravity corresponding to a mark which is to be taken as the lowest point of the stem is ascertained as follows. The volume of the instrument up to that cross-section is calculated. The specific gravity of a liquid in which the instrument stands at that mark is evidently $W/V\rho_w$, where W is the weight of the instrument, V the volume of the instrument and ρ_w the density of water.

The specific gravity of a liquid in which the instrument stands at a distance x above the lowest mark is $W/(V+xc)\rho_w$, where c is the cross-section of the stem. Let V be equal to lc so that l is a length of the stem which would be equal in volume to V. Then the specific gravity of the liquid corresponding to the point at distance x from the lowest point is equal to $W/(l+x)c\rho_w$. If we denote the specific gravity corresponding to x by y we have the equation $(l+x)y=W/c\rho_w$. If, then, from an origin O along the axis of x we lay off a distance $l+x$, and from the point so found, and at right angles to that, lay off a distance $y=W/(l+x)c\rho_w$, and do this for different values of x we shall get a series of points forming a curve. The ordinates of that curve are the specific gravities of the liquids corresponding to the different values of x. The curve is clearly an equilateral hyperbola referred to its asymptotes as axes of co-ordinates and is shown in Fig. 219. It is only necessary, then, to draw an equilateral hyperbola for the constant parameter $W/c\rho_w$, and we have the complete graduation of the instrument.

Values of y are taken expressed by simple numbers, and differing from one another in arithmetical progression, and the points

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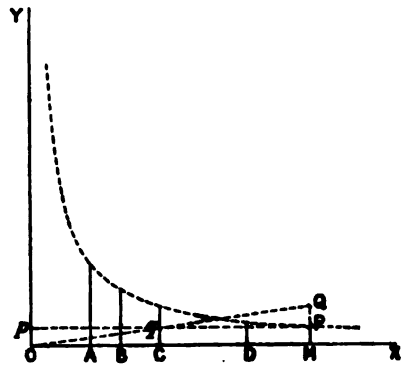
corresponding to these are marked by the cross lines on the scale along the stem. As will be seen from the curve the distances AB, BC, CD, between the lines representing values of y in arithmetical progression, are closer together, the smaller the value of x, that is the higher the specific gravity. The graduation of the stem, therefore, becomes more and more open towards the top of the instrument.

Let OM represent $l+$ *length of stem from the lowest division to highest.* The ordinate MP at M is the least specific gravity shown by the instrument. To find the point on the stem for any specific gravity which the instrument can measure, the following construction can be used. Draw from P a line Pp parallel to OM. Lay off along MP the specific gravity for which the point is required, let it be MQ. Join OQ, intersecting Pp in q, and let the ordinate through q meet OX in C. MC is the distance of the point along the stem from the top division of the hydrometer.

FIG. 219.



An instrument could be made to have any required range by making the stem sufficiently long, but it would be very unwieldy, consequently a succession of instruments is arranged for by adjusting the ballasting weight and volume of the body of the instrument, so that where the range of one stem leaves off at the top the range of the next begins at the bottom.

In the form of variable immersion hydrometer constantly used called "Twaddell's hydrometer," the numbers marked at the divisions of the stem are not themselves the specific gravities corresponding to these marks. If n be one of the numbers the specific gravity is $1+5n/1000$ or $1+n/200$.

429. **Hydrometer of Constant Immersion. Nicholson's Hydrometer.**—A constant immersion hydrometer consists of a hollow body with fine stem above, and weighted below to give stability, but carries at the top of the stem a pan to receive weights. There is only one mark on the stem to which the instrument is sunk in all liquids by means of weights placed in the scale-pan above. (See Fig. 220.)

If W be the weight of the hydrometer, and W_w the weight required in the scale-pan to sink the instrument to the mark in water, the weight of water displaced by the instrument when so sunk is $W+W_w$.

Let now W_l be the weight which must be placed in the scale-pan to sink the hydrometer to the mark in a liquid, the specific gravity

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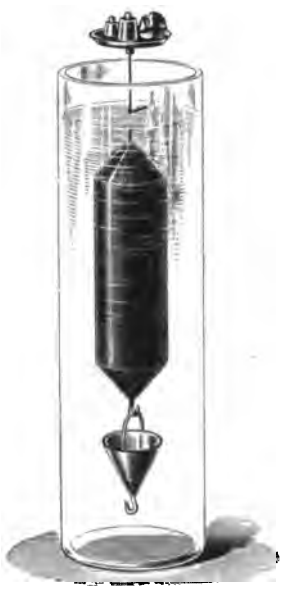
of which is to be determined. The weight of liquid displaced by the hydrometer is $W + W_l$. Hence, if G be the specific gravity,

$$G = \frac{W + W_l}{W + W_w}. \qquad (33)$$

The instrument is clearly, up to a certain limit $[G = W/(W + W_w)]$, applicable to liquids specifically lighter than water.

It is possible also to find the approximate weight of a body (which is not too heavy) with this hydrometer, by placing it in the upper scale pan with additional weights to sink the instrument to the mark. The body is then replaced by weights which sink the hydrometer again to the mark.

FIG. 220.



In Nicholson's hydrometer the instrument just described has added to it a scale pan or basket at the lower end, which enables the specific gravity of a body weighed as just described to be determined. The body after being weighed in the upper scale pan is transferred to the lower, and the additional weight required in the upper pan to sink the instrument to the mark is added.

Let the weight of the instrument be W, of the body W_b, and the weights which must be added in the upper scale pan when the body is in the upper and lower scale pan respectively be W_1, W_2. The weight of water displaced in the first case is $W + W_b + W_1$, in the second it is $W + W_2 + W_b$.

The weight of water displaced by the body in the second case is therefore $(W + W_2 + W_b) - (W + W_b + W_1) = W_2 - W_1$. The specific gravity of the body is therefore

$$G = \frac{W_b}{W_2 - W_1}. \qquad (33)$$

The sensibility of Nicholson's hydrometer is greater the thinner the stem at the mark which is adjusted to be at the liquid surface. For clearly any additional small weight placed in or taken off the scale pan will cause a larger depression or rise of the instrument, since the volume immersed or withdrawn is equal to the cross-section of the stem multiplied by the depression or rise in question. The thickness of the stem is in many of the ordinary instruments absurdly too large. An instrument which the author had constructed is

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wholly of glass and platinum (except the upper scale pan) and has a stem only 1 millimetre in diameter.

The capillary elevation of the water round the stem of the hydrometer produces a downward force upon it; the error due to this is minimised by making the stem as small as possible.

430. **Period of Vertical Oscillation of a Hydrometer.**—The vertical oscillation of such a hydrometer is easily investigated if the motion of the liquid be neglected. Let the instrument floating with the mark at the surface be depressed a further distance x. The additional volume immersed is xs if s be the cross-section. The upward force acting is now in absolute units $(W+\rho xs)g$, if ρ be the density of the liquid, and this, besides balancing gravity, gives an upward accelerating force ρxsg. The upward acceleration is $\rho xsg/W$. Hence we have for the period T of oscillation,

$$T=2\pi\sqrt{\frac{W}{\rho sg}}. \tag{34}$$

The length of the equivalent simple pendulum is thus $W/\rho s$, and is greater the smaller s and the smaller ρ. A very sensitive Nicholson's hydrometer set up in this way has been used by Messrs. Milne and Gray for observations of the vertical motions of the ground in earthquakes. The instrument having a very long period of oscillation does not respond to the quickly varying vertical motions of its support, and gives a steady point relatively to which these motions can be observed.

431. **Pressure in Gases (Boyle's Law).**—A gas, as we have seen, is a fluid that diffuses into any space presented to it, even if that space be already occupied by another gas. Also two portions of the same gas are really diffusing into one another. An equilibrium state, however, is set up in which changes of pressure are balanced at all places in the gas, which is, however, subject to the elementary laws of the pressure in a fluid under gravity.

We must defer to Vol. II. the full discussion of the laws of gases, including the phenomena of diffusion, inasmuch as these involve the conception and measurement of temperature. We may, however, deal shortly with the subject of gaseous pressure under the condition of constancy and uniformity of temperature, or that of non-communication of heat, applying in the latter case the (here assumed) relation between pressure and volume.

We have assumed at § 426 above the law of Boyle, which states that the temperature being kept constant, the pressure in any portion of gas is inversely as the volume which the portion occupies. This law was established by Boyle in an experiment of which we transcribe here his own description. It is to be premised that the question was then being discussed whether the support of the mercury in the barometer, or, as it was then called, the Torricellian tube, was due to the pressure of atmospheric air. To form the Torricellian tube the process followed was that which is used to the present day in making a mercury barometer.

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A clean glass tube of about 3 ft. long and over $\frac{3}{8}$ inch in internal diameter is taken, and is closed at one end. Held with the closed end down it is filled with mercury to within an inch or so of the top, and with the finger of the operator closing the open end, it is inverted. The bubble of air from the unoccupied space runs up carrying with it most of the small bubbles of air at the sides of the tube. It is then inclined back and the air returns to the open end, still further freeing the mercury column from air. This is repeated two or three times, then the unoccupied space is filled with mercury, the finger pressed against the open end, the tube inverted and the open end inserted, before the finger is withdrawn, below the surface of mercury contained in an open vessel. The top of the mercury column is then found to settle to a height of about 30 inches, or about 76 centimetres, more or less, above the level of the mercury in the open vessel.

FIG. 221.

The construction of the instrument can be more perfectly carried out by boiling the mercury in the tube, so as to expel the air. This must be done, however, with great care. First, only about two or three inches of the tube are filled with mercury, heat is then applied by a lamp at the lower or closed end while the tube is held in an inclined position. The mercury is made to boil first at the bottom, and mercury vapour and air are expelled, then the lamp is moved a little higher up to cause the mercury to boil higher up, and so on to within $\frac{3}{4}$ of an inch of the surface. To avoid oxidation of the mercury, the boiling is not allowed to reach the surface. Then a few more inches are added, and the boiling is carried on as before until the tube is all but full. After the mercury has been left to cool undisturbed, the tube is filled up and inverted in the open vessel or cistern as described.

That the column was supported by atmospheric pressure was asserted by Torricelli and proved by the celebrated Blaise Pascal, who carried in 1648 a barometer up a tower in Paris, 30 metres high, and found the height of the column to fall 3 millimetres. Pascal suggested the experiment which was made the same year, of carrying a mercury barometer up the mountain of the Puy de Dôme, when it was observed that in proportion as the pressure fell off with elevation the height of the column diminished.

432. **Boyle's Experiments on the "Spring of Air."**—We shall discuss the barometric principle and various forms of barometer more fully below. We give now Boyle's own account of his remarkable experiments "touching the measure of the force of the spring of air compressed and dilated."

"We took then a long glass-tube, which, by a dexterous hand and the help of a lamp, was in such a manner crooked at the bottom, that the part turned up was almost parallel to the rest of the tube, and the orifice of this shorter leg of the siphon (if I may so call the whole

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instrument) being hermetically sealed, the length of it was divided into inches (each of which was subdivided into eight parts) by a streight list of paper, which containing those divisions, was carefully pasted all along it. Then putting in as much quicksilver as served to fill the arch or bended part of the siphon, that the mercury standing in a level might reach in the one leg to the bottom of the divided paper, and just to the same height or horizontal line in the other; we took care, by frequently inclining the tube, so that the air might freely pass from one leg into the other by the sides of the mercury (we took, I say, care) that the air at last included in the shorter cylinder should be of the same laxity with the rest of the air about it. This done, we began to pour quicksilver into the longer leg of the siphon, which by its weight pressing up that in the shorter leg, did by degrees streighten the included air: and continuing this pouring in of quicksilver till the air in the shorter leg was by condensation reduced to take up half the space it possessed (I say, possessed, not filled) before; we cast our eyes upon the longer leg of the glass, on which was likewise passed a list of paper carefully divided into inches and parts, and we observed, not without delight and satisfaction, that the quicksilver in that longer part of the tube was 29 inches higher than the other. Now that this observation does both very well agree with and confirm our hypothesis, will be easily discerned by him that takes notice what we teach; and Monsieur Paschal and our English friend's experiments prove, that the greater the weight is that leans upon the air, the more forcible is its endeavour of dilatation, and consequently its power of resistance (as other springs are stronger when bent by great weights). For this being considered, it will appear to agree rarely-well with the hypothesis, that as according to it the air in that degree of density and correspondent measure of resistance, to which the weight of the incumbent atmosphere had brought it, was able to counter-balance and resist the pressure of a mercurial cylinder of about 29 inches, as we are taught by the Torricellian experiment; so here the same air being brought to a degree of density about twice as great as that it had before, obtains a spring twice as strong as formerly. As may appear by its being able to sustain or resist a cylinder of 29 inches in the longer tube, together with the weight of the atmospherical cylinder, that leaned upon those 29 inches of mercury; and, as we just now inferred from the Torricellian experiment, was equivalent to them.

FIG. 222.

"We were hindered from prosecuting the trial at that time by the casual breaking of the tube. But because an accurate experiment of this nature would be of great importance to the doctrine of the spring of air, and has not yet been made (that I know) by any man; and because also it is more uneasy to be made

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than one would think, in regard of the difficulty as well of procuring crooked tubes fit for the purpose, as of making a just estimate of the true place of the protuberant mercury's surface; I suppose it will not be unwelcome to the reader, to be informed, that after some other trials, one of which we made in a tube whose longer leg was perpendicular, and the other, that contained the air, parallel to the horizon, we at last procured a tube of the figure exprest in the scheme; which tube, though of a pretty bigness, was so long, that the cylinder, whereof the shorter leg of it consisted, admitted a list of paper, which had before been divided into 12 inches and their quarters, and the longer leg admitted another list of paper of divers feet in length, and divided after the same manner. Then quicksilver being poured in to fill up the bended part of the glass, that the surface of it in either leg might rest in the same horizontal line, as we lately taught, there was more and more quicksilver poured into the longer tube; and notice being watchfully taken how far the mercury was risen in that longer tube, when it appeared to have ascended to any of the divisions in the shorter tube, the several observations, that were thus successively made, and as they were made set down, afforded us the ensuing table:

"A TABLE OF THE CONDENSATION OF THE AIR.

A.	A.	B.	C.	D.	E.
48	12	00	Added to $29\frac{1}{8}$ makes	$29\frac{2}{16}$	$29\frac{2}{16}$
46	$11\frac{1}{2}$	$01\frac{7}{16}$		$33\frac{6}{16}$	$33\frac{6}{16}$
44	11	$02\frac{15}{16}$		$31\frac{15}{16}$	$31\frac{2}{16}$
42	$10\frac{1}{2}$	$04\frac{4}{16}$		$33\frac{8}{16}$	$33\frac{1}{7}$
40	10	$06\frac{9}{16}$		$35\frac{9}{16}$	35
38	$9\frac{1}{2}$	$07\frac{14}{16}$		37	$36\frac{15}{16}$
36	9	$10\frac{2}{16}$		$39\frac{5}{16}$	$38\frac{5}{8}$
34	$8\frac{1}{2}$	$12\frac{8}{16}$		$41\frac{10}{16}$	$41\frac{9}{17}$
32	8	$15\frac{1}{16}$		$44\frac{3}{16}$	$43\frac{11}{16}$
30	$7\frac{1}{2}$	$17\frac{15}{16}$		$47\frac{1}{16}$	$46\frac{3}{5}$
28	7	$21\frac{3}{16}$		$50\frac{5}{16}$	50
26	$6\frac{1}{2}$	$25\frac{3}{16}$		$54\frac{5}{16}$	$53\frac{10}{13}$
24	6	$29\frac{14}{16}$		$58\frac{3}{16}$	$58\frac{3}{5}$
23	$5\frac{3}{4}$	$32\frac{3}{16}$		$61\frac{5}{16}$	$60\frac{8}{15}$
22	$5\frac{1}{2}$	$34\frac{12}{16}$		$64\frac{1}{16}$	$63\frac{6}{11}$
21	$5\frac{1}{4}$	$37\frac{5}{16}$		$67\frac{1}{16}$	$66\frac{4}{7}$
20	5	$41\frac{9}{16}$		$70\frac{11}{16}$	70
19	$4\frac{3}{4}$	45		$74\frac{2}{16}$	$73\frac{11}{16}$
18	$4\frac{1}{2}$	$48\frac{12}{16}$		$77\frac{14}{16}$	$77\frac{2}{3}$
17	$4\frac{1}{4}$	$53\frac{11}{16}$		$82\frac{12}{16}$	$82\frac{4}{17}$
16	4	$58\frac{2}{16}$		$87\frac{14}{16}$	$87\frac{2}{3}$
15	$3\frac{3}{4}$	$63\frac{15}{16}$		$93\frac{1}{16}$	$93\frac{1}{5}$
14	$3\frac{1}{2}$	$71\frac{5}{16}$		$100\frac{7}{16}$	$99\frac{2}{7}$
13	$3\frac{1}{4}$	$78\frac{11}{16}$		$107\frac{3}{16}$	$107\frac{7}{13}$
12	3	$88\frac{7}{16}$		$117\frac{9}{16}$	$116\frac{4}{8}$

AA. The number of equal spaces in the shorter leg, that contained the same parcel of air diversely extended.

B. The height of the mercurial cylinder in the longer leg, that compressed the air into those dimensions.

C. The height of the mercurial cylinder, that counter-balanced the pressure of the atmosphere.

D. The aggregate of the two last columns, B and C, exhibiting the pressure sustained by the included air.

E. What that pressure should be according to the hypothesis, that supposes the pressures and expansions to be in reciprocal proportion."

Boyle took a thin tube, open at both ends, pressed it vertically

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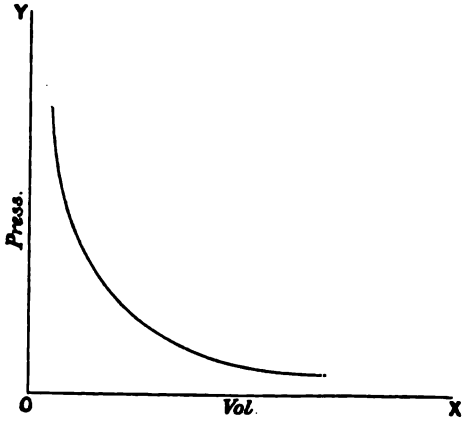
downwards into a deep cistern of mercury until only about an inch protruded, and sealed up the open end with sealing-wax. He had thus a quantity of air which, under atmospheric pressure, filled about an inch of the tube. He then raised the tube to a somewhat higher level, so that the air could occupy a greater length of the tube, and noticed that the mercury rose in the tube above the level in the cistern. The pressure in the air in the tube was then, by § 397, equal to atmospheric pressure less $g\rho h$, where h was the height of the top of the column of mercury in the tube above the mercury-surface in the cistern, and ρ the density of mercury. By raising the tube, therefore, to different distances and measuring the values of h, Boyle made a series of experiments at pressures below atmospheric, which confirmed his former results.

FIG. 223.



If pressures be taken as ordinates of a curve of which the corresponding volumes of a mass of air are abscissæ, the state of the gas is represented by points on a rectangular hyperbola (Fig. 223), which, therefore, is the graphical expression of Boyle's law. A curve, thus showing the connection between the pressures and corresponding volumes of a gas at constant temperature, is called an *Isothermal* curve.

FIG. 224.

The ordinary experimental arrangement for verifying Boyle's law, both for pressures above and pressures below atmospheric, is that shown in Fig. 223. The tube on the left is closed and contains air, that on the right is open and is the pressure measuring tube. Both tubes can be altered as to level, and the pressure applied to the air over and above atmospheric pressure is equal to the height of the level of the mercury surface in the right-hand tube above the surface in the left.

433. **Verification of Boyle's Law. Experiments of Regnault and Amagat.**—The exactness with which Boyle's law is followed by gases formed the subject of an important research by Regnault, and later by Amagat, who extended Regnault's investigation to a much greater range of pressures.

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It is essential that the air compressed should be maintained always at the same temperature. This was carefully secured in Regnault's experiments by means of a water-bath surrounding the space in which the gas was compressed. The apparatus was a Boyle's tube, the short limb of which was 3 metres long, and had been exactly calibrated. Two marks upon this indicated volumes which were in the ratio of 1 to 2. The other limb was about 36 metres long, and was so arranged as to enable different pressures to be applied, and to give an exact measure of these pressures. The tube was so constructed that the limb in which the gas was compressed could be connected at will to a reservoir of gas, and filled with the gas at any required pressure. The gas chamber in the tube was first filled with the gas at a pressure in the reservoir which was noted (the first pressure taken was atmospheric), then the gas was compressed into half the space, and the pressure required observed. The tube was then filled with gas at about two atmospheres, and the volume again reduced to approximately half under an observed pressure. The same experiment was made with gas initially at 4 atmospheres pressure and so on.

This method of proceeding obviated the difficulty of making experiments at very high pressures by compressing, as in Boyle's own mode of experimenting, the air initially at atmospheric pressure into a space so small as to render its measurement with anything like accuracy nearly, if not altogether impossible.

The general result was to show that the product pv fell off slightly for air, nitrogen, and carbonic acid as the pressure was increased; that is, these gases were more compressible at constant temperature than according to Boyle's law, while the contrary was the case for hydrogen. Regnault, however, could not obtain pressures higher than about 45 atmospheres, and the result just given was found by Amagat (who experimented at much higher pressures) to be a very incomplete statement of the facts.

Amagat's apparatus was similar to Regnault's. His compression-tube, however, was placed in a gallery of a coal-pit, where of course the temperature was nearly constant, while his pressure-tube, made of steel, passed up the shaft of the pit and was about 1000 feet long. A number of side branches, at each of which was a tap on the branch and another on the main tube, were provided at intervals along the pipe so that the pressure could be fixed by opening the side tap and closing that on the main tube. Mercury was forced up from below by a force pump until it ran out at the branch fixing the pressure.

Careful experiments were first made on nitrogen with pressures varying from about 30 up to 420 atmospheres. These experiments then enabled the behaviour of other gases to be compared with that of nitrogen by compressing them in closed tubes side by side under the same source of pressure, as was first done by Pouillet.

The general results of Amagat's experiments are given for nitrogen and hydrogen in Figs. 225, 225′, in which the ordinates are values of pv, and the abscissæ are values of the pressure in atmospheres. It will be seen that the gradual fall in the value of pv in

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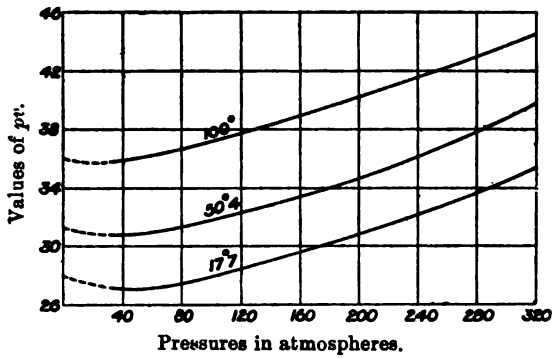
the case of nitrogen vanishes (in the curve at temp. 17·7° C.) at a pressure of about 40 atmospheres, and that for higher pressures the gas is less compressible than according to Boyle's law. Thus at the pressure of minimum value of pv Boyle's law is exactly fulfilled.

FIG. 225.

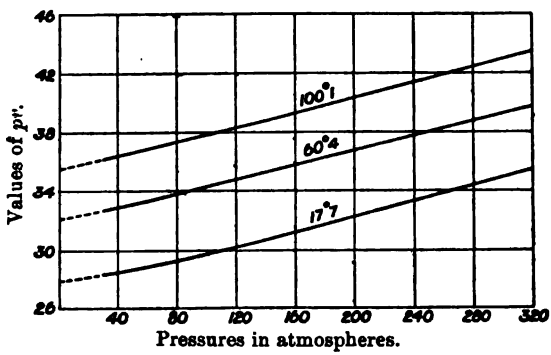


The point at which Boyle's law is exactly fulfilled, it will be observed, occurred at a lower and lower pressure the higher the temperature for which the curve was obtained.

Similar results were observed for air, carbonic acid, marsh-gas, and ethylene.

The diminution of pv to a minimum and subsequent increase

FIG. 225′.



with rise of pressure is very strikingly shown by ethylene and carbonic acid, but a full discussion of the curves is reserved for Part II. of this work, where also will be found an account of the classical researches of Andrews on carbonic acid.

For hydrogen Amagat's curves for all temperatures were straight lines sloping upwards, as shown in Fig. 225′, thus showing less

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compressibility than according to Boyle's law. The great similarity of the hydrogen curves to the high-pressure parts of the nitrogen curves should be noted.

At high pressures, where the curves for nitrogen are nearly straight lines, and for hydrogen so far as the observations go, the equation of the curves is very nearly

$$p(v-b)=c. \tag{35}$$

No very reliable results as to fulfilment of Boyle's law at low pressures have been obtained. At very low pressure there is great difficulty caused by condensation of the gas on the walls of the tube. The general result is to show very slight, if any, deviation from Boyle's law.

434. **Dalton's Law of a Mixture of Gases.**—Another law which includes that of Boyle holds for a mixture of gases. If a number of gases occupy together a certain volume at a certain temperature, the pressure is the sum of those which the gases would give if they occupied separately the whole space at the given temperature. This law is only approximately true, but it may be regarded as being nearly enough so for all practical purposes. This law also will be dealt with in Part II.

435. **Isothermals of a Gas.**—If Boyle's law be regarded as accurately true for a gas it will be seen, as remarked in § 432, that the *isothermal curve*, as it is called, that is, the curve obtained by taking the pressures as ordinates and the corresponding volumes as abscissæ, is a rectangular hyperbola of which the axes of zero pressure and zero volume are the asymptotes. Different curves are obtained for different temperatures; in fact the parameter c in the equation $pv=c$ of the curve is directly proportional to the absolute temperature. [See Vol. II.] Fig. 222 shows the form of the curve. The parameter is the square of the ordinate of the point in which the axis of symmetry, the line bisecting the right angle XOY, cuts the curve.

436. **Sounding Machine. Depth Gauge.** — In Lord Kelvin's sounding machine, now much used for obtaining rapid soundings on approaching the shore in thick weather, a tube coated inside with chromate of silver and closed at one end is let down with the sinker, with the open end downwards. The sea-water entering it compresses the air, and to the height to which it rises discolours the tube, and so leaves a register of the depth. The tube applied to a rod properly graduated enables the depth to be read off in fathoms in a moment.

The graduation of the rod will furnish an example of Boyle's law. Initially the pressure of the air is atmospheric, and the height in fathoms of a column of sea water which gives at its base atmospheric pressure being H, immersion to a depth D raises the pressure in the ratio $(H+D)/H$. If l be the length of the tube, and h the height to which the sea-water rises in it, we have for the ratio of the initial and final volumes of the air $l/(l-h)$. By Boyle's law we have $(H+D)(l-h)=Hl$, and therefore $D=Hh/(l-h)$ is the depth in

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fathoms. To graduate the rod it is only necessary to mark on it at points at different distances h from the end against which the open end of the tube is to be laid the corresponding values of D given by the formula, $Hh/(l-h)$. This gives the depth, of course, of the top of the water column in the tube, but this is accurately enough for all practical purposes the depth of water at the ship.

In this discussion capillary action is neglected, but it is allowed for in graduating the rod. Capillarity of course affects the form of the water surface in the tube and prevents the free surface from becoming horizontal when the tube, following in the wake of the ship, is inclined to the vertical. The height to which the water rises is thus always well defined.

The action of this tube is the same as that of the diving-bell, formerly much used for examining ships and other objects under water. As the bell, which was made heavy and of low centre of gravity, sank mouth downwards in the water, air was compressed into the upper part. It is easy to calculate by Boyle's law the pressure for any depth.

Into this space, however, air was continually pumped from a boat above to support the respiration of the occupants of the bell, so that the pressure was not necessarily that given by the calculation made above for the sounding machine. Now, however, divers almost universally employ the Fleuss diving dress, an air-tight covering into which air is pumped by a tube from above. The diver is therefore free within the limits of the tube to move about on the wreck or sea-bottom.

437. **Theory of Pipette.**—A tube open at both ends is dipped into a liquid to a depth h, the finger is then placed on the upper end and the tube is raised; it is required to find what length of column is raised in the tube. Let x be this length, and l the length of the tube; the pressure in the air above it is $P-g\rho x$ if P be atmospheric pressure and ρ the density of the liquid. The volume of the air has been increased in the ratio of $l-x$ to $l-h$. Thus we have by Boyle's law $(P-g\rho x)(l-x)=P(l-h)$, which gives for x the quadratic equation

$$g\rho x^2-(P+g\rho l)x+Ph=0. \qquad (36)$$

The solution of the problem is the smaller root of this equation.

The same principle may be applied to explain the action of the pipette in which the liquid is made to rise (either by dipping it sufficiently deeply or by suction applied to the upper end) into a wider middle part. In the above solution capillarity is neglected.

438. **Barometer. Measurement of Atmospheric Pressure.**—The construction of a barometer has been described in § 431 above; in practice it is usual to employ various aids to exact determination of the pressure. Before proceeding to speak of these, we may notice here a very erroneous mode of explaining the action of the barometer employed in some popular lectures, as well suited to the intelligence of elementary or young students! The statement made is that the weight of the atmospheric column standing on the surface of the mercury in the open vessel outside the tube balances the weight of

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the mercury column in the tube. The absurdity of this "explanation" is manifest when it is considered that the weight of the column is quite independent of the relation between the area of the surface exposed to the atmosphere and the area of the cross-section of the tube. The area of the surface of the mercury in the open vessel or "cistern" might be increased from a square inch or two to half an acre without appreciably affecting the height of the column supported. But the weight of the atmosphere supported would be increased just as many times as the former area of surface is contained in the new. The statement about the balance of the two weights is therefore pure nonsense.

FIG. 226.

The equality, of course, is between the hydrostatic pressure, or force per unit of area, applied to the surface of the mercury by the air, and the pressure at the same level in the mercury within the tube.

The pressure in atmospheric air varies from hour to hour and from day to day, but in all its fluctuations (except on extraordinary occasions) it never departs more than at most 5 or 6 per cent. from its average value. But any diminution x of the height of the top of the column means a rise of level of the mercury in the cistern, of amount xa/A where a, A are the cross-sectional areas of the column at the top and of the surface of the mercury in the cistern. If x is read off from a scale at the upper part of the tube, the true fall of height *of the column* is $x(1+a/A)$. If the height of the column is known for say the position of the top of the column when $x=0$, the height after the fall x of level of the top is $h-x(1+a/A)$. The quantity xa/A is called sometimes the *correction for capacity*. If x be a rise, of course the formula is $h+x(1+a/A)$.

By making the graduation divisions shorter than the true length in the ratio of $A/(A+a)$ the fall can be read off correctly at once from the scale at the top of the column. Marine barometers are graduated in this way, and moreover a length of the tube intermediate between the top and bottom is made of narrow bore to prevent violent oscillation of the mercury from being produced by the rolling of the ship.

A form of barometer was proposed by Huyghens for the purpose of increasing the range of motion of the top of the pressure column. The mercury was to carry on top a column of water, as shown in Fig. 226. The tube was to be widened at the top of the mercury, and the top of the column of water was to be comparatively narrow. A less volatile liquid, such as glycerine or sulphuric acid, would be much more suitable than water, the vapour pressure of which in the space above the column is considerable and varies rapidly with temperature.

Let h be the height of the mercury, h' the height of the column of liquid above it, ρ, ρ' the densities of the liquids. Let P be the pressure, then in gravitation units we have $P=\rho h+\rho' h'$. Let the

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pressure change to P, and the heights of the columns to h_1, h'_1. We have then $P_1 = \rho h_1 + \rho' h'_1$. Hence

$$P_1 - P = \rho(h_1 - h) + \rho'(h'_1 - h').$$

But if A be the area of the top of the mercury column, a that of the column of liquid above, we have (neglecting the capacity connection) $A(h_1 - h) = a(h'_1 - h')$, so that the increase in height of the liquid is given by $h'_1 - h' = (h_1 - h)A/a$. The whole rise in height of the liquid column is thus—

FIG. 227.

$$h'_1 - h' + h_1 - h = (h_1 - h)\frac{A + a}{a}. \qquad (37)$$

and the increase in height of the mercury is magnified in the ratio $(A + a)/a$.

The form of barometer shown in Fig. 227 is called the *siphon barometer*. The cistern is merely the tube turned up. The lower part is bent back to allow the scale necessary at top and bottom to be in one vertical.

It is usual, however, to employ as a standard a barometer, in which the level of the mercury in the cistern is brought to a position fixed relatively to the instrument. The form frequently employed is that known as Fortin's. The cistern is made with a flexible bottom which can be raised or lowered by turning a screw below the instrument forward or backward, and so the volume of the cistern is diminished or increased. A brass case surrounds the tube and widens out below into an open case surrounding the cistern, the sides of which are of glass, so that the surface of the mercury can be seen. The arrangement of the cistern, &c., will be made out from Fig. 228 (a).

From a horizontal cover over the cistern (Fig. 228) projects downward an ivory point p, which, when the barometer is in a vertical position, the mercury is brought to touch by moving up or down the flexible bottom. This contact is very exactly brought about by observing the upward turned image of the ivory point in the surface of the mercury, and just annulling the apparent gap between the opposed extremities.

The vertical position of the barometer is obtained by hanging it from a ring, properly placed by the maker at the top of the brass case and noting that the vertical screw at the bottom is at the centre of a fixed horizontal ring c below the cistern. Three horizontal screws in the ring then can be made to clamp the instrument in position. A window in front and another behind in the brass case enables the top of the mercury column to be seen.

The level of the mercury is read off on a scale, the zero of which is exactly at the mercury surface when thus adjusted. To get the reading numerically exact a *vernier* is employed, which enables the divisions of the scale to be subdivided; a description of this will be

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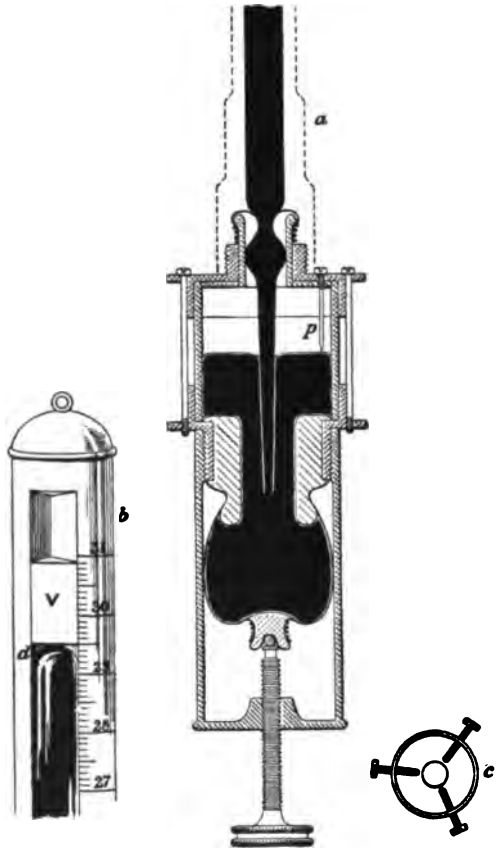
found in chap. xvi. This vernier is engraved on the piece **V** (Fig. 228 *b*), carried by a tube which can be moved up or down by a screw, to a position in which the top of the mercury column is just seen to be on a level with the front and back edges of the bottom of the sliding vernier tube, as shown at *d* (Fig. 228 *b*).

FIG. 228.



439. **Corrections of the Barometer. Standard Atmospheric Pressure.**—The height of the column as thus read requires correction for reduction to the height which would be shown by an ideal barometer uninfluenced by temperature or capillarity under a certain standard value of gravity.

The chief corrections are those for temperature. As the temperature rises the mercury expands, and a column of a given height corresponds to a smaller pressure; but the tube also expands, and hence, as the graduation is on the tube, the height as read off is less

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than the true height, if the temperature is above that at which the scale is correct. Therefore if, for example, the standard temperature is 0° C. for both mercury and scale, and the actual temperature be higher, there is a subtractive correction for the effect of expansion of the mercury and an additive one for the expansion of the tube. It is usual to reduce to a column of mercury at the temperature of melting ice. The height of column read off must be reduced to that which would be obtained from a correct measuring rule, which the scale engraved on the brass case is not, if the temperature is above or below that at which the graduation-spaces have their correct lengths.

Let the temperature be $t°$ C., and the height read off h units. Then mercury expands for 1° rise of temperature by ·000181 of its volume. The density of the mercury will therefore become $\rho = \rho_0/(1 + \cdot 000181t)$ if ρ_0 be the density at 0° C., and

$$g\rho h = g\rho_0 h/(1 + \cdot 000181t).$$

The height thus corrected for the expansion of the mercury is therefore $h/(1 + \cdot 000181t)$, or very approximately $(1 - \cdot 000181t)h$.

Again, if the scale be correct at 0° C., each unit of the brass scale has expanded from 1 to $1 + \cdot 000019t$, and therefore the height h read off is truly

$$(1 + \cdot 000019t)h.$$

The height, therefore, as it would be shown by a barometer containing ice-cold mercury and having a correctly graduated unexpanding scale, is very nearly $(1 - \cdot 000181t + \cdot 000019t)h$, or

$$(1 - \cdot 000162t)h.$$

If, as is generally the case, the scale is correct at some other temperature than 0° C., say $\theta°$ C., the corrected height will have the value

$$\{1 - \cdot 000181t + \cdot 000019(t - \theta)\}h,$$

or $(1 - \cdot 000162t - \cdot 000019\theta)h$. The correction will be zero exactly in this case when

$$t = -\frac{\cdot 000019\theta}{\cdot 000162}.$$

If the temperature is expressed by the Fahrenheit scale, in which 180 degrees measure an interval which is *very nearly* identical with the interval between 0° C. and 100° C., and the temperature 0° C. is exactly the temperature 32° F., the corrected value of h is

$$\{1 - \cdot 0001006(t - 32) - \cdot 0000106(\theta - 32)\}h,$$

in which t and θ are now temperatures on the Fahrenheit scale.

Besides the correction for temperature there is one for the variation of gravity. If the observations have been made at the mean

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sea-level in latitude λ, they may be corrected to the indications of a barometer at sea-level in some standard latitude L.

As a result of pendulum observations made all over the world we have for the ratio of the value g_λ of gravity at latitude λ to its value at latitude L the equation

$$\frac{g_\lambda}{g_L} = \frac{1 - \cdot 0026 \cos 2\lambda}{1 - \cdot 0026 \cos 2L}. \tag{38}$$

If, as is sometimes done, the standard latitude is latitude 45°, the equation is

$$\frac{g_\lambda}{g_{45}} = 1 - \cdot 0026 \cos 2\lambda. \tag{39}$$

In centimetre-second units the value of g in latitude 45° is 980·6 nearly.

If the observations are made at a height h cms. above the surface,

$$\frac{g_\lambda}{g_{45}} = 1 - \cdot 0026 \cos 2\lambda - \cdot 0000000031h. \tag{40}$$

Hence to reduce to latitude 45° we have to multiply by the factor on the right-hand side of this equation. The value of g_{45} in centimetre-second units is very approximately 980·6.

If h and R be taken in centimetres, the term $\cdot 0000000031h$ represents $2h/R$ (R = earth's radius), which, as the reader may verify, expresses the falling off of gravity due to height h above the earth's surface.

Still another correction is necessary for capillarity. This depends upon the internal diameter of the tube and the height of the curved top of the mercury or *meniscus*, as it is called. According as the mercury column is rising or falling the top of the column is more or less curved than the average, and this has a sensible effect on the amount of the capillary depression. A table of corrections is given in books of physical data; but the correction may be taken as quite non-existent for a mercury column over three-quarters of an inch in diameter, and as a standard the use of such a barometer is desirable.

A final small correction for the pressure of mercury-vapour in the space above the mercury is necessary. This may be made by adding (when the height is taken in cms. and the temperature is t° C.) $\cdot 0001t$ to the height.

It should be noticed here that for many purposes in physics what is called a *standard atmospheric pressure* is employed. This is defined in various ways. It may be taken as the pressure given at a place in latitude 45° at the sea-level by a column of mercury 76 centimetres high, and throughout at the temperature of melting ice.

Sometimes, instead of "at a place in latitude 45°," "at Paris" is used in the definition.

440. **Variation of Pressure with Height in the Atmosphere. Height of the Homogeneous Atmosphere.**—At first we suppose the

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temperature to be uniform and the value of g to be independent of the height. If ρ be the density and p the pressure of the air at any height x above sea-level, the pressure at a point dx higher is $p - g\rho dx$. Hence $dp = -g\rho dx$. But by Boyle's law $p/\rho = p_0/\rho_0$, and therefore $\rho = p\rho_0/p_0$ where p_0, ρ_0 are the pressure and density, we shall suppose, at the sea-level. Hence $dp = -g\rho_0 p dx/p_0$. Integrating we obtain, since $p = p_0$ when $x = 0$,

$$\log p - \log p_0 = -\frac{g\rho_0}{p_0}x, \tag{41}$$

or, which is the same thing,

$$p = p_0 e^{-\frac{g\rho_0 x}{p_0}}. \tag{42}$$

(See also § 394.)

The last equation shows that as the height increases in arithmetical progression the pressure diminishes in geometrical.

The quantity $p_0/g\rho_0$ is called the height of the "homogeneous atmosphere" for the place where the pressure is p_0 and the density ρ_0. Denoting this by H_0, we have

$$p = p_0 e^{-x/H_0}. \tag{43}$$

441. **Determination of Heights by the Barometer.**—It is to be remarked here that a height H of the "homogeneous atmosphere" is given for any point at which the pressure is p and the density ρ by the formula $p/g\rho$. This by Boyle's law is clearly the same for all points in the same vertical, and is moreover independent of variations of atmospheric pressure as shown by the barometer.

So far the temperature has not been supposed to vary, but any variation of temperature affects the density ρ without affecting the pressure if the air is free to expand. Thus the density when the temperature is raised from 0° C. to t° C. is diminished (see Part II.) in the ratio $1/(1 + t/273)$, and thus the height of the homogeneous atmosphere H at 0°C. becomes $H(1 + t/273)$ at t° C.

It will be shown under *Sound* that, if V be the velocity a body would acquire in falling under constant gravity g, equal to that at the place where H is determined, through a height $\frac{1}{2}H$, the velocity of propagation of sound is approximately $V\sqrt{\gamma}$, where γ is the ratio of the specific heat of air, pressure constant, to the specific heat of air, volume constant.

It is to be noticed that according to this view the velocity of sound in air is independent of the pressure. This however does not seem to be the case, as it has been shown by Witkowski* that an increase of pressure from 1 to 100 atmospheres increases the velocity of sound by about 7 per cent.

The heights of places above the sea-level can be determined with considerable accuracy by carrying a barometer to the places and determining the pressure at the places (correcting if necessary for

* *Bulletin Intern. de l'Académie des Sciences de Cracovie.* Mars 1899.

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alterations of temperature). The height x is given by equation (41), which may be written

$$x = \frac{p_0}{g\rho_0}\left(\log p_0 - \log p\right) = H \log \frac{p_0}{p}, \tag{44}$$

where H is the height of the homogeneous atmosphere at $0^\circ C$. H is approximately 8000 metres for air at the freezing-point of water.

The quantities p_0, ρ_0 may apply to any station taken as that of reference; x is then the height above the reference level of the station at which the pressure is p and the density ρ.

If the temperature of the lower station be $t^\circ_1 C.$, that of the upper $t^\circ_2 C.$, ·004 be taken as the co-efficient of expansion of ordinary air, x_1, x_2 be the upper heights of the stations, and the observed pressures be p_1, p_2, we have, taking $(t_1 + t_2)/2$ as the temperature of the air, and reducing to latitude 45°, and the freezing temperature of water as shown in § 439, but neglecting the variation of g with height,

$$x_2 - x_1 = 18400(1 + \cdot 0026 \cos 2\lambda)\{1 + \cdot 002(t_1 + t_2)\}\log_{10}(p_2/p_1). \tag{45}$$

Here logarithms to the base 10 are used and $18{,}400 \log_{10}(p_2/p_1)$ represents $8000 \log_e(p_2/p_1)$.

442. **Aneroid Barometer.**—For the determination of heights by this method and in general for the observation of atmospheric pressure by travellers the *aneroid* barometer is almost exclusively employed. The part which responds directly to changes of pressure is a shallow cylindrical box of metal, exhausted of air. The top of the box is strengthened by corrugations of the metal, and rises and falls with variations of air pressure. These motions, magnified by a system of levers, are communicated to an index which ranges round a dial showing pressures and also heights corresponding. The mechanism is best made out from an actual instrument.

FIG. 229.

443. **Convective Equilibrium of Temperature in the Atmosphere.**—It is not the case that the condition of uniformity of temperature imposed on the calculation above is fulfilled. It is found that there is generally a slight fall of temperature with increase of height. According to Lord Kelvin (Mem. Manch. Phil. Soc., March. 1865), the equilibrium of temperature is one brought about by rapid motion of air, and communication of heat has little effect, so that the condition is approximately the adiabatic one expressed by $pv^\gamma = c$, or $p/\rho^\gamma = p_0/\rho_0{}^\gamma$.

According to this condition we shall have, besides $dp = -g\rho dx$,

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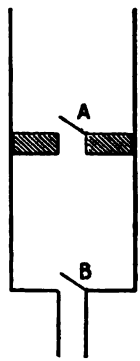
$p/p_0 = \rho^\gamma/\rho_0^\gamma$. Substituting from the second of these in the first and integrating we get

$$\frac{\gamma}{\gamma-1}\left(\frac{\rho^{\gamma-1}}{\rho_0^{\gamma-1}}-1\right) = -\frac{x}{H_0}, \tag{45'}$$

where H_0 is the height of the homogeneous atmosphere for the ground (for H is no longer the same for points in the same vertical). This gives for $\rho = 0$, $x = H_0\gamma/(\gamma-1)$, that is, there is a theoretical limit to the height of the atmosphere, namely (if $\gamma = 1{\cdot}4$) $3{\cdot}44 \times H_0$. But for air at the freezing temperature, at Paris, $H_0 = 8000$ metres very nearly. Hence the theoretical height of the atmosphere is about 28,000 metres, that is 28 kilometres, or very nearly $17\frac{1}{2}$ miles.

FIG. 230.



There are good reasons for supposing that this limit of height does not exist, and it is certain that the adiabatic condition is not exactly fulfilled. The subject of the equilibrium and permanence of the atmosphere of a planet will be dealt with under the *Kinetic Theory of Gases* in Part II.

It will be shown in Part II. also that this convective equilibrium, as it is called, gives a slow uniform fall of temperature with increase of height.

444. **Applications of Boyle's Law.* Air Pumps. Theoretical Exhaustion.**—We have scarcely space left to discuss the more important applications of Boyle's law and will therefore do no more than shortly describe the action of pumps of different kinds and the theory of a balloon.

The air-pump seems to have been invented by Otto von Guericke, and to have been much used by Boyle in many, for that time, very remarkable experiments. Von Guericke, however, first filled the space to be rendered vacuous with water and then pumped out the water with an ordinary suction-pump, and to Hauksbee, Boyle, and Hooke seems due the credit of inventing the modern form of air-pump, for in general construction the ordinary barrel-pump has changed but little since Boyle's day. A cylindrical barrel is open at one end, and at the other end communicates by a pipe with the vessel to be exhausted of air. In this barrel a well-fitting piston is moved backwards and forwards by a rack worked by a crank. In the piston is a valve opening outwards, and at the junction of the communicating pipe with the barrel is another valve opening into the barrel. The valves are shown diagrammatically in Fig. 230, and have different forms in practice. They are sometimes flaps of oiled silk held flat to a plate on two opposite sides, so that air escapes under them on the other two sides when these are slightly raised, and sometimes conical

* Boyle's works will repay perusal by the curious reader. Much space is taken up however in the papers devoted to pneumatics in repelling absurd though dialectically acute objections brought by Hobbes of Malmesbury to the new experiments and doctrines regarding gases.

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stoppers fitting seats from which they are moved, and to which they are returned, either by air pressure or by rods actuated at the proper instants by the mechanism driving the pump.

To understand the action, it is only necessary to suppose the piston to be driven from the outer end of the barrel towards the other. The air having to find its way through the narrow opening of the valve, if that is not closed, is compressed in front of the advancing piston. The valve A is opened, and the valve B is closed. The piston is thus brought down to the farther end of the barrel, and it begins to return. The advance in the opposite direction causes the valve A to close, and the piston therefore leaves behind it a space nearly empty of air. The valve B now opens and air passes from the vessel to the barrel, and when the piston has passed to its first position the air which formerly filled the vessel is now shared between the vessel and the barrel, so that if V, v be the volumes of these, the density has been diminished in the ratio of $V/(V+v)$.

The stroke of the piston just described is now repeated. As it begins A opens and B closes, and the air filling the barrel is transferred in the first half of the stroke to the atmosphere. At the second half-stroke A closes and B opens, and the air in the vessel is again shared between the vessel and the barrel.

Its density is reduced to $V/(V+v)$ of its former amount, that is to $\{V/(V+v)\}^2$ of its initial amount, and so of course also is its pressure.

This process is continued to n strokes, and if we suppose the exhaustion to proceed as here described, and p_0, ρ_0, p_n, ρ_n to represent the pressure and density initially and after n strokes respectively, we shall have

$$p_n = \left(\frac{V}{V+v}\right)^n p_0, \quad \rho_n = \left(\frac{V}{V+v}\right)^n \rho. \tag{46}$$

445. **Effect of Untraversed Space. Limits to Action of Pump.**—If there is untraversed space S at the bottom of the pump barrel, then when the piston moves from that position in the first stroke, there is a space S filled with air at atmospheric pressure. When the piston has been withdrawn to the other end the air which fills the vessel and barrel, that is the volume $V+v$, and is at density ρ_1 at the end of the first stroke, has mass $(V+v)\rho_1$ which is equal to the mass $(V+S)\rho$ if ρ be atmospheric density. At the end of the second stroke the density is ρ_2, and we have $(V+v)\rho_2 = V\rho_1 + S\rho$, so that $\rho_2 = \rho\{(V^2+VS)/(V+v)^2 + S/(V+v)\}$. In this manner we obtain finally

$$\begin{aligned}\rho_n &= \rho\frac{V^{n-1}(V+S)}{(V+v)^n} + \left\{\left(\frac{V}{V+v}\right)^{n-2} + \left(\frac{V}{V+v}\right)^{n-3} + \ldots + 1\right\}\frac{S\rho}{V+v} \\ &= \left(\frac{V}{V+v}\right)^n\left(1-\frac{S}{v}\right)\rho + \frac{S}{v}\rho.\end{aligned} \tag{47}$$

Untraversed space can be got over as in Kravogl's pump, by filling it up with mercury, or, as in some more recent pumps, with

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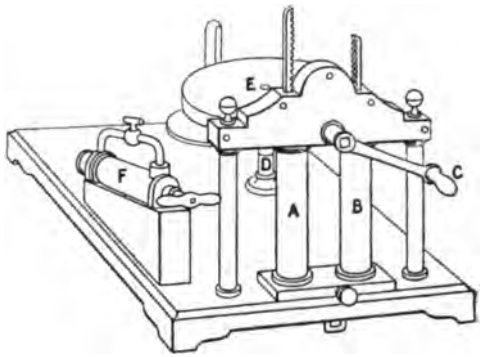
oil previously freed from air. For the highest vacua, however, recourse is always had to some form of mercury air-pump. (See § 447.)

This theory is of very little practical use as finally the pressure of the air may not enable the valves to operate, and unless they are actuated by the mechanism moving the pistons the pump may cease to increase the exhaustion. Another cause of limitation is the air given out by the oil of the valves to the exhausted chamber on the one side and taken in from the other.

The exhaustion is measured by a gauge of the kind shown in Fig. 232. Initially at atmospheric pressure the mercury fills the closed limb completely and stands at a lower level in the other.

FIG. 231.

FIG. 232.



When the exhaustion has come down to a pressure measured by the initial difference of level of the mercury columns the gauge begins to act and the mercury descends in the closed limb and rises in the open more and more towards equality of level.

The instrument is often made with two barrels, the pistons in which move in opposite directions at the same instant, and thus increase the rapidity of the exhaustion. It is stated that the object of this is to give a balance of pressure of the atmosphere on the top of the pistons. Little advantage however is gained from this in ordinary pumps, owing to friction. In a form of pump made by Babinet and otherwise ingeniously arranged, the barrels are of glass, the bodies of the pistons of brass; and after the pump has worked some time so much friction is developed by the unequal expansion produced by heating that it is almost impossible to drive the pump at all. A diagram of the double pump is shown in Fig. 232.

446. **Pumps for Compression of Air.**—If the action of the valves in Fig. 230 be reversed so that the piston valve and the valve at the bottom both open downwards air will be forced into the receiver by the downward motion of the piston, and will be prevented from returning by the closing of the valve *B*, while the piston is returning to the top of the barrel and filling the barrel with atmospheric air by the opening of the valve *A*, in preparation for another stroke.

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A simpler form of compression pump is shown in Fig. 233. Here the valve in the piston is dispensed with, and a hole is left in the side of the barrel as shown at A. When the piston is drawn back it passes the hole at A, through which then the barrel is filled with

FIG. 233.

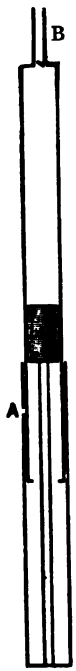
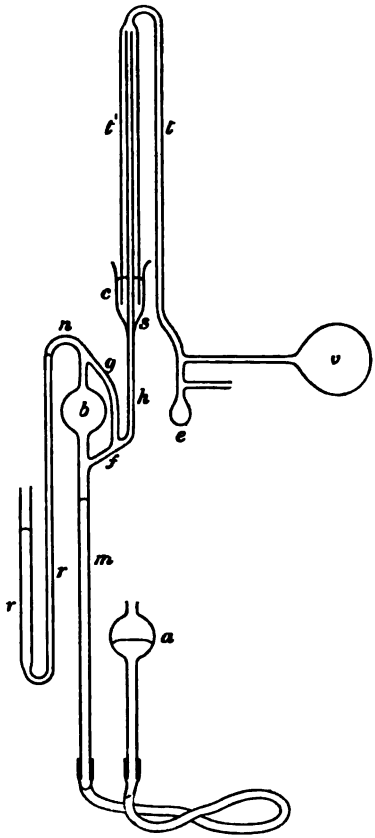


FIG. 234.



air at atmospheric pressure. This is forced into the receiver when the piston makes the next half-stroke.

This is really the action of an ordinary bicycle pump. The valve B is attached to the tyre and consists of a piece of rubber tube projecting from the air tube of the tyre, and *closed except for a hole in one side*. This is surrounded by a metal tube to which the pump is screwed, and the rubber tube is closed against this except when air is being forced in by the pump. This valve is the more tightly closed the higher the pressure of air within the tyre.

The degree of compression obtained by n strokes may be calcu-

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lated in a similar way to that used above for the exhausting pump. The formulæ are modified also by untraversed space, and the theory is rendered only roughly applicable by leakage which can never be avoided in practice.

Compression pumps are of great service in engineering. Compressed air is used for drilling in mines, for pneumatic dispatches, for filling caissons and diving bells; is employed in building piers, and repairing vessels, for driving machinery, and for many other purposes. For the efficiency of compression pumps, and compressed air machinery generally, reference must be made to treatises on applied mechanics.

447. **Mercury Air-pumps. Töpler's Pump.**—Mercury air-pumps are used for the high vacua required for incandescent lamps, and for Röntgen ray tubes, &c. Two forms may be here shortly described, Geissler's displacement pump, as modified by Töpler, and Sprengel's pump.

Töpler's pump shown in Fig. 234 has no stop-cocks, and is fairly convenient for use, though it takes up a considerable amount of space. The vessel to be exhausted is v and is connected (if of glass by blowpipe sealing, or by a joint surrounded by mercury if this is impossible) to the pump by a branch t, t' in the form of an inverted U. The left-hand limb t' of this is wider than the other and is placed over the tube s, and rests with its mouth in a cup c of mercury through which s passes. The height of the part of s surrounded by t' is greater than the barometric height. Mercury is placed in the bend of the tube r, which when the bulb b is exhausted forms a siphon barometer, for the bend n is sufficiently high to prevent mercury from getting over from r.

The mercury cistern a is connected by a strong rubber tube with the vertical main tube m, which is shielded from any stress due to the swaying of the mercury-filled tube by a box of plaster of Paris, in which the joint and a part of the rubber tube are embedded.

The action of the pump is as follows:—The cistern a is raised sufficiently to cause the mercury to rise to the top of b, the air in which is compressed into the long limb of r and bubbles through the mercury in the bend. As the mercury rises through b it passes also into the side tube f and cuts off communication with the vessel v, and then divides passing some way up the tube h, the rest running round the tube g into the mercury which is rising through the bulb.

The cistern is now lowered and the mercury sinks, clearing the bulb b, into which air is prevented from returning by the rising of the mercury in r from the bend. The mercury descends also in h and g, and as soon as the junction of these two tubes has been passed the air from v rushes over and is shared between v and b. Mercury at the same time rises from the cup c between s and t' and prevents ingress of air from the atmosphere to v.

The cistern is again raised and the air now in b is swept out and forced out through r as before; the cistern is then lowered and the air left in v after the first stroke is shared between v and b.

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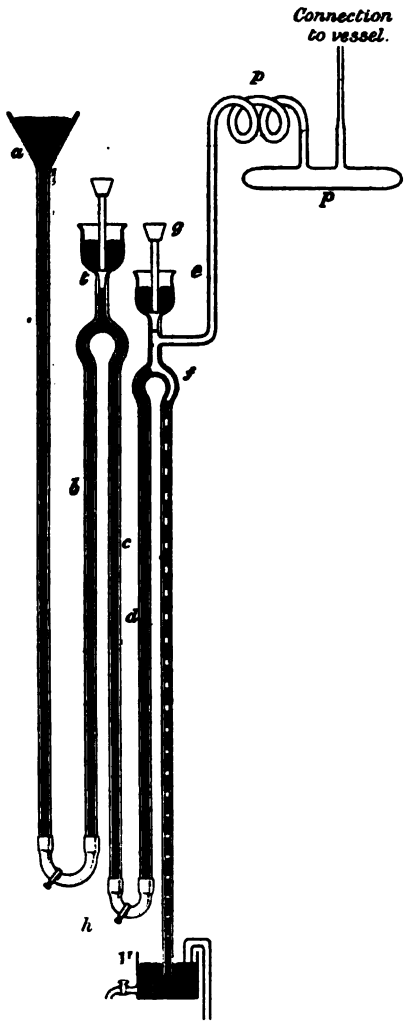
This process is repeated until the exhaustion of v has proceeded far enough. The vessel v may be kept free from aqueous vapour arising from moisture in the mercury by a drying material such as phosphoric acid placed in the bulb e. And a trap for mercury vapour may be arranged by stuffing a portion of the tube leading to t with gold leaf.

FIG. 235.



The extent of the exhaustion may be estimated by observing the pressure (which can be reckoned by the height of the bulb a above the mercury in r) required to compress the air which fills the bulb b into a bubble at the bend n.

448. **Sprengel Pump. Gimingham's Modification.**—Another form of pump much used in the manufacture of vacuum tubes and incandescent lamps is that of Sprengel, as modified mainly by Gimingham, and represented in Fig. 235. Mercury from a raised cistern flows down the tube a and up b, at the top of which any air bubbles contained in it pass into a trap t, then it passes down the tube c and up d, and at the bend it falls in drops into the "fall tube" f, which is only about 2 millimetres in diameter. These drops as they pass down the fall tube carry between them bubbles of air which is drawn by the tube e above from the vessel to be exhausted. The height of the top of the fall tube above the mercury in the delivery vessel v is greater than that of the barometer.

The supply tubes are connected below by strong black rubber tubing strengthened by canvas. These are furnished with pinchcocks to regulate the supply of mercury.

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If the pump has been properly made the fall of the mercury continues to carry the air out in shorter and shorter bubbles, until the highest possible vacuum is attained. The approach to this state is indicated by the metallic hammering sound of the mercury falling in the vacuum, the formation of an unbroken mercury column in the fall tube, and the dropping of the mercury on the top of the column without enclosing visible bubbles of air.

The thistle funnel above the fall tube has a carefully ground neck in which has been ground a stopper *g*. Mercury poured into the funnel round the stopper provides a perfect seal against the ingress of air. The object of the funnel is to provide a satisfactory method of cleaning the fall tube, which is done by stopping the supply of mercury by the pinch-cock leaving the supply tube up to the fall bend full, then running down from the thistle funnel some strong sulphuric acid which has been boiled with sulphate of ammonium to remove volatile matters. The fall tube is allowed to stand for an hour or so full of acid, which is then withdrawn. To avoid the transference to the pump of acid sent in this way to *v*, the mercury should be drawn off from *v* by a siphon reaching near the bottom so that only clean mercury is drawn off and transferred to *a*.

Pumps are made with two or three fall tubes. This is sometimes done by allowing the mercury supplied by one tube, as in Fig. 235, to separate at the fall bend into a stream for each tube, but as the efficiency of the pump depends on the proper regulation of the fall down each tube it is well to duplicate or triplicate the arrangement of supply tube bend with clearing funnel, and fall tube. The supply-tubes are all connected with a single supply from *a* at *h*. Branches from *e* pass to the several fall bends.

The tube *p* contains phosphoric anhydride for drying, as already described for the Töpler pump, and a mercury vapour trap is provided in the same way as before.

For further particulars regarding these pumps the reader may refer to the *Incandescent Lamp and its Manufacture* by G. S. Ram, and for a valuable account of precautions to be used in making the pump he should consult Threlfall's *Laboratory Arts*.

The mercury may be returned to the cistern *a* from *v* by a very ingenious arrangement due to the Rev. F. J. Smith, F.R.S. (*Phil. Mag.*, July 1892) and shown in Fig. 236. A supply of compressed air is obtained by means of a water-jet *L* which draws in air at *G*. The air collects in *H* while the water leaves by the tube *MN*, which is about 8 feet high. The air is supplied from *H* through the drying chamber *Q* to an elevator tube *RCB*, and forces the mercury up to *B* where it falls into *a*.

The arrangement is worked by the pressure energy of the water in the mains; and the water escaping at *N* (and the head given it) may be utilised by receiving the water in a tank there placed from which water may be drawn for any required purpose.

449. **Common Suction Pump.**—The barrel air-pump of Hauksbee is simply the common suction pump applied to the transference of air.

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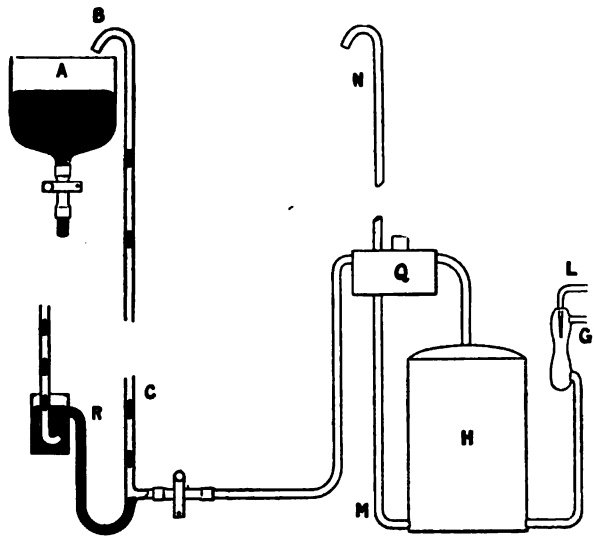
Indeed, the the common pump acts in the first instance as an air-pump. Let the tube communicating with the barrel by the valve B in Fig. 230 be a long vertical tube inserted at its lower open end in a liquid, the surface of which is under atmospheric pressure. We call this the *suction pump*. When the piston makes a stroke, air is withdrawn from the space above the liquid in the tube, and the consequent diminution of pressure causes the liquid to rise to a higher level for the fulfilment of the condition that the pressures outside and inside the tube at the same level in the liquid should be the same. Repeated strokes will, if the suction pipe be not too long, bring the liquid up to the suction pipe and through the valve

FIG. 236.



B, finally through the valve A, until a column stands above the piston. At each upward stroke then the valve A closes, and a quantity of liquid is raised to the level of a spout attached below the top of the barrel and discharged.

In the first stroke let the liquid rise above the supply level a height x_1. If H denote here the height of the column of the barometer made with the liquid being pumped, the pressure in the suction pipe after the first stroke is diminished in the ratio of $H - x_1$ to H. Hence if ρ_0 be the initial density of the air in the pipe and ρ_1 the density after the first stroke, Boyle's law gives

$$\rho_1 H = \rho_0 (H - x_1).$$

By the stroke the air which was contained in the suction pipe

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and the untraversed space in the barrel is made to occupy this space, the whole traversed space in the barrel, and a reduced length of the suction pipe. If we call the cross-section of the suction pump s, that of the barrel S, the height of the top of the suction pipe above the level of the supply a, and the heights of the highest and lowest positions of the lower side of the bucket above the suction pipe valves b, c, we have

$$\rho_1\{Sb + s(a - x_1)\} = \rho_0(Sc + sa).$$

Hence eliminating the ratio ρ_1/ρ_0 between the two equations found we obtain

$$x_1 = H\frac{S(b-c) - sx_1}{Sb + s(a - x_1)}. \tag{48}$$

The liquid will therefore rise to the top of the suction pipe in the first stroke if $x_1 = a$, that is $a/H = S(b-c)/(Ha + Sb)$.

In general, after n strokes have been made the equations connecting the densities ρ_{n-1}, ρ_n of the air and the heights x_n, x_{n-1} of the liquid in the suction pipe before and after the nth stroke, stand

$$\begin{aligned} \rho_n(H - x_{n-1}) &= \rho_{n-1}(H - x_n) \\ \rho_n\{s(a - x_n) + Sb\} &= \rho_{n-1}s(a - x_{n-1}) + \rho_0 Sc \\ &= \rho_{n-1}s(a - x_{n-1}) + \rho_n\frac{H}{H - x_n}Sc, \end{aligned} \tag{49}$$

since $\rho_n H = \rho_0(H - x_n)$.

This gives a quadratic equation for x_n in terms of x_{n-1}, of which the positive root is the value of x_n.

If $x_n = x_{n-1}$ the liquid has ceased to rise higher in the suction pipe, and we have by (49), since now $\rho_n = \rho_{n-1}$,

$$x_n = H\left(1 - \frac{c}{b}\right). \tag{50}$$

Hence the liquid will not rise to the top of the suction pipe if $a > x_n$ that is if

$$a > H\left(1 - \frac{c}{b}\right), \quad \text{or} \quad H < \frac{a}{1 - \frac{c}{b}}. \tag{51}$$

If there be no untraversed space c the pump will just raise the water to the top of the suction pipe if $a = H$: if there be untraversed space the limiting equation is $a = H(1 - c/b)$.

An ordinary suction pump often refuses to act at all owing to the bucket not fitting the barrel sufficiently well. Any want of air-tightness will of course make the above theory inapplicable. But the bucket may be made air-tight and the pump brought into action when otherwise it could not be used, by pouring water into the barrel above the bucket.

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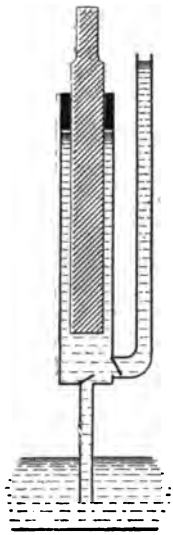
450. **Force Pump.**—In the force pump, Fig. 237, the piston is a valveless plunger; but a valve opening outwards is placed at the junction of the side tube, by which the liquid is to be raised to the required level. The liquid is thus driven into the side tube by the downward stroke and is raised from the supply in the upward. The pumps of a mine must therefore be force-pumps placed near the bottom of the shaft, which necessitates the employment of long communicating rods for driving them if that is done by an engine at the mine-head.

FIG. 237.



451. **Balloons.**—The rise of a balloon in the air affords an example of the law of Boyle, and also of the principle of Archimedes. It consists of a large bag of silk inflated with hydrogen or coal-gas, and carrying a car for the aeronauts with ballast, food, instruments, &c. The gas being specifically lighter than the air, a lifting force equal to the weight of air displaced by the gas and the material of which the balloon and its accessories (including the aeronauts and passengers) are composed is exerted upon it by Archimedes' principle. If this is greater than the total weight of the balloon and everything connected with it, the balloon has an acceleration of amount equal to the excess of force divided by the inertia of the balloon and its appendages.

Let W be the total weight of the silk, car, people carried, &c., W' the weight of air displaced by them, P the weight of the gas, ρ the density of the air, and ρ/n that of the gas, then the weight of air displaced by the gas is nP, and the volume U of gas is nP/ρ. We have therefore for equilibrium

$$W + P = W' + nP,$$

and

$$U = \frac{nP}{\rho} = \frac{n}{n-1}\,\frac{W - W'}{\rho}, \text{ or } P = \frac{W - W'}{n-1}. \qquad (52)$$

Thus if $W - W' = 1$, and the gas be hydrogen, so that $n = 14$, $P = 1/13$ that is a weight $(W - W')$ of 1 ton would be lifted by 1/13 of a ton of hydrogen, which at atmospheric pressure and the freezing point of water would occupy about 30,000 cubic feet.

For coal-gas n used to be taken as 2 roughly, and since a pound of air occupies at atmospheric pressure and temperature about 12·5 cubic feet, 1000 cubic feet of coal-gas will give a lifting force of about 40lbs., which is a practical rule used in ballooning. With coal-gas as now often made n is about 3.

If there be a slight excess of the upward force $(n-1)P$ over the total weight $W - W'$, that is if there be a positive ascensional force of amount $(n-1)P - (W - W')$, the balloon will ascend. This is the pull

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on the rope mooring a captive balloon. If not fully inflated to begin with, the balloon will expand as it ascends until the pressure inside becomes equal to the diminished external pressure, and the only quantity to alter will be W', which, however, is comparatively small. The volume of gas will become V, and the density of the air will become $U\rho/V$ while n remains unaltered. Thus W' will become $W'U/V$. The ascensional force becomes then $(n-1)P-(W-W'U/V)$, which is practically the same as before.

To provide a safety-valve against undue expansion of the gas, sudden passage into the sun's rays and consequent increase of internal over external pressure, and possible bursting of the silk envelope, the neck of the balloon is left open to some extent, and so gas escapes as the balloon rises.

Whatever the amount of gas which may have escaped from the balloon, the ascensional force at height z at which the density of the air is ρ_z and the volume of the gas is V is $V\rho_z(n-1)/n-(W-W'\rho_z/\rho)$. This is zero when

$$\rho_z=\frac{W\rho}{(n-1)Q+W'} \tag{53}$$

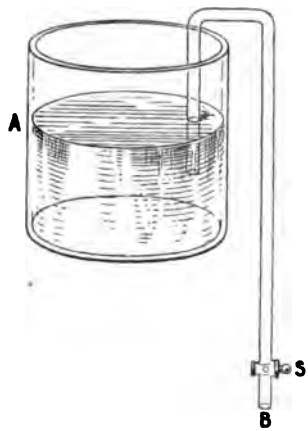
where $Q=V\rho/n$, the mass of gas which would fill the balloon of volume V at the ground.

The mass of gas in the balloon is now $V\rho_z/n=Q\rho_z/\rho$, that is

$$QW/\{(n-1)Q+W'\}$$

so that if the balloon was filled to volume V at starting the loss has been $P-QW/\{(n-1)Q+W'\}$, or if $P=Q$ at starting, $Q[1-W/\{(n-1)Q+W'\}]$.

FIG. 238.



A free balloon can be made to ascend by throwing out ballast to cause it to move upward, or to descend by opening a valve at the top to allow gas to escape. But it is always in its motion either rising or falling, and to steady this upward and downward motion, when the balloon is low enough, a long rope is made to trail after it on the ground. If, then, the balloon rises, part of the rope is lifted off the ground, and a downward pull is applied, and if the balloon descends the part of the rope in the air is shortened, and the downward pull thereby diminished.

For a complete account of the theory of the balloon, the reader should refer to Greenhill's *Hydrostatics*, where also a Bibliography of the subject will be found.

452. **Siphon.**—The instrument called the *siphon* used for transferring liquids from one vessel to another may be described here. It

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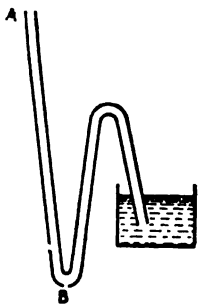
consists as ordinarily used of a tube bent as in Fig. 238, and placed with one limb in the liquid to be transferred as shown. Let us suppose that there is a stopcock S in the limb outside the vessel below the level A. Let S be closed, with the siphon full of liquid. If the height of S above the mouth of the tube be not too great (the barometric height for the liquid is the limit) the liquid below the stopcock will be held in position by the pressure of the atmosphere. There is then atmospheric pressure at A and at B. The pressure below the stopcock is less than atmospheric pressure. If, then, the stopcock be opened, a flow of liquid through it will take place, and this will be continuously kept up until the level of the liquid in the vessel has fallen to B, or the whole of the liquid has run out.

It is to be noticed that the liquid in the vessel and in the siphon tube (as well as the vessel, siphon, &c.) is immersed in the atmosphere.

FIG. 239.



Let us consider a siphon transferring a fluid of density ρ from one vessel to another, when the whole arrangement is immersed in another fluid of density ρ' and both limbs of the siphon are immersed in the liquids. Let h be the height of the upper surface above the level of the lower, and z the distance of the tap from the lower surface; and let the tap be closed. If p_1 be the pressure in the surrounding medium at the lower surface, and p the pressure at the upper $p_1 - p = g\rho' h$. The pressure just below the tap is $p_1 - g\rho z$ and just above $p + g\rho(h - z)$. The excess of the pressure above the tap over the pressure below it is therefore $p - p_1 + g\rho h = g(\rho - \rho')h$.

When the tap is opened, therefore, flow takes place through it in the direction from above to below if $\rho > \rho'$. If, however, $\rho < \rho'$, the flow takes place in the opposite direction and the fluid is transferred from the lower to the upper vessel. Thus hydrogen can be siphoned upward from one vessel to another in air. The flow is zero of course when $h = 0$, that is when the two surfaces are on the same level.

To start a siphon it is necessary to fill it with the fluid to be transferred, close the open ends of the limbs and place it in position, then unclose the ends. The flow will at once begin, and continue at a rate decreasing to zero until the level has fallen to the mouth of the limb in the vessel from which the transfer is taking place, or the difference of levels is reduced to zero.

Ordinarily the fingers are used to close the ends of the siphon, but for transferring many liquids (acids for example) a siphon shaped as in Fig. 239 is used. The siphon is placed in position empty, and is started by suction applied at A, the mouth at B being stopped by a pad applied to it. As soon as the fluid has reached the pad the orifice is opened and the flow continues.

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CHAPTER XI.

GRAVITATIONAL ATTRACTION.

453. **Potential due to Uniform Spherical Shell.**—The subject of gravitational forces between different particles has been considered to some extent in chap. iv., and we here resume the discussion as a preliminary to the chapter which follows on Astronomical Dynamics. The results of the discussion will also be applicable to questions which will arise in connection with electricity and magnetism, and will therefore economise space in the treatment of those subjects.

We have proved at § 198 above that a uniform spherical shell has potential at any external point P equal to kM/r, where k is the so-called gravitation constant, M the mass of the shell, and r the distance of P from the centre. It follows from this that the force of attraction exerted by the shell on a particle of unit mass at P is kM/r^2. For the work done in increasing r by an amount dr, that is, in displacing the particle against the attraction of the shell through that distance, is $kM\{1/r - 1/(r+dr)\}$, which is $kMdr/r^2$. But this is equal to the force of attraction on the particle towards the centre of the shell multiplied by dr, and therefore that force is equal to kM/r^2. It will be convenient to take as the positive direction of the force at P the direction outward from the centre. If F denote the force thus reckoned we have $F = -kM/r^2$. Also from the process of derivation just employed it is evident that if V denote kM/r

$$F = \frac{\partial V}{\partial r}. \tag{1}$$

We call F the *intensity of the field of force* at the point P, that field being the whole space surrounding the attracting matter. It is a result of experience that the different parts of a body attract a particle with forces which exist independently, and therefore the total resultant force on a particle is the resultant of the forces which the different parts thus independently exert. This is called the principle of superposition of attractions. It is assumed in the theoretical discussion of the subject, and the agreement of the results with experience justifies the assumption.

454. **Potential due to Solid Sphere.**—We shall apply this principle now to the case of a solid sphere of matter, the distribution of which

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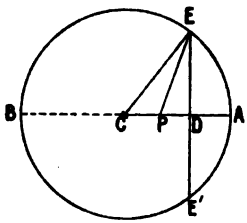
is symmetrical about the centre of the sphere, in other words, a sphere which may be regarded as made up of spherical concentric shells of density uniform over each shell, but not necessarily uniform from shell to shell. Since the intensity of the field at P due to each shell is the same as the intensity which would be produced at P by a particle of mass equal to the shell placed at the centre, the field intensity at P due to the whole sphere is the same as if the whole mass were collected at the centre, that is $-kM/r^2$ where M now denotes the whole mass of the sphere. This theorem, as well as that of the uniform shell, was first given by Newton.

FIG. 240.



Further the potential at any point within a uniform spherical shell of attracting matter is the same at all points. Let the shell be infinitely thin, of radius a, and of mass σ per unit of area. Consider the potential at an internal point P (Fig. 240) produced by a narrow zone EE' of the shell taken perpendicular to the radius CPA. Let CP be denoted by b and PE by f, and the angle ECP by θ. Let the breadth of the zone be $ad\theta$; its radius DE is $a\sin\theta$, and therefore its mass is $2\pi\sigma a^2\sin\theta d\theta$. The potential of this at P is $2\pi k\sigma a^2\sin\theta d\theta/f$. Hence for the potential V at P of the whole shell we have

$$V = 2\pi k\sigma a^2 \int_0^\pi \frac{\sin\theta d\theta}{f}$$

But $f^2 = a^2 + b^2 - 2ab\cos\theta$, so that $fdf = ab\sin\theta d\theta$, or $\sin\theta d\theta = fdf/ab$. This substituted in the equation for V gives

$$V = 2\pi k\sigma \frac{a}{b}\int_{a-b}^{a+b} df = 4\pi k\sigma a. \qquad (2)$$

The potential is therefore independent of the position of P within the shell.

If the shell be not infinitely thin, but constitute a distribution of matter symmetrical about the centre of internal radius a' and external radius a we obtain for this case,

$$V = 4\pi k \int_{a'}^{a} \rho x dx \qquad (3)$$

where ρ is the volume-density at distance x from the centre.

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If ρ be uniform this gives

$$V = 2\pi k\rho(a^2 - a'^2), \tag{4}$$

which again is independent of the position of P if within the shell.

The uniformity of potential thus shown to exist in the internal space shows that the field intensity is there zero.

The potential at any point P within the solid sphere can now be found. Let the sphere be supposed divided into two parts by a concentric spherical surface described through P. If M be the total mass of the internal portion of the sphere, and a' its radius, the potential at P due to this portion of the matter is kM/a'. For the total potential V at P we have therefore

$$V = 4\pi k \int_{a'}^{a} \rho x dx + k\frac{M}{a'}.$$

In all these cases the field intensity at the point P with in the attracting matter is to be calculated by finding the value of kM/a'^2. But here a slight difficulty may present itself to the reader. In order that the field intensity might be experimentally determined at a point within the mass it would be necessary to place there a particle and measure the force exerted upon it. This would necessitate the existence of a small space within the body for the accommodation of the particle, the force on which due to the matter elsewhere was to be observed. In adapting then the result to the case of a body in which no such cavity exists, allowance would have to be made for the matter removed from the cavity in which the point considered is situated, and as this matter is of finite density, and is all very close to the point considered it becomes of importance to inquire what the effect of the removal would be. By the result we may suppose the cavity of a spherical shape, and the point considered to be situated at its centre. Further, the cavity may be taken so small that the density throughout it of the matter which would fill it according to the constitution of the body may be regarded as uniform. This is always possible when the variation of density is continuous. By the result obtained above the potential at the centre of a uniform sphere due to its own matter is $2\pi k\rho a^2$, which vanishes when a is made indefinitely small. The effect therefore of making a small spherical hollow within a body of continuous density has no influence upon the potential at a point within the hollow.

455. **Solid Angles.**—The following definitions will be of service in calculations of potentials and attractions. The *solid angle* subtended at any point P by a closed curve C, such as the edge of an unclosed surface, is measured in the following manner. Lines are drawn from the given point to the points of the closed curve and form a conical surface which intercepts an area upon a sphere of unit radius described from the given point as centre: the area so intercepted is the measure of the solid angle.

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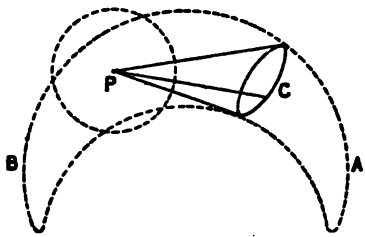
In this definition there is an ambiguity. The conical surface divides the spherical surface into two parts, one of area S, the other of area $4\pi - S$, and it has not been settled which is to be taken as the solid angle subtended at P by C. This must be decided by what is, in the problem in hand, regarded as the surface bounded by C, which is the common boundary of two caps forming a closed surface. Let the cap A, Fig. 40, be the surface bounded by C, and let the other cap B be drawn so that the closed surface encloses P. All straight lines that can be drawn from P can be divided into two groups or pencils, a group G passing through points of S, and a group G' passing through points of $4\pi - S$. Now either every line of G or every line of G' passes through a point of A, but both these statements cannot be true. If the former is the case S is the solid angle, if the latter $4\pi - S$ is the solid angle.

FIG. 241.



It is clear that the solid angle subtended at the centre of a sphere by a curve enclosing an area on the sphere equal to one fourth the spherical surface is π, that the solid angle subtended by a circle at any point in the plane of the circle and within it is 2π, and at any point in the plane of the circle and without it is zero, and that the solid angle subtended by a closed surface at any point within the surface is 4π, and at any point without is zero.

Now take on any surface an area so small that it may be regarded as plane, and draw from any point lines to every point of the periphery of the element. These lines will form a cone of small solid angle which is cut more or less obliquely by the plane area. Let dS be the area, supposed small in every dimension, then every line of the cone will make approximately the same angle with a normal drawn outwards as regards the cone from the area.

Let this angle be θ, the projection of the area at right angles to a generating line of the cone is $dS.\cos\theta$. The projection $dS.\cos\theta$ may be regarded as the area intercepted by the cone on the surface of a sphere of radius r described from the vertex as centre. But since the surfaces of spheres are proportional to the squares of their radii, the solid angle of the cone is $dS.\cos\theta/r^2$, if r be the distance of any point of dS from the vertex of the cone. We may denote this small solid angle for the cone by $d\phi$; hence for the solid angle ϕ subtended at any point which is the vertex of a conical surface on which lies the bounding edge of the surface, we have

$$\phi = \int dS \cos\theta / r^2 \qquad (5)$$

where the integration is extended over the whole surface.

A useful case is the following:—A cone is drawn from the centre of a sphere and intercepts on the sphere an area bounded by a small

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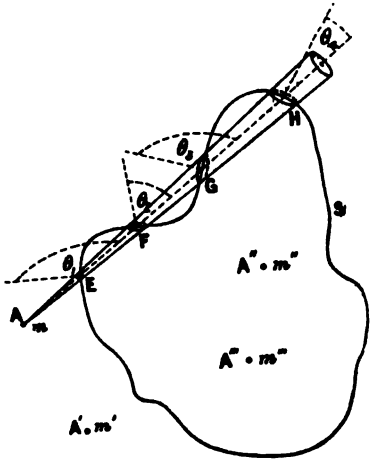
circle the radius of which subtends at the centre of the sphere an angle α. The area of the part of the spherical surface of which this circle is the edge is $2\pi r^2(1-\cos\alpha)$ as the reader may verify. If α is taken less than $\pi/2$ we have the area of the part of the sphere within the small circle; if α is taken greater than $\pi/2$ we have the area of the part which lies outside the circle. The solid angle subtended by the area is therefore $2\pi(1-\cos\alpha)$. Thus if $\alpha=0$, $\pi/2$ or π, the solid angle is 0, 2π, or 4π, as it ought to be.

456. **Surface-Integral of Normal Force.**—Now, returning to the attracting sphere, describe from its centre a spherical surface passing through the external point P. The field intensity F at P acts normally outwards from the spherical surface. Multiplying the area $4\pi r^2$ of the sphere by F we get, by § 454, $-4\pi kM$. This product of field intensity by the area of the surface of the sphere is an example of a general theorem of attraction which we shall now demonstrate.

FIG. 242.



Let a closed surface be drawn in the field of force, let F be the field intensity at an element dS of the surface, and θ the angle which F makes with the outward-drawn normal to dS. The product $FdS\cos\theta$ or $dV/dn.dS$ is the integral of normal force over dS, or, as it is sometimes called, the flux of force across dS. The sum of such products taken for any part of the closed surface is called the surface-integral of normal force over that part of the surface, or the flux of force across it.

The theorem to be proved may be stated as follows. *The surface-integral $\int dV/dn.dS$ of normal force taken over a closed surface in a field of force due to matter attracting according to the inverse square of the distance is equal to -4π times the quantity of matter within the surface, multiplied by the gravitation constant.*

Consider a closed surface S (Fig. 242) and let the matter producing the field be particles, of masses m, m', m'', ..., situated at points A, A', A'', ..., some within and some without the closed surface. Draw from A as vertex a small cone, joining A by straight lines to the different points of a small closed curve s. This cone will intersect at E, F, G, H, small elements of the surface. Call the area of the element at E dS_1, that at F dS_2, that at G dS_3, and that at H dS_4, and let r_1, r_2, r_3, r_4 be the distances of the middle points of these areas from A, and the field intensities at E, F, G, H are respectively $-km/r_1^2$, $-km/r_2^2$, $-km/r_3^2$, $-km/r_4^2$. Now draw normals *outwards*

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from these elements of surface at E, F, G, H, at points at which the elements are intersected by a straight line which may be regarded as the axis of the small cone. Let the angles which these normals make with the positive direction of that line be θ_1, θ_2, θ_3, θ_4.

The normal force outwards at dS_1 is $-km\cos\theta_1/r_1^2$, and the product of this by the area dS_1 of the portion of the surface intercepted by the cone is $-kmdS_1\cos\theta_1/r_1^2$. In this way we get for the elements dS_1, dS_2, dS_3, dS_4, the four products

$$-kmdS_1\cos\theta_1/r_1^2,\ -kmdS_2\cos\theta_2/r_2^2,\ -kmdS_3\cos\theta_3/r_3^2,\ -kmdS_4\cos\theta_4/r_4^2$$

These are the contributions given by the surface elements intercepted by the cone to the surface integral of normal force due to the attracting particle m_1 at A. But any of the expressions $dS\cos\theta$ is numerically the projection of the element dS on a plane at right angles to the axis of the cone, and therefore $dS\cos\theta/r^2$ is numerically equal to the area intercepted by the cone on a sphere of unit radius described from A as centre. The sign of the expression $-kmdS\cos\theta/r^2$ is negative or positive according as $\cos\theta$ is negative or positive, that is in the figure according as the cone emerges from or enters the surface at dS. Thus the expressions are all of the same numerical value, and if there be an even number of them, as is always the case when the apex of the cone is external to the closed surface, the sum of the expressions is zero. Now cones can evidently be drawn from A so as to intercept the whole of the closed surface without excess or defect, and if the apex be external to the closed surface the sum of the contributions to the surface integral of normal force made by the elements of surface will be zero.

If, however, we take a point A'' inside the closed surface, and draw a double cone with its apex at that point, the total number of emergences of the cone from the surface will always be two greater than the entrances, and so the surface integral obtained will by (5) have the value $-4\pi km''$, that is the product of the area of the sphere of unit radius described from A'' as centre and the quantity $-km''$. The same thing holds for any other particle at any other point within the surface, and the normal forces at different points in the surface due to different particles can of course be added together to give the resultant normal force at each point. Thus if we take at each element of the surface the normal force due to the whole distribution, of which part is inside the surface and part is outside, we get for the surface integral of the normal force $-4\pi kM$, where M is the total quantity of attracting matter enclosed by the surface. This theorem is given by Gauss, and is a particular case of a more general theorem due to Green. From it we can derive some important consequences.

457. **Deductions from Theorem of Surface Integral of Normal Force.**—It will be observed that in the proof of this theorem those proved in § 454 are not involved. Consider then a continuous distribution of matter symmetrical about a centre C, that is to say, a uniform spherical distribution, or a distribution made up of uniform concentric

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shells as already described. From C as centre describe a sphere of radius r surrounding the spherical distribution. By the symmetry of the arrangement the field-intensity at every point of this spherical surface must be the same, and be directed normally to the surface. Let its value taken positively outwards be F. The surface integral of normal force for the spherical surface thus drawn is $4\pi r^2 F$, and by the theorem this must be equal to $-4\pi kM$, where M is the total quantity of matter within the surface. Hence we have the equation

$$F = -kM/r^2. \tag{6}$$

But this is clearly the field-intensity which would be produced by a mass M concentrated at the centre of the sphere. Thus we have an independent proof of Newton's theorem—that the attraction of a spherical distribution on an external particle is the same as it would be if the whole quantity of matter were collected at the centre of the sphere.

Next consider a spherical distribution bounded by two concentric spherical surfaces, and let the distribution be symmetrical about the centre of the sphere. Within the internal space describe a concentric spherical surface. By symmetry the field-intensity at every point of this surface must have the same value, and be directed along the normal to the spherical surface. Calling it F as before, and the radius of the sphere r', we have for the integral of normal force over the spherical surface $4\pi r'^2 F$. But this by the theorem is $-4\pi k$ times the quantity of matter within the surface, which is zero. That is, there is no field-intensity within the spherical distribution.

It follows from these results that the potential at an external point due to a spherical distribution of matter, is the same as if the whole mass were collected at the centre, while the potential at an internal point is the same at all such points.

458. **Potential at Internal Point of Spherical Shell.**—To find the potential at an internal point, we may proceed as follows. Consider an infinitely thin uniform spherical shell of total mass m and radius x. The potential at an external point close to its surface is km/x. Now the potential at a point close to the surface, but internal to the shell, must have this same value, since $\Sigma dm/r$ cannot differ for the two cases. The potential at an external point close to the surface is the potential at every point within the shell, since, F being zero at every internal point, there is no variation of potential within the shell.

Now to find the potential at an internal point due to a shell of finite thickness and of density which is a function of the distance from the centre, we proceed as follows. Take any thin concentric shell of radius x and thickness dx. If the density be ρ the mass of the shell is $4\pi x^2 \rho dx$, and the potential at every point on the surface or within it is therefore $4\pi k\rho x dx$. For the potential V at every point in the space internal to the thick shell we have as in (3)

$$V = 4\pi k \int_{a'}^{a} \rho x dx,$$

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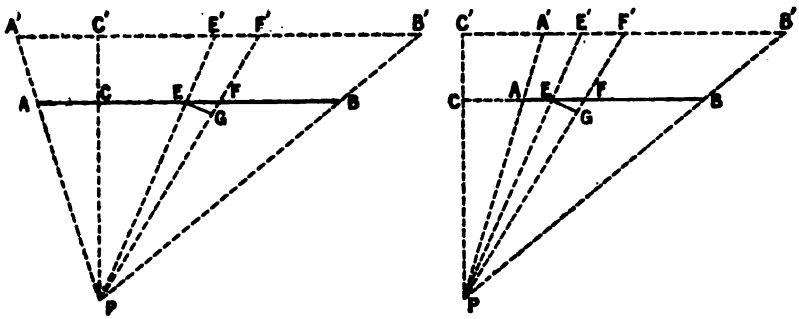
where a' and a are the internal and external radii of the shell. If ρ be uniform, we obtain for the potential $2\pi\rho k(a^2 - a'^2)$. If the density be inversely as the distance from the centre, that is, if $\rho = C/x$, the potential will be $4\pi kC(a - a')$, and so in other cases.

459. **Potential due to Straight Uniform Rod.**—We shall now consider a number of cases of practical importance, and take first the problem of the potential at a point P due to a uniform distribution of matter along a straight line AB.

Let a perpendicular drawn from P to AB meet that line in C and let the length of $PC = x$. Also let the distance CE be denoted by s and the length of the element EF of the line by ds, and its mass

FIG. 242'.



by σds, so that σ is the linear density, or mass per unit length, of the distribution. The potential at P due to the element EF is $k\sigma ds/EP$, or if we put r for EP it is $k\sigma ds/r$.

This value for the potential at P produced by the element can be put into another form from which some results can be immediately deduced. Let EG be a perpendicular let fall from E on FP, and let the angle CPE be denoted by α. Then the change of α corresponding to the element EF may be denoted by $d\alpha$, and we have $rd\alpha/ds = \cos\alpha$. Thus for ds/r we have the value $d\alpha/\cos\alpha$. This depends on the value of $d\alpha$ and α. Hence, if we suppose PE, PF produced to $E'F'$ where $E'F'$ is parallel to EF, we see that if $E'F'$ were an element of a linear distribution of the same density σ, the potential produced by it at P would be the same as that produced by EF. This holds for successive elements of the linear distribution $A'B'$, and therefore the potential at P due to a uniform linear distribution AB is equal to that due to a parallel distribution $A'B'$ in the plane APB, of the same density, and lying between the two lines PA, PB produced if necessary.

460. **Potential of Triangular Lamina at Vertex.**—From this theorem it follows that if from a plane lamina of uniform mass per unit of area two areas AB', $A_1B'_1$, be taken, bounded by parallel lines AB, $A'B'$, A_1B_1, $A'_1B'_1$, and the lines PA'_1, PB'_1, the potentials

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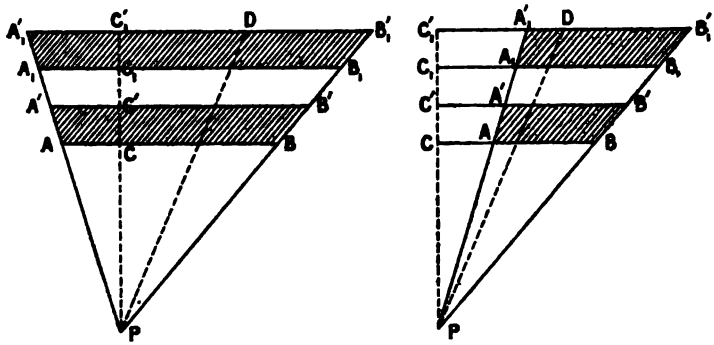
which these areas produce at P are proportional to their breadth. For they may be split up into uniform rods all of the same infinitesimal breadth, each of which produces the same potential at P.

The potential at P of such an area AB' is thus vc if c be the breadth CC' of the area and v be the potential due to an area of unit breadth terminated by the lines PA, PB. It is to be noticed that PC may be zero.

All these theorems are true, it will be seen, if the density of the linear or laminar distributions is not uniform, provided it depends only on the angle α which any line PD drawn in the plane of the distributions makes with a fixed line in the plane, say PC.

FIG. 243.



*461. **Calculation of Potential of Uniform Rod.**—Now returning to the uniform linear distribution we have for the potential due to EF the value $k\sigma ds/r$ $(=k\sigma d\alpha/\cos\alpha)$. Hence, if V be the potential due to the whole rod

$$V = k\sigma \int \frac{ds}{r},$$

where the integration is taken along the rod from A to B. But $r=(s^2+x^2)^{\frac{1}{2}}$, where x is the distance PC of the rod from P, so that

$$V = k\sigma \int_{CA}^{CB} \frac{ds}{\sqrt{(s^2+x^2)}}.$$

Now $d\{\log(s+\sqrt{s^2+x^2})\}/ds = 1/\sqrt{s^2+x^2}$, and therefore, when A and B are, as in the second diagram of Fig. 242, both on the same side of C,

$$V = k\sigma \log \frac{CB+\sqrt{CB^2+PC^2}}{CA+\sqrt{CA^2+PC^2}} = k\sigma \log \frac{CB+PB}{CA+PA}. \qquad (7)$$

If A lies on the other side of C from B, the sign of the length of CA is to be changed in (7). The radicals are to be taken positive.

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The reader may at once infer from (7) the attraction at P due to a uniform rectangular plate in the same plane. It is the ratio to dx of the work done in carrying a strip of breadth dx from the side nearer to a unit particle at P to the side farther from P, that is of the change in V.

If α_1, α_2 denote the angles CPA and CPB, V becomes

$$V = k\sigma \log \frac{\cos\alpha_1(1+\sin\alpha_2)}{\cos\alpha_2(1+\sin\alpha_1)}, \tag{8}$$

and since, as has been seen, $ds/v = d\alpha/\cos\alpha$, we have incidentally the result that (provided $\cos\alpha$ does not vanish between $\alpha = \alpha_1$ and $\alpha = \alpha_2$)

$$\int_{\alpha_1}^{\alpha_2} \frac{d\alpha}{\cos\alpha} = \log \frac{\cos\alpha_1(1+\sin\alpha_2)}{\cos\alpha_2(1+\sin\alpha_1)}. \tag{9}$$

Let now the line extend to both sides of the point C as shown in the first diagram of Fig. 242′. Then if we put a, b for the *lengths* of CA, CB we have

$$V = k\sigma \log \frac{\sqrt{b^2+x^2}+b}{\sqrt{a^2+x^2}-a}. \tag{10}$$

If both a and b be very great in comparison with x this becomes

$$V = k\sigma \log \frac{b+x^2/2b+b}{a+x^2/2a-a} = k\sigma \log \frac{(4b^2+x^2)a}{x^2 b}, \tag{11}$$

or if x^2 be neglected in comparison with $4b^2$,

$$V = 2k\sigma \log \frac{2\sqrt{ab}}{x}. \tag{12}$$

462. **Field Intensity due to Uniform Rod.**—In the case just considered of a long rod extending on both sides of C the component of force in the direction from P to C is $-\partial V/\partial x$ or $2k\sigma/x$, and is thus inversely as the distance PC or x. The attraction of the rod on a unit particle at P is thus the same as that which would be exerted by a particle of mass 2σ situated at the point C, if the attraction varied as the inverse first power of the distance.

The attraction of the finite straight uniform distribution AB (Fig. 244) upon a particle at P is the same as that of a circular arc CD, centre P, of the same linear density σ, and touching AB in L. For consider the element EF. The attraction due to it on a unit particle at P is $k\sigma ds/r^2$. But $ds = rd\alpha/\cos\alpha = r^2 d\alpha/x$. Hence, $k\sigma ds/r^2 = k\sigma x d\alpha/x^2$. But $\sigma x d\alpha$ is the mass of the corresponding element GH of the circular arc and x is the distance of the element from P. Hence, the attraction of GH is the same as that of EF, and the same thing holds for every other element of the straight line and the corresponding element of circle.

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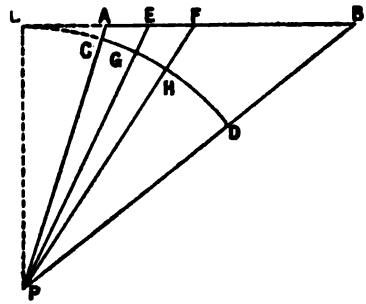
463. **Lines of Force and Equipotential Surfaces of Uniform Rod.**—It follows that the direction of the field intensity at P is along the line bisecting the circular arc CD, that is along the line bisecting the angle APB. If, then, different positions of P be taken, all lying on an ellipse described in the plane of the paper with A, B as foci, the direction of the field-intensity at every such point is that of the normal at the point to the ellipse. It is not difficult to prove that the locus of P is an ellipse if the ratio

$$(LB+PB)/(LA+PA)$$

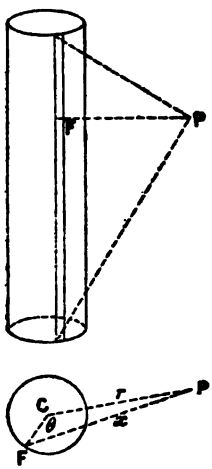
[see (7)] is a constant. The potential due to AB is therefore the same at all points lying on the ellipse as it is at P.

FIG. 244.



Also the direction of the field-intensity at P, since it bisects the angle APB, is that of the tangent to a hyperbola drawn through P with A, B as foci. A particle of unit mass, under the influence only of the attraction of AB would thus, if placed at any point, move off along the hyperbola passing through the point with acceleration equal to the attraction. The hyperbola is thus an example of a *line of force;* that is, a line drawn in the field, such that the field-intensity at every point of it is in the direction of the tangent at that point. We shall discuss such curves more fully later.

FIG. 245.



It will be seen from what has been stated, and it will be referred to later when we discuss *equipotential* surfaces, that the locus of points for which the potential V, due to the uniform linear distribution, has a constant value, is a prolate ellipsoid of revolution described round AB as axis with A, B as foci. Also it will be proved that a line of force drawn through any point of an equipotential surface is perpendicular to the surface. We have an example of this in the result just obtained, that the direction of the field-intensity at P bisects the angle between the lines AP, PB.

*464. **Potential and Field-Intensity due to Long Cylindrical Shell.**—The result expressed in (12) may now be used to give the potential at any point due to a long right cylindrical shell of matter of uniform infinitesimal thickness and uniform surface density.

Consider an infinitely long straight filament F (Fig. 245) of the cylinder at distance x from P, and let c be the radius of the shell, r

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the distance of P from its axis, and θ the angle between the radius CF drawn to the filament and CP in the normal section through P, shown in the lower diagram. Then if $cd\theta$ be the breadth of the filament, τ its thickness, and ρ the volume density of the material, the mass of the filament per unit of length, or σ, is $\rho\tau cd\theta$. By (12) the potential of the filament supposed to extend to distances a, b from F in the two directions is $2k\rho\tau cd\theta\log(2\sqrt{ab}/x)$. But $x^2 = r^2 + c^2 - 2rc\cos\theta$, and therefore the potential of the filament is

$$2k\rho\tau cd\theta\log\{2\sqrt{ab}/\sqrt{(r^2+c^2-2rc\cos\theta)}\}.$$

Hence the potential of the cylindrical shell is

$$V = 2k\rho\tau c\{\log(2\sqrt{ab})\int_0^{2\pi} d\theta - \tfrac{1}{2}\int_0^{2\pi}\log(r^2+c^2-2rc\cos\theta)d\theta\}$$

$$= k\rho\tau c\{4\pi\log(2\sqrt{ab}) - \int_0^{2\pi}\log(1+\frac{r^2}{c^2}-2\frac{r}{c}\cos\theta)d\theta - 2\log c\int_0^{2\pi}d\theta\}. \quad (13)$$

The first integral is $4\pi\log(r/c)$ or zero, according as $r >$ or $< c$. In the former case therefore (which is that here considered) we have

$$V = k\rho\tau c\{4\pi\log(2\sqrt{ab}) - 4\pi\log\frac{r}{c} - 4\pi\log c\} = 4\pi k\rho\tau c\log\frac{2\sqrt{ab}}{r}. \quad (14)$$

The potential is therefore the same as that produced at P by a single filament of mass per unit length $2\pi\rho\tau c$ situated at the axis of the cylinder; that is, a filament of mass per unit length equal to the total mass per unit of length of the cylinder.

The field-intensity at P is $\partial V/\partial r$, and is $-4\pi k\rho\tau c/r$.

If the point P be inside the shell the analysis is the same as before, but we have now $r < c$. The first integral in (13) is then zero, and we have

$$V = 4\pi k\rho\tau c\log\frac{2\sqrt{ab}}{c}. \quad (15)$$

Since this is independent of r the potential is constant inside the cylinder; that is, the component of field-intensity in any cross-section is zero.

*465. **Potential due to a Cylindrical Shell of finite Thickness.**—We can now find the potential at any external point due to a right cylindrical shell of finite thickness, say that between two coaxial cylindrical surfaces of radii c_1, c_2. This may be divided into thin coaxial shells for each of which the result stated in (14) holds. Hence for such a shell we have only to replace in (14) $2\pi c\tau$ by $\pi(c_2^2 - c_1^2)$.

$$V = 2\pi k(c_2^2 - c_1^2)\rho\log\frac{2\sqrt{ab}}{r}. \quad (16)$$

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Again, if P be within the shell at the distance from the axis $c' > c_1$ and $< c_2$, we have for the matter internal to the cylinder of radius c'

$$V_1 = 2\pi k\rho(c'^2 - c_1^2)\log\frac{2\sqrt{ab}}{c'}, \tag{17}$$

and for the matter external to the same cylinder

$$V_2 = 4\pi k\rho\int_{c'}^{c_2} cdc\log\frac{2\sqrt{ab}}{c}. \tag{18}$$

Therefore

$$V_2 + V_1 = 2\pi k\rho\{c_2^2\log\frac{2\sqrt{ab}}{c_2} - c_1^2\log\frac{2\sqrt{ab}}{c'} + \tfrac{1}{2}(c_2^2 - c'^2)\}. \tag{19}$$

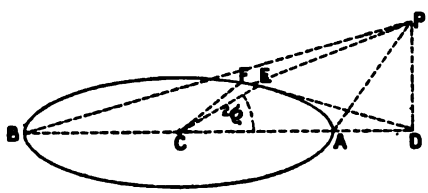
If $c_1 = 0$; that is, if the cylindrical space *internal* be completely filled with matter of density ρ, we have for (16) and (19)

$$V = 2\pi k\rho c_2^2\log\frac{2\sqrt{ab}}{r}. \tag{20}$$

$$V = 2\pi k\rho\{c_2^2\log\frac{2\sqrt{ab}}{c_2^2} + \tfrac{1}{2}(c_2^2 - c'^2)\}. \tag{21}$$

*466. **Potential due to a Uniform Circular Distribution.**—It is of some importance to find the potential at any point P of a uniform

FIG. 246.



circular distribution of matter. Let σ be the linear density of the distribution, c the radius of the circle, AB the intersection with the circle of a plane normal to that of the circle and passing through P and C. Let the perpendicular distance PD from the circle be h. Let 2ϕ be the angle ECA and consider an element $2cd\phi$ of the circle at E. Join E to D. Then denoting EP by r we have $r^2 = h^2 + ED^2$. But if f denote CD, $ED^2 = f^2 + c^2 - 2cf\cos 2\phi$, and therefore

$$r^2 = f^2 + c^2 + h^2 - 2cf\cos 2\phi.$$

If PA, PB be denoted by a, b respectively we have

$$a^2 = f^2 + c^2 + h^2 - 2cf, \quad b^2 = f^2 + c^2 + h^2 + 2cf,$$

and

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$$r^2 = (f^2 + c^2 + h^2)(\cos^2\phi + \sin^2\phi) - 2cf(\cos^2\phi - \sin^2\phi) = a^2\cos^2\phi + b^2\sin^2\phi.$$

Thus we obtain

$$V = 2k\sigma c \int_0^\pi \frac{d\phi}{(a^2\cos^2\phi + b^2\sin^2\phi)^{\frac{1}{2}}}. \tag{22}$$

It is to be observed that the value of this integral is not affected by interchanging a and b. For if $\psi + \pi/2$ be put for ϕ the integral becomes

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\psi}{(b^2\cos^2\psi + a^2\sin^2\psi)^{\frac{1}{2}}} = \int_0^\pi \frac{d\psi}{(b^2\cos^2\psi + a^2\sin^2\psi)^{\frac{1}{2}}},$$

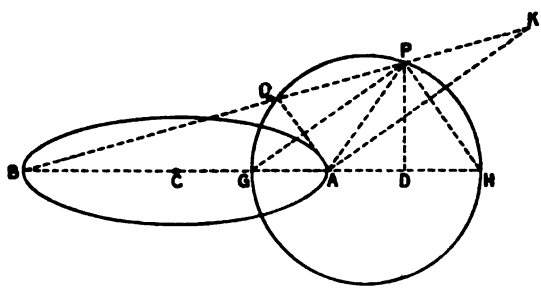
the same integral as before with a, b interchanged. Hence

$$V = 2\frac{k\sigma c}{a} \int_0^\pi \frac{d\phi}{(\cos^2\phi + \frac{b^2}{a^2}\sin^2\phi)^{\frac{1}{2}}} = 2\frac{k\sigma c}{b} \int_0^\pi \frac{d\phi}{(\cos^2\phi + \frac{a^2}{b^2}\sin^2\phi)^{\frac{1}{2}}}, \tag{23}$$

so that the integral here may be taken as either $f(b/a)$ or $f(a/b)$, where $f(b/a)$ denotes a certain function of b/a.

Now through P describe a circle in the plane ABP and inter-

FIG. 247.



secting the plane ABE of the circular distribution in two points G, H, which divide AB internally and externally in the ratio a/b as shown by the construction in Fig. 247. The lines AQ, QB joining *any* point in this circle with AB are in the ratio a/b. Hence the value of V will be found by calculating the integral for the point G or the point H in the plane of the circle. The problem is thus reduced to finding the potential due to the circular distribution at a point in its own plane.

We choose the point of internal division G of AB and put r for

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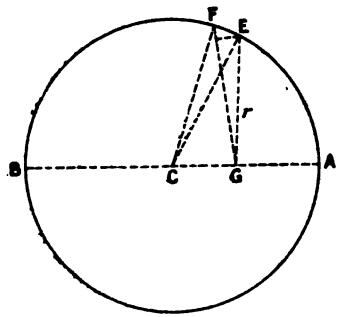
EG and s for CG, which, of course, is known. By what has already been proved, if 2ϕ denote the angle ECG, we have, if V_G be the potential at G,

$$V_G = 2\frac{k\sigma c}{c-s}\int_0^\pi \frac{d\phi}{\left(\cos^2\phi + \frac{(c+s)^2}{(c-s)^2}\sin^2\phi\right)^{\frac{1}{2}}}. \tag{24}$$

FIG. 248.



Let θ denote the angle AGE. By Fig. 248,

$$\frac{EF}{r} = \frac{d\theta}{\sin EFG} = \frac{d\theta}{\cos CFG}.$$

But $\sin CFG/\sin\theta = s/c$, so that $\cos CFG = \frac{1}{c}\sqrt{(c^2 - s^2\sin^2\theta)}$. Hence

$$V_G = 2k\sigma c\int_0^\pi \frac{d\theta}{(c^2 - s^2\sin^2\theta)^{\frac{1}{2}}} = 2k\sigma c\int_0^\pi \frac{d\theta}{\{c^2\cos^2\theta + (c^2 - s^2)\sin^2\theta\}^{\frac{1}{2}}}. \tag{25}$$

Thus by (24) and the equation just found

$$V = \frac{2k\sigma c}{c-s}f\left(\frac{c+s}{c-s}\right) = 2k\sigma f\left(\frac{\sqrt{c^2-s^2}}{c}\right). \tag{26}$$

In the discussion above $b > a$ and $b/a = (c+s)/(c-s)$. Denote $c - s$ by a_1, $c + s$ by b_1, and form a succession of quantities, $a_2, b_2, a_3, b_3, a_4, b_4$, . . . in the following manner:

$$a_2 = \sqrt{a_1 b_1}, \quad b_2 = \frac{a_1 + b_1}{2}, \quad a_3 = \sqrt{a_2 b_2}, \quad b_3 = \frac{a_2 + b_2}{2},$$

. . . . . .

Then $b_1 - a_1 - (b_2 - a_2) = \frac{1}{2}b_1 + \sqrt{a_1 b_1} - \frac{3}{2}a_1 > 0$ since $b_1 > a_1$. Hence $b_1 - a_1 > b_2 - a_2$, and we have the series of inequalities

$$b_1 - a_1 > b_2 - a_2 > b_3 - a_3 > b_4 - a_4 > \ldots$$

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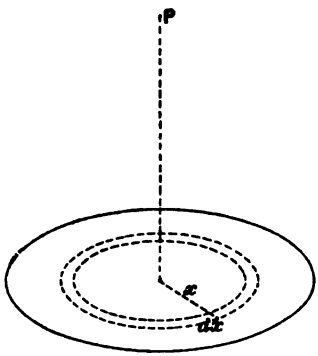
The differences thus continually diminish. Since this process can be carried as far as we please, we can take n so large that $b_n - a_n$ can be made smaller than any quantity that can be named. Thus we obtain to any required degree of approximation from (26), since it gives $f(a_1/b_1)/b_1 = f(a_2/b_2)/b_2 = \ . \ . \ . \ . \ = f(a_n/b_n)/b_n$,

$$V_G = k\sigma \frac{2\pi c}{a_n}. \tag{27}$$

This process of calculation is due to Gauss,* who named the limiting value a_n the *arithmetico-geometric mean* of the quantities, a, b. It is practically an application of what is known as *Landen's transformation*, by which an elliptic integral of the first kind given in terms of one modulus can be changed into an integral of the same kind in terms of another modulus. The value of V can be found at once by the known series for the complete elliptic integral of the first kind; but its value can also be found by successive approximation in the simple manner indicated above.

FIG. 249.



467. **Potential at a Point on the Axis of a thin Circular Disc of Attracting Matter.**—Let P be the point on the axis, x the radius of a narrow concentric ring of the disc of breadth dx. If σ be the surface density (matter per unit area) of the disc, the matter on the ring is $2\pi\sigma x dx$, and its potential is $2\pi k\sigma x dx/\sqrt{(x^2+h^2)}$, where h is the distance of P from the plane of the disc. Hence, if a be the radius of the disc,

$$V = 2\pi k\sigma \int_0^a \frac{x dx}{\sqrt{(x^2+h^2)}}.$$

But $d(\sqrt{x^2+h^2})/dx = x/\sqrt{x^2+h^2}$, and therefore, taking the limiting values, we have

$$V = 2\pi k\sigma(\sqrt{a^2+h^2} - h). \tag{28}$$

If h be small in comparison with a this equation becomes

$$V = 2\pi k\sigma a. \tag{29}$$

The attraction of the disc on a unit particle at P is the work done against the attraction in carrying a unit particle through an infinitesimal increase of distance δh divided by that distance. For the work thus done is $-\delta V$, and this is $-F\delta h$. Hence in the limit the field-intensity is

* See his paper, *Determinatio Attractionis*, &c. Werke, Bd. iii. s. 357.

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$$F = \frac{dV}{dh} = -2\pi k\sigma\left(1 - \frac{h}{\sqrt{a^2+h^2}}\right), \tag{30}$$

so that if h be very small

$$F = -2\pi k\sigma. \tag{31}$$

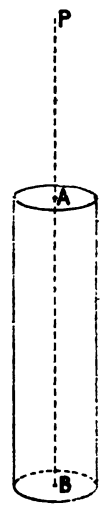
This last result is of great importance and will be of frequent application in what follows.

468. **Field-Intensity at a Point on the Axis of a Circular Disc of Finite Thickness.**—If the disc, instead of being thin, be of finite thickness, and have uniform volume-density ρ, then putting $\sigma = \rho dh$ we get for the attraction of a thin slice, perpendicular to the axis, of thickness dh,

$$-F = 2\pi k\rho\, dh\left(1 - \frac{h}{\sqrt{a^2+h^2}}\right).$$

Hence, if the disc be comprised between the values $PA = h_1$, $PB = h_2$ of h as shown in Fig. 250, we have for the field-intensity at P due to the whole cylinder

FIG. 250.



$$F = -2\pi k\rho\int_{h_1}^{h_2}\left(1 - \frac{h}{\sqrt{h^2+a^2}}\right)dh$$

$$= -2\pi k\rho\{h_2 - \sqrt{h_2^2+a^2} - (h_1 - \sqrt{h_1^2+a^2})\}. \tag{32}$$

If a be very great in comparison with both h_2 and h_1

$$F = -2\pi k\rho(h_2 - h_1), \tag{33}$$

as might have been inferred from the result (31) already obtained.

If h_2 be very great and h_1 be very small both in comparison with a

$$F = -2\pi k\rho a. \tag{34}$$

If h_2 be very great in comparison with a and h_1 be finite

$$F = -2\pi k\rho(\sqrt{h_1^2+a^2} - h_1). \tag{35}$$

469. **Field-Intensity on Axis of Long Cylinder coincides with Potential of Thin Disc.**—If σ be substituted for ρ this (with changed sign) becomes the expression (28) already obtained for the potential produced by a thin disc of surface density σ, at a point P on its axis, and distant h_1 from its plane. Therefore the field-intensity at P due to an infinitely long solid cylinder of volume-density ρ, the axis of which produced passes through P, is numerically the same as the potential at P due to a thin disc coinciding with the end of the cylinder, and of surface density numerically equal to ρ. The two results (29) and (34) are the expression of

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the same theorem for the particular case of h_1 very small in comparison with a. The theorem is of course evident without analysis.

470. **Potential and Field due to a Line-Distribution of Matter.**—A similar theorem holds for the force and potential produced by a thin uniform rod at a point P in its own line. Let the linear density of the rod (mass per unit of length) be μ, then the potential of a length dx at distance x from P is $k\mu dx/x$. Hence, the potential of the linear distribution is

$$V = k\mu \int_{h_1}^{h_2} \frac{dx}{x} = k\mu \log \frac{h_2}{h_1}, \tag{36}$$

where h_1, h_2 are the distances of the near and further ends A, B of the rod from P. V is therefore infinite if $h_1 = 0$, that is, if P is at the end A of the rod.

If a unit particle at P were withdrawn from the rod through a distance dh_1, the work done by external force would be $-Fdh_1$, and V would be diminished by an equal amount dV. But since $dh_2 = dh_1$

$$-dV = k\mu dh_1 \left(\frac{1}{h_1} - \frac{1}{h_2}\right).$$

The quantity on the right is positive, and is the excess of the potential at P due to a mass $k\mu dh_1$ at the end A of the rod over the potential at P due to the same mass at the farther end of the rod. That is, the work done by withdrawing the particle at P through the distance dh_1 from the end A_1, or the work done in withdrawing the rod through the same distance from the particle, is equal to the work done in removing the mass μdh_1 from A to B.

For the field-intensity at P we have

$$F = -k\mu \left(\frac{1}{h_1} - \frac{1}{h_2}\right); \tag{37}$$

that is, the field-intensity at P is equal to the difference of the potentials produced at P by equal particles of mass μ situated at the ends of the rod.

If $h_2 = \infty$ the work done in increasing h_1 by dh_1 is $k\mu dh_1/h_1$, and the field-intensity due to the infinitely long linear distribution is given by

$$F = -\frac{k\mu}{h_1}, \tag{38}$$

which is the potential produced at P by a particle of mass μ situated at the near end A of the rod.

Several of the results given above are of use in the mathematical theory of Capillary Action.

*471. **Gauss's Theorem of Average Potential over Spherical Surface.**—We now prove a very important theorem, due to Gauss, of

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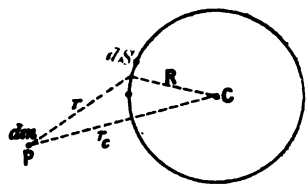
which many applications will be made in what follows. It may be stated thus:

The mean potential over a spherical surface of radius R described in a gravitational field of force is equal to the potential at the centre of the sphere, provided the sphere does not inclose any part of the matter to which the field is due.

Let dS be a small element of the surface of such a sphere, r its distance from any particle dm of the attracting matter at a point P. The potential at dS due to dm is kdm/r; let this be multiplied by dS, and the sum $kdm\int dS/r$ of the quantities be taken for all the elements of the spherical surface. This sum divided by $4\pi R^2$ is what is defined as the mean potential over the sphere due to the particle dm at P. But clearly kdS/r is the potential at P due to matter of unit surface density situated on dS, and the total potential at P due to matter thus distributed over the whole sphere is equal to the potential which would be produced at P if the whole quantity on the surface were situated at the centre. This whole quantity is $4\pi R^2$; therefore the potential at P in this case, that is $k\int dS/r$, is $4\pi kR^2/r_c$, where r_c is the distance of the centre C of the sphere from P. Hence $kdm\int\{dS/r\}/4\pi R^2$ is kdm/r_c, that is the potential at C due to dm at P. The theorem thus holds for the part of the potential due to dm at P, and therefore by the principle of superposition holds for the potential produced by the whole external distribution.

FIG. 251.



*472. **More General Theorem of Potential over Spherical Surface.**—This is a particular case of a more general theorem also due to Gauss. Let *any* sphere be drawn in the field of the distribution. The potential at an element dS of the surface due to dm, at a distance r from dS, is kdm/r; the product of this by dS, $kdSdm/r$, may be taken also as the product by dm of the potential, kdS/r, produced at the position of dm by a distribution of surface density unity over the element dS of the sphere.

Now we may take the sum of such products for the whole distribution and the whole sphere in either of two ways: (1) by finding $k\int dm/r$, the potential at dS produced by the whole distribution, and then taking the sum of the products $kdS\int dm/r$ for the whole sphere; or (2) by calculating first $k\int dS/r$, the total potential produced by the elements of the spherical surface at the position of dm, and then finding the sum of the products $kdm\int dS/r$ for the distribution. The two methods obviously lead to the same result. We take the calculation of $\int dS/r$. For all points external to the sphere $k\int dS/r = 4\pi R^2/r_c$, if R be the radius of the sphere, and r_c the distance of the point considered from the centre of the sphere. Also for points internal to the sphere $\int dS/r = 4\pi kR^2/R = 4\pi kR$, since it is the potential at a point within a spherical surface produced by a

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uniform distribution of density unity on the surface, and, as we have already seen, is equal to the potential at the centre.

Hence, supposing the matter of which dm is an element, divided into two parts, the one external, the other internal to the spherical surface, we get

$$\int V dS = 4\pi k R^2 \int \frac{dm_e}{r_c} + 4\pi k R M_i, \tag{39}$$

where V denotes the potential at dS due to the whole distribution, dm_e an element of the external distribution, and M_i the total matter internal to the sphere.

If M_i be zero we obtain

$$\frac{1}{4\pi R^2}\int V dS = k \int \frac{dm_e}{r_c}, \tag{40}$$

the theorem of mean potential already demonstrated.

Now there are two ways in which M_i may vanish: (1) there may be no matter at any internal point, which is the only case to be considered with ordinary gravitating matter; (2) there may be just as much negative matter at certain internal points as there is of positive matter at others. This case is of importance in the theory of electricity, and need not be here further dealt with.

If dm_e be zero, that is, if there be no matter external to the surface,

$$\int V dS = 4\pi k R M_i,$$

or

$$\frac{1}{R}\int V dS = 4\pi k M_i. \tag{41}$$

In the last form of the equation we have the theorem that if the surface integral of potential over the spherical surface be divided by the radius of the sphere, the result is equal to $4\pi k M_i$, that is, the surface integral of *inward* normal force over the surface.

If M_i be also zero (41) states, that the surface integral of potential over any sphere not surrounding any part of the distributions is zero. But the potential is then zero at every point.

473. **Deductions from Gauss's Theorem.**—It immediately follows from the theorem of the mean potential (40) that the potential due to a distribution of gravitating matter cannot have a maximum or a minimum value at any point of the field of force unoccupied by attracting matter; for, if such a point existed, it would be possible to describe about it as centre a sphere so small that the potential at each point of the surface should, in the case of a maximum, be less than the potential at the centre, and in the case of a minimum, be greater than the potential at the centre. In the former case the mean potential over the spherical surface would be less than the potential at the centre, in the latter case it would be greater; in

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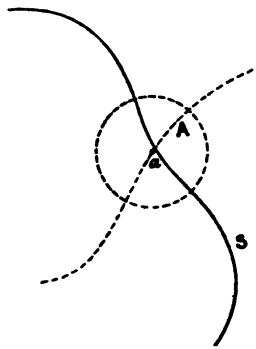
either case Gauss's theorem of the mean potential would be violated. Thus, as stated, no such point can exist.

Another important consequence of Gauss's theorem is the following: If the potential throughout any region of the field have a constant value, then it must have the same value at every point of the field that can be reached by a path drawn from the region in question without passing through any part of the attracting matter. For let a be a point just within the boundary S (Fig. 252) of the region through which this path passes: it is possible to describe round a as centre a sphere of radius so small that the sphere does not include any part of the attracting matter. This sphere will be approximately half within and half without the region specified. The potential at the centre is the uniform potential of the region, as is likewise the potential at each point of the portion of the sphere included within the region. Now if the potential in the part of the sphere outside the region be not the same as that of the region, there must be variation of potential along lines drawn through a, from across the boundary into this portion of the sphere. The sphere may be made so small that the potential at every point of the part of it, A, which lies outside the region shall be either greater or less than, if it is not equal to, that within the region. If it is greater, the mean potential over the surface is greater than that at the centre; if it is less, the mean potential over the surface is less than that at the centre, and in either case Gauss's theorem is violated. Therefore the potential at any point within the part of the sphere which lies outside the region cannot differ from the potential within the region which is thus extended so as to include the whole sphere. The region can then be further extended along the path by taking another sphere with centre at a point in the path just within the extended boundary, and so on until the whole space which can be reached as described is included in the region of constant potential.

FIG. 252.



474. **Indirect Consequences of Gauss's Theorem.**—Other consequences less direct are the following: The potential at any point within a closed surface due to any distribution of matter whatever external to the surface cannot be greater than the maximum or less than the minimum potential at the surface. For if it were greater than the maximum the theorem that the potential cannot have a maximum value in free space would be contradicted. Similarly the potential cannot be less than the minimum at the surface.

It follows from this that if the potential due to external matter be constant over the closed surface, the potential at every internal point has the same value as the potential at the surface.

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A similar theorem sets limits to the potential in space *external* to a closed surface when the potential is produced by matter within the surface, and of amount equal to zero. If there is no matter at any internal point the potential at every point is, of course, zero; but we may consider for a moment the case in which just as much negative matter is enclosed as there is of positive. A quantity dm of negative matter is such that if it existed it would produce a potential at distance r from it $= -kdm/r$, it would attract matter of the same kind as itself, and repel and be repelled by matter of the opposite kind. A quantity of negative matter is equal to a quantity of positive matter when the repulsion between these is numerically equal to the attraction which would be exerted if either were removed and replaced by a quantity of matter equal to and of the same kind as that which is allowed to remain.

The surface-integral of potential over the spherical surface enclosing the distribution $M_i = 0$ is in this case by (39) also zero, so that at some points of the space V is negative, at others positive. If successive concentric spheres of larger and larger radius are considered, it is plain that the potentials, both negative and positive, will become of smaller and smaller numerical value; and therefore the potential at any point external to any one of the surfaces cannot have a greater negative or a greater positive potential than that which is to be found on the surface. Hence the extreme values of the potential on the surface are superior limits for the extreme values at points in external space.

*475. **Calculation of Field-Intensities from Potential.**—The potential at any point P due to a number of particles situated at different points $A, B, C, \ldots$, is $k\Sigma(m/r)$ where r is the distance of P from the point at which the representative mass m is situated. If a, b, c be the co-ordinates of the latter point, and x, y, z those of P relatively to rectangular axes drawn from any origin,

$$r^2 = (x-a)^2 + (y-b)^2 + (z-c)^2,$$

and therefore $\partial r/\partial x = (x-a)/r$, $\partial r/\partial y = (y-b)/r$, $\partial r/\partial z = (z-c)/r$. Hence we have $\partial(1/r)/\partial x = -\{\partial r/\partial x\}/r^2 = -(x-a)/r^3$, and $\partial(1/r)/\partial y = -(y-b)/r^3$, $\partial(1/r)/\partial z = -(z-c)/r^3$.

Now if X, Y, Z be the component field-intensities in the direction of x, y, z at P, the work done by external force in increasing x to $x+\delta x$ is by the definition of the potential $-\delta V$. But this work is very approximately $-X\delta x$, and is so the more nearly the smaller δx is made. In the limit $X = \partial V/\partial x$. Similarly for the displacements in the direction of y and z. Thus

$$X = \frac{\partial V}{\partial x}, \quad Y = \frac{\partial V}{\partial y}, \quad Z = \frac{\partial V}{\partial z}.$$

By the values of $\partial(1/r)/\partial x$, &c., calculated above, these equations may be written

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$$\left.\begin{aligned} X &= \frac{\partial V}{\partial x} = -k\Sigma\frac{m(x-a)}{r^3} \\ Y &= \frac{\partial V}{\partial y} = -k\Sigma\frac{m(y-b)}{r^3} \\ Z &= \frac{\partial V}{\partial z} = -k\Sigma\frac{m(z-a)}{r^3}. \end{aligned}\right\} \quad (42)$$

*476. **Laplace's Equation of the Potential.**—Let now P be a point outside the attracting matter. There is no doubt of the continuity of the components of field intensity at P, and hence $\partial X/\partial x$, $\partial Y/\partial y$, $\partial Z/\partial z$ have there finite values, as will indeed be seen from their calculated values. Now

$$\partial(x-a)/r^3\}/\partial x = \{r^3 - 3(x-a)r^2\partial r/\partial x\}/r^6 = \{r^2 - 3(x-a)^2\}/r^5,$$

and so for the other quantities. Hence

$$\left.\begin{aligned} \frac{\partial X}{\partial x} &= \frac{\partial^2 V}{\partial x^2} = -k\Sigma\left\{m\frac{r^2-3(x-a)^2}{r^5}\right\} \\ \frac{\partial Y}{\partial y} &= \frac{\partial^2 V}{\partial y^2} = -k\Sigma\left\{m\frac{r^2-3(y-b)^2}{r^5}\right\} \\ \frac{\partial Z}{\partial z} &= \frac{\partial^2 V}{\partial z^2} = -k\Sigma\left\{m\frac{r^2-3(z-c)^2}{r^5}\right\}. \end{aligned}\right\} \quad (43)$$

Adding these equations we see that since $(x-a)^2+(y-b)^2+(z-c)^2=r^2$ the sum of the quantities on the right vanishes. Hence

$$\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}=0, \quad (44)$$

or, as it is usually written,

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=0. \quad (45)$$

The symbol $\nabla^2 V$ is generally used in this country for the left-hand side of this equation, so that ∇^2 denotes the operator

$$\partial^2/\partial x^2+\partial^2/\partial y^2+\partial^2/\partial z^2.$$

On the Continent ΔV is used in the same sense.

This is called Laplace's equation of the potential. The solution of the general problem of finding the fields of force in different cases consists in finding a value of V satisfying this equation through a given space, and fulfilling certain expressed conditions at its bounding surface or surfaces.

*477. **Poisson's Equation.**—A more general equation [(46) below] due to Poisson, and coinciding with Laplace's equation in free space, holds at points within attracting matter, provided the density varies continuously at the point.

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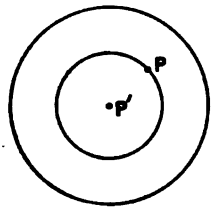
Let ρ be the volume-density of the matter at P, and $\rho + \delta\rho$ the density at a distance δs from it. If then, however small and in whatever direction δr is taken, the ratio $\delta\rho/\delta s$ be finite, the density ρ is said to vary continuously at P. It also varies continuously at points very near to P.

FIG. 253.



Take then a point P' very close to P, and describe round it as centre a sphere of radius τ equal to the distance of P' from P. P lies on the surface of this sphere, and the attraction on a unit particle at P exerted by the matter contained within the sphere is $\frac{4}{3}\pi k\rho\tau$. Let the co-ordinates of P' be $x-\xi$, $y-\eta$, $z-\zeta$; the component attraction along the axis of x is $\frac{4}{3}\pi k\rho\tau\xi/\tau = 4\pi k\rho\xi/3$. Hence the component field-intensity due to the matter in the small sphere is, if taken in the direction of x increasing, $-4\pi k\rho\xi/3$. Similarly the component field intensities at P in the positive directions of y and z are $-4\pi k\rho\eta/3$, $-4\pi k\rho\zeta/3$. At a point P_1, the co-ordinates of which relatively to P are $d\xi$, $d\eta$, $d\zeta$, the components X, Y, Z of *total* field-intensity can only differ from those at P by amounts $dX = -4\pi k\rho d\xi/3$, $dY =$ &c., since P_1 is on the external surface and P on the internal surface of the thin spherical shell between the two points. Hence, identifying $d\xi$, $d\eta$, $d\xi$ with dx, dy, dz we get

$$\left.\begin{aligned} &\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z} = -4\pi k\rho \\ \text{or}\quad &\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}+4\pi k\rho = 0. \end{aligned}\right\} \quad (46)$$

478. **Equipotential Surfaces. Points of Equilibrium. Contouring a Gravitational Field.**—A surface drawn in a field of gravitational force at every point of which the potential has the same value is called an *equipotential surface*. Since the potential has only one value for any point P in the field, no distinct equipotential surfaces can cut one another. It is possible, however, for an equipotential surface due to a possible distribution to intersect itself, and for an equipotential surface to consist of two distinct sheets. For example, Fig. 254 shows the section by the plane of the paper of the equipotential surfaces due to two equal quantities of matter concentrated at the points A, B. The "figure of eight" curve is the line of section by the paper of an equipotential surface intermediate between the closed surfaces, each consisting of a single sheet, which lie outside it, and the surfaces each consisting of two closed sheets surrounding A and B respectively.

The point marked P in Fig. 254 is one at which the field-intensity is zero, and is called a *point of equilibrium*. The field-intensity at two points a, b of the surface, on the two sides of P, are outwards

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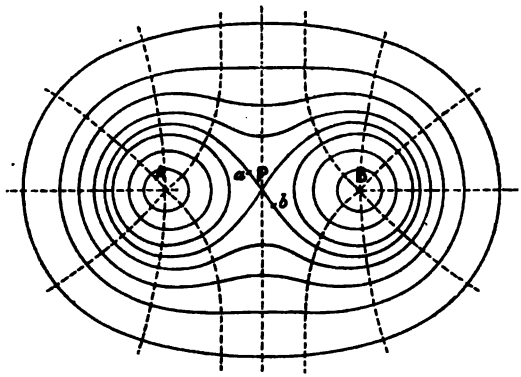
from the closed sheet in each case, and therefore it changes sign at P in passing through zero. In gravitational fields such points exist where two equipotential surfaces, due to two separate parts into which the distribution is divided, are in contact, and the values of dV/dn are

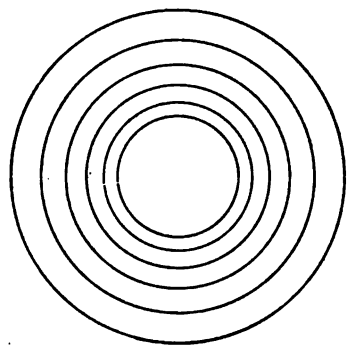
FIG. 254.



equal and opposite for the two. Thus we have a line of equilibrium in the case of two infinitely long, straight, parallel and uniform rods. The surfaces of equal potential for the two rods taken separately are right circular cylinders, and there is a line in the plane of the rods and parallel to them at which the field-intensity is zero. A full discussion of points and lines of equilibrium in electrical fields is given in Maxwell's *Electricity*, vol. i., chap. vi.

Successive equipotential surfaces surrounding a uniform spherical distribution, or its equivalent for points without the sphere, a central particle, are drawn in Fig. 255. They are simply concentric spherical surfaces. They have been drawn for potentials diminishing by equal differences as the distances outward from the centre increase, and therefore their radii are in harmonic progression, that is as $1/n$, $1/(n-1)$, $1/(n-2)$.

FIG. 255.



The variation of distance between the successive equipotential surfaces indicates the change in the value of the field-intensity. For if at any place δr be the distance from the surface of potential V to the surface of potential $V - \delta V$, the work done in carrying a unit particle from the former surface to the second against the attractive force is $P\delta r$, where P is the average force which must be applied over

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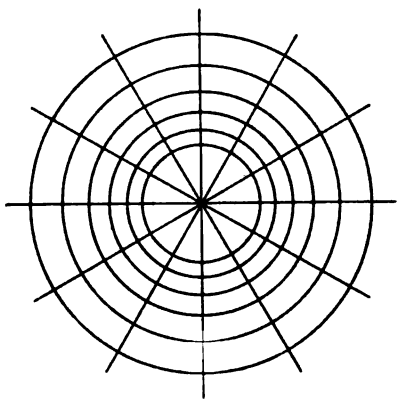
the displacement δr. The actual field-intensity F is, however, in the opposite direction, and when the two surfaces are brought indefinitely close $F = dV/dr$, that is at any equipotential surface the field-intensity is equal to the rate at which the potential increases outward from the surface. Hence the field-intensity at any point between two close equipotential surfaces is inversely proportional to the perpendicular distance there between the surfaces.

We have, therefore, the important result that if a gravitational field be *contoured* by constructing equipotential surfaces for successive equal differences of potential the field-intensity at different points can be compared at once by the eye. Where the surfaces are closer together the field-intensity is greater, where they are wider apart the field-intensity is smaller.

479. **Lines of Force.**—As we have just seen, the component field-intensity in any direction is equal to the rate of increase of the potential in that direction. Hence, as there is no variation of potential along an equipotential surface, there is no component of field-intensity along the surface in any direction. The resultant field-intensity at any point of the field is therefore directed along a normal to the equipotential surface at any point. This leads us to a definition of a *line of force* in a gravitational field.

FIG. 255′.



Let successive equipotential surfaces be drawn in the field for indefinitely small differences of potential. Then let (what may here be assumed as always possible) a system of lines be drawn such that each line cuts every surface of the family perpendicularly. Each of these lines is called a *line of force*. If a free particle of unit mass acted on only by the attractive force of the distribution of matter move in the field, it will move along a line of force, and its acceleration at each instant will equal the field-intensity.

The lines of force for any field of which equipotential surfaces are known may therefore be drawn at once by drawing a line starting from any point of an equipotential surface and continuing it outwards and inwards, cutting the surfaces always at right angles until the line passes off to infinity or arrives at a point of convergence or divergence of different lines of force.

The radial lines in Fig. 255′ are lines of force lying in the plane of the paper for the uniform spherical distribution. The equipotential surfaces for another case, that of two equal point-charges of

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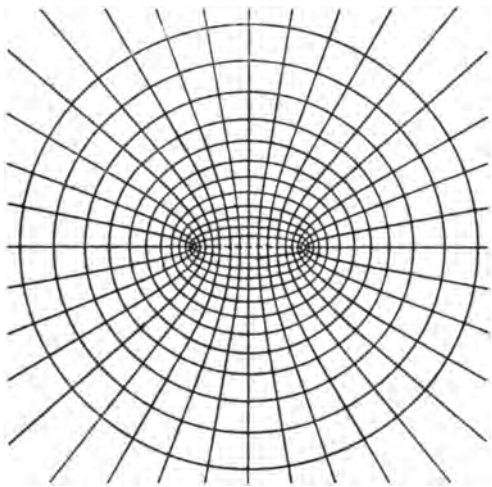
matter, is shown in Fig. 254, and the direction of a few of the lines of force are shown by dotted lines.

Fig. 256 shows lines of force and sections of the equipotential surfaces for a uniform rod of attracting matter coincident with the dotted line in the centre of the diagram. As has been shown, the line of force at any point bisects the angle between the lines drawn

FIG. 256.



from the point to the extremity of the rod, and therefore the sections of the equipotential surfaces by a plane through the rod are confocal ellipses having the extremities of the rod as foci. The surfaces are therefore confocal ellipsoids of revolution.

The lines of force lie in planes through the rod, and are a system of hyperbolas confocal with the ellipsoids.

480. **Graphical Construction of Equipotential Curves.**—A family of equipotential curves due to an attracting system, of which the matter is concentrated at two centres, may be drawn by the following simple method, which seems to have been given first by Clerk Maxwell. It is illustrated by Fig. 257 for the case of quantities of matter, $m_1 = 5$ units, $m_2 = 20$ units, situated at the points A and B respectively.

A series of potentials proceeding by equal differences is first chosen. The radii of the circles which shall have these potentials are then calculated for the charge at each centre separately.

If r_1, r_2 be the radii for the centres A, B of the circles at which m_1, m_2 would separately produce a potential V, clearly $r_1/r_2 = m_1/m_2$.

Now let these circles be drawn round the centres A, B, and let each series be numbered 1, 2, 3, ... from the innermost outwards. The two series of circles will intersect. Take any point of intersection:

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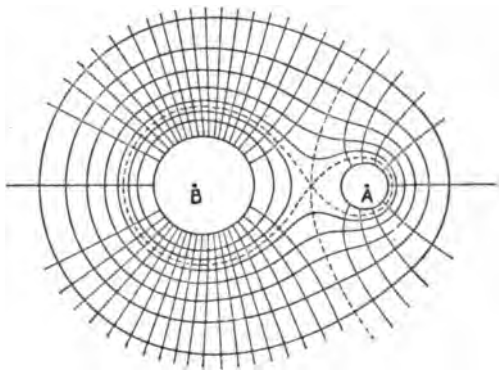
let h, k be the numbers of the two circles which there intersect, the potential at that point is proportional to $h+k$. Immediately adjoining will be an intersection of the circle $h+1$ of one series with the circle $k-1$ of the other. The potential here is of the same value as before, since the potential of the point is proportional to $h+1+k-1$ or $h+k$. Join the two points of intersection by a short line, and repeat this process until a complete curve has been drawn.

This method is applicable in a modified way when the number of centres exceeds two. The equipotential curves for a pair of centres are drawn as just described and are numbered off by their potentials, that is the sum of the numbers of the intersecting circles from which they are drawn. A set of circles is then drawn for a third centre, of radii corresponding to those for the first two, but on the scale deter-

FIG. 257.



mined by the point-charge at the centre. These are then numbered like the others. They intersect the equipotential curves, and the equipotential curves for the three centres are obtained by drawing curves through the points of equal potential given by the intersections of circles and curves which have an equal sum of numbers.

If these curves be then numbered off by their potentials, the equipotential curves for four centres can be obtained by combining in like manner with the curves the circles for the fourth centre, and so on for any number of centres.

Lines of force will start from every centre at which a point-charge is situated, and they can be drawn by starting from such a point, drawing the line at first radially outwards (since the equipotential surfaces very close to a point-charge must be circles round it) and thereafter carrying the line forward at right angles to the successive equipotential curves.

481. **Conjugacy of Equipotential Curves and Lines of Force.**—Equipotential curves and lines of force in a plane diagram are, as we have seen, two sets of curves, the individual members of each of which cut orthogonally the curves of the other set. It is possible,

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therefore, to regard *either* set as equipotential curves and the other as the corresponding lines of force. This conjugate property is of very great importance in many cases. For example, the closed equipotential curves in Figs. 254 and 258 due to the charges situated at A, B are exactly the lines of magnetic force due to electric currents flowing in straight conducting wires passing through A and B perpendicular to the paper. The current-intensities at A, B, have the ratio (equality in Fig. 254, 1 to 4 in Fig. 257) of the charges supposed, in the gravitational case here considered, to be situated at A, B.

The lines of force of the gravitational case in Figs. 254, 257 are for the electric currents curved (cylindrical) surfaces each passing through one of the straight wires of which A B represent the sections by the plane of the paper. These are the equipotential surfaces of the magnetic field given by the currents.

This subject will be discussed more fully in connection with electricity and magnetism, but the reader should notice here the intimate bearing on many different parts of physics of the theory of what has been called the *Newtonian potential*, the potential of forces of attraction (and repulsion) which obey the law of the inverse square of the distance.

482. **Tubes of Force.**—Upon an equipotential surface drawn in a field of force let a small closed curve be marked out. The lines of force which pass through the points of the curve mark out a tubular surface, which is called a *tube of force*. Let a closed surface be taken, made up of a portion of a tube of force enclosed between two equi-

FIG. 258.

potential surfaces met by the tube. The field-intensity is perpendicular to the ends of the tube; let its values at the two ends be F_1, F_2, the areas of its ends dS_1, dS_2. The surface integral of normal force outwards over the closed surface is $-F_1dS_1+F_2dS_2$, since there is no component perpendicular to the sides of the tube. If there be no matter within the portion of the tube considered the surface integral of normal force over the tube is zero, and therefore

$$F_1dS_1=F_2dS_2. \tag{47}$$

As the tube widens therefore the value of F falls off, so that the product of the mean field-intensity by the area dS is constant all along the tube.

If the area dS be so chosen that $FdS=1$ the tube is called a *unit tube*. The surface integral of normal force taken over a closed surface which surrounds any part of the distribution is clearly equal to the number of unit-tubes which cross the surface from within over the number which enter it from without, and thus as

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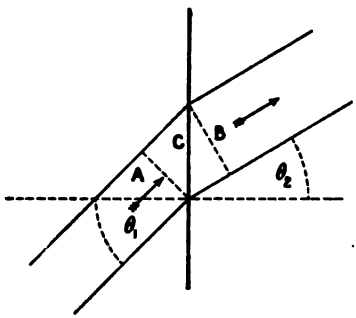
we have seen in § 456, is $-4\pi k M_i$, if M_i denote the quantity of matter within the surface.

The number of unit-tubes here referred to is often in electrical applications referred to as the number of *lines of force* crossing the surface. When the phrase "number of lines of force" is used in such a connection, it is really *unit-tubes* that are meant, if the idea to be conveyed has any definiteness.

483. **Discontinuity of Normal Force at Surface-Distribution of Matter.**—Let now a tube of force cross a surface on which is distributed matter of surface density σ. The direction of the tube will not be necessarily the same on the two sides of the surface. If it be inclined to the surface on one side, the angle of inclination on the other side will be altered by the action of the matter on the surface.

FIG. 259.



Take a closed surface consisting of two sections A, B (Fig. 259) of the tube near the surface on the two sides, and let F_1, F_2 be the field intensities at A, B. If dS_1, dS_2 be the areas of the sections, and dS the area of the surface element C intercepted by the tube $-F_1 dS_1 + F_2 dS_2$ is the surface integral of normal force over the closed surface, and we have

$$F_2 dS_2 - F_1 dS_1 = -4\pi k \sigma dS. \tag{48}$$

But if the normals drawn from the surface on the two sides make angles θ_1, θ_2 with the lines of force of the tube, as shown in Fig. 259, $dS_2 = dS\cos\theta_2$, $dS_1 = dS\cos\theta_1$, and hence we obtain

$$F_2\cos\theta_2 - F_1\cos\theta_1 = -4\pi k\sigma. \tag{49}$$

$F_2\cos\theta_2$ is the normal force from the surface on the side B, $F_1\cos\theta_1$ the normal force towards the surface on the side A. If we denote by N_1, N_2 the normal forces *from* the surface on the two sides we have $N_1 = -F_1\cos\theta_1$, $N_2 = F_2\cos\theta_2$, so that (49) becomes

$$N_1 + N_2 + 4\pi k\sigma = 0. \tag{50}$$

The components of field-intensity taken normal to the surface-element dS on the two sides are thus, because of the term $4\pi k\sigma$ in (49) or (50), discontinuous. The tangential components of F_1, F_2 are, however, continuous, that is

$$F_1\sin\theta_1 = F_2\sin\theta_2. \tag{51}$$

The tube of force, therefore, suffers a species of refraction.

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*484. **Graphical Representation of Potential and Force for Thick Spherical Shell.**—It is instructive to represent graphically the potential and force in the fields of simple distributions. Fig. 260 taken from Thomson and Tait's *Natural Philosophy* shows these quantities for the case of a thick spherical shell of uniform density ρ, and internal and external radii a, a'.

Since the field is symmetrical about the centre of the shell the force is radial at every point. Hence if a spherical surface concentric with the shell, and of radius r be described, we have by § 453 for any part on that surface.

$$-\frac{dV}{dr} = k\frac{M}{r^2}, \tag{52}$$

where M is the total quantity of attracting matter within the surface.

Again for this case, as the reader may prove, Poisson's equation has the form

$$\frac{d^2V}{dr^2} + \frac{2}{r}\frac{dV}{dr} = -4\pi k\rho, \tag{53}$$

so that

$$\frac{d^2V}{dr^2} = -4\pi k\rho - \frac{2}{r}\frac{dV}{dr},$$

and d^2V/dr^2 can be calculated from dV/dr easily by this equation.

There are three possible cases (1) $0<r<a$, (2) $a<r<a'$, (3) $a'<r<\infty$. For these the values of M, V, $-dV/dr$, d^2V/dr^2 are as in the following table:

	$o<r<a.$	$a<r<a'.$	$a'<r<\infty.$
M	0	$\frac{4\pi\rho}{3}(r^3-a^3)$	$\frac{4\pi\rho}{3}(a'^3-a^3)$
V	$2\pi k\rho(a'^2-a^2)$	$\frac{4\pi k\rho}{6r}(3a'^2r-r^3-2a^3)$	$\frac{4\pi k\rho}{3r}(a'^3-a^3)$
$-\frac{dV}{dr}$	0	$\frac{4\pi k\rho}{3r^2}(r^3-a^3)$	$\frac{4\pi k\rho}{3r^2}(a'^3-a^3)$
$\frac{d^2V}{dr^2}$	0	$-\frac{4\pi k\rho}{3r^3}(r^3+2a^3)$	$\frac{8\pi k\rho}{3r^3}(a'^3-a^3)$

The variations of V, $-dV/dr$, d^2V/dr^2 are shown in Fig. 260. It will be seen that V, represented by the curve LMN, is constant, having the value OL, from O to A, that is within the hollow space, falls off slowly at first then more rapidly from A to B, that is within the shell, and finally approaches zero asymptotically as r increases towards infinity.

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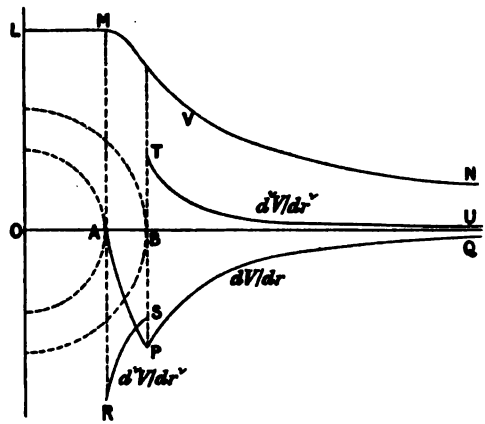
The value of dV/dr is shown by the curve APQ. Thus dV/dr increases numerically from A to B, starting with the value 0, and ending with BP, then the direction of the curve suddenly changes and dV/dr falls off asymptotically towards zero as r approaches infinity.

Finally d^2V/dr^2 is zero within the hollow space, and is represented for other points by the curves RS, TU. There is a discontinuity in

FIG. 260.



the value of d^2V/dr^2, at the sharp point of the curve for dV/dr, as shown by the sudden change of the ordinate from the negative value BS to the positive BT.

*485. **Direct Problem of Potential.**—We are unable to deal here at length with the solution of the problem of finding the potential for different given surface-distributions (what is sometimes called the Kelvin-Dirichlet problem), but we shall offer some general observations and deal with a few important cases. Consider a closed surface S, consisting it may be of two or more mutually exclusive detached surfaces. The problem to be solved is in one of its forms that of finding a function V which, as well as its first and second space-rate of variation, is single-valued and continuous throughout the space external to S, satisfies Laplace's equation throughout that space, is zero at every point at an infinite distance from the attracting matter, and has (1) certain assigned values at the surface or surfaces S, or (2) gives certain assigned values of dV/dn, the rate of variation of V along the normal from the surface to the field, at every point of S, or (3) has certain assigned values at some of these surfaces, and gives certain assigned values of dV/dn at the others. A continuous single valued function satisfying Laplace's equation in a certain region of space is said to be *harmonic* in that region.

The problem just stated is the problem most generally to be solved, but the region throughout which V is to be found in some cases is the simply connected (see § 486) space within one or more

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closed surfaces for which either the value of V or the value of dV/dn is given for every point.

It is to be observed that the variation of V from the value it has at any point of the space considered to the value which it takes at the surface, is to be taken along a path drawn from one point to the other without passing across the boundary of the space considered.

The question as to whether this problem has a solution has been treated by several writers, among whom are Lord Kelvin, Lejeune Dirichlet, E. Neumann, and Poincaré. A very valuable discussion of the subject is given in M. Poincaré's treatise entitled *Théorie du Potentiel Newtonien.*

***486. Uniqueness of Solution of Potential Problem. Connectivity of Spaces.**—The question whether a function V does or does not exist in all cases we shall not here consider. In the cases with which we shall be concerned in the present work, the question is answered by the discovery of the function. What is therefore of much greater importance is the theorem, which is easily proved, that if one solution is obtained it is the only solution. This theorem may be stated as follows. Let V be a value of the potential (that is of course a function of the co-ordinates) which satisfies Laplace's equation throughout a given *simply connected* space, and fulfils the required conditions at the surface, and let if possible V' be *another* function which does the same things. We shall prove that throughout the space $V = V'$.

A simply connected space is one in which if any closed path be drawn the path can be contracted to zero without passing anywhere out of the space. Such a space is that within a spherical surface or within a surface to which a spherical surface can be deformed without bringing one part of the surface into contact with another, or the whole space extending to infinity everywhere outside one such surface, or outside two or more distinct surfaces of this kind, or the space between one such surface and any number of such surfaces external to one another but contained within the first.

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An example of a space which is not simply connected is that within the surface of an anchor-ring. A closed curve can obviously be drawn within such a space which cannot be contracted to zero without passing outside the space—for example, the circular axis of the ring is such a curve. The space may be rendered simply connected by drawing across the ring at any place a diaphragm which no closed curve is to cross. All closed curves can then evidently be shrunk to zero without passing out of the space. The anchor ring space is said to be *doubly connected.* Again, an anchor ring with one side branch as in Fig. 261 requires two diaphragms to be rendered singly connected. The connectivity of a space is always designated by a number greater than the number of diaphragms which are necessary to render the space simply connected. The space within the double ring is thus triply connected. It can be reduced to a simply connected space by diaphragms across the ring at A, B.

Let U denote the difference $V - V'$ of the two functions which

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fulfil all the conditions stated for a solution of the problem of the potential. Then U is also a solution which gives zero values of the potential at the bounding surface. Consider the integral

$$\int\left\{\left(\frac{\partial U}{\partial x}\right)^2+\left(\frac{\partial U}{\partial y}\right)^2+\left(\frac{\partial U}{\partial z}\right)^2\right\}d\varpi$$

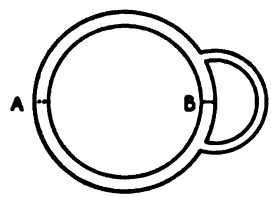
(where $d\varpi$ denotes an element of space) taken throughout the space to which the solutions apply. Integrating by parts we transform it to

$$\int U\frac{dU}{dn}dS-\int U.\nabla^2 U d\varpi, \tag{54}$$

where the first integral is taken over the bounding surface and $U.dU/dn$ in it denote respectively the value of U, and the rate of variation of U per unit distance along the normal drawn outward from the surface; the other integral is taken throughout the enclosed space.

Wherever on the bounding surface the potential is given there $U=O$, for there V and V' coincide in value, and where dV/dn has its value assigned there dU/dn is zero. Hence the

FIG. 261.



surface integral is zero. Also since throughout the space $\nabla^2 V=0$, and also $\nabla^2 V'=0$, $\nabla^2 U=0$ also holds throughout the space. Hence the second integral on the right vanishes, and the integral considered is zero throughout the whole space. Hence everywhere in the space $\partial U/\partial x=\partial U/\partial y=\partial U/\partial z=0$, that is since U is zero at the bounding surface it must also be zero throughout the whole space. Hence everywhere $V=V'$, and there is only one solution.

The necessity for simple connectivity of the space arises from that of making the surface integral on the right perfectly definite. The integral on the left is not equal to the two integrals on the right for a multiply connected space if only the actual bounding surface of the space is considered in the first integral on the right. The surfaces given by the two sides of each diaphragm required for reduction to simple connectivity must be introduced to render the equation applicable.

*487. **Green's Function.**—The solution of the direct problem of the potential can be effected in a number of important cases by a

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method due to Green in which certain functions invented by that mathematician play a very important part.

Consider the space external to a closed surface S, and let P be any point in that space, Q a fixed point the distance of which from any element E of the surface is r. Then k/r is the potential produced at E by a unit particle at Q. Now consider a function G of the co-ordinates which satisfies Laplace's equation throughout the space considered, is zero at infinity, and is equal to $-k/r$ at every element E. Then if U denote $G+k/r$, U is a function which is zero at every point of the surface, and is harmonic (§ 485) and finite throughout the external space except at Q where it is infinite.

The function G is what is properly called *Green's function*, though sometimes the name is given to the function U. G is equal and opposite to the potential at each element E produced by the unit charge at Q. [In electricity it is the potential at each point of S and at every external point, due to the induced electrical distribution called into existence on a conducting surface coinciding with S, and maintained at zero potential in presence of a unit positive charge at an external point Q.]

A certain reciprocal property is possessed by Green's functions which may here be demonstrated. Let G be the value of Green's function at a point P, when the unit particle is situated at Q, and G' the value of Green's function at Q when the unit particle is situated at P, then $G=G'$.

Since G is harmonic throughout the external space it must be the potential due to a distribution of matter either on or within the surface S. Let, if possible, a surface-distribution over S which would produce the potential $-k/r$ at a point just external to any other element E' of the surface be assigned, and let σ be the matter per unit of area on the element dS at E. Then we should have

$$k\int\frac{\sigma dS}{E'E}=-\frac{k}{r},$$

where $E'E$ is the distance from E' to E, and r the distance $E'Q$, and the integral is taken over the whole surface S. But this surface distribution must produce not only the potential G at each element of the surface but at every external point. For the potential is harmonic at every external point and has the assigned surface values, and as we have seen there cannot be two potential functions which do this.

Now the potential at P due to this surface-distribution is $k\int\sigma dS/EP$, where as stated σ is the density at E of the surface distribution which just counteracts for every element of the surface the potential due to the unit particle at Q. But if σ'_1 denote the density at E' of the surface distribution which would produce the potential $-k/EP$ at each element E, that is, just counteract the potential due to a unit particle at P, we have $-1/EP=\int\sigma_1'dS/EE'$. Hence the potential at P just found for the other case is

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$$k\int \sigma dS\left\{\int \frac{\sigma_1' dS'}{EE'}\right\}.$$

But this may be written also

$$k\int \sigma_1' dS'\left\{\int \frac{\sigma dS}{EE'}\right\},$$

and $k\int \sigma dS/EE'$ is evidently the potential at E' due to the charge which just counteracts the potential due to unit charge at Q, for it has the value $-k/E'Q$. Hence the reciprocal proposition is proved,

*488. **Idea of Negative Matter.**—This proof of the proposition rests on the assumption that a surface distribution can be found which gives over the surface the potential due to a unit charge situated at any point Q. We shall now endeavour to justify this assumption.

In the preceding discussion use is really made of the notion already referred to of negative matter, an idea which we shall have to employ in dealing with problems in electrostatics. Now it is possible (as we shall see in vol. iii.) to have a metallic surface of any form at zero potential in presence of any external distribution of electricity, and therefore the distribution on the surface just counteracts at the surface the potential due to the external distribution. This distribution reversed would produce at each point of the surface a potential equal to that produced there by the external distribution. A distribution of gravitational matter following the same law of surface-density would give a potential at the surface proportional to that due to an external distribution of matter corresponding to the external electrical distribution. Hence the surface-distribution is physically possible.

*489. **Closed Surface-Distribution producing Internal Potential due to External Matter.**—Further it is to be observed that since the potential at the surface due to the external distribution and the counteracting surface-shell of matter, is zero, the potential at every point within the surface is zero also. The surface-distribution thus not only counteracts the effect of the external one at the surface, but also does so for every internal point. This counteracting surface-shell is therefore a distribution which, if reversed in sign, gives a potential at every point within the surface equal to that produced by the external distribution. It remains to determine the density at each element of the surface.

Since the potential due to the (unreversed) surface and external distributions has a constant (zero) value within the surface, there is no field-intensity within the surface. There is therefore no component, normal or tangential, close to the surface inside, and the field-intensity just outside is, by the results obtained in § 484 for a tube of force crossing a surface distribution, normal to the surface and of amount $-4\pi k\sigma$, where σ is the surface-density at the point where the normal is drawn. If then dV/dn be the field-intensity

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outwards from the surface due to both surface and external matter the density required is given by the equation

$$\frac{dV}{dn} = -4\pi k\sigma. \tag{55}$$

If then dV/dn can be determined the surface density can be found.

The potential at external points due to this surface-shell is a harmonic function and vanishes at infinity, and since it produces a specified system of values at the surface, it is the only potential function and the distribution is the only distribution which can be found to do so.

We shall find (see § 492) Green's function in one or two simple cases, but further use of it as a method of finding distributions will be deferred until the need for them arises in connection with electricity.

*490. **Surface Distribution Equivalent to Internal Distribution.**—A very important result regarding equivalence of distributions may, however, be noticed here. Consider an equipotential surface of any distribution, which may, to take the most general case, be partly external partly internal to the surface. The field-intensity at every point of it is normal to the surface, and has the value dV/dn. We have seen, § 456, that

$$\int \frac{dV}{dn} dS = -4\pi k M_i, \tag{56}$$

where M_i is the whole quantity of matter within the surface. If now the matter within the surface be distributed over the equipotential surface with surface density σ, so that at each point $-4\pi k\sigma = dV/dn$, the value of the potential at the surface will not be altered, and the external field will be the same as before.

For suppose that the internal distribution is removed: the surface will in general cease to be one of equal potential. Let then such a distribution, if necessary, of negative matter, or partly of positive and partly of negative matter, be made over the surface as brings the surface throughout to zero potential. Since there is no internal distribution the potential throughout the internal space is also zero.

Now distribute over the surface a quantity of matter in such a way as to bring each point of the surface to the potential it had originally. The potential at each point of the external space close to the surface is now what it was with the original distribution, that is the distribution now on the surface produces at every such point just the potential that the original internal distribution produced. Let this be V_1. To V_1 corresponds a value V_1' of the potential produced by the surface distribution at each point of external space. But it has been proved that corresponding to a given value V_1 of the potential at a surface due to a distribution upon or within the surface, there can only be one value of the potential at external points. Hence the

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potential at all external points, and also the field of force, is the same for the surface as for the original distribution.

Now consider a tube of force crossing the surface distribution. Since the field-intensity within is zero, the tube starts normally outwards from the surface, and we have $-dV/dn = 4\pi k\sigma$. Further, since as we have just seen dV/dn, the field-intensity just outside the surface, is the same as it was before, we must have

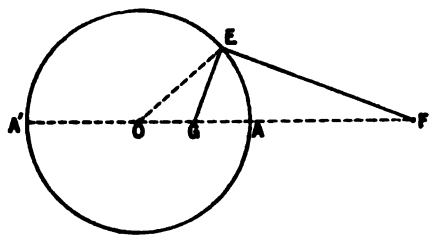
$$\int \frac{dV}{dn} dS = -4\pi k \int \sigma dS = -4\pi k M, \tag{57}$$

as stated in the theorem.

This theorem is a case of a more general one. If any surfaces whatever, open or closed, and whether equipotential or not, be described in a field of gravitational force, it is possible to find one, and only one distribution of matter over these surfaces which shall produce at every point of them, and at every point of space separated by those of the surfaces which are closed or infinite, the same potential as is produced in the actual case. The proof of this theorem cannot be entered on here, and would involve a discussion of most of the difficulties of the Kelvin-Dirichlet problem. The reader may refer to the work of M. Poincaré referred to above, and to a paper of his in the *American Journal of Mathematics* for 1898.

491. **Spherical Distribution Equivalent to External Point-Charge. Electric Images. Centrobaric Distributions.**—An important example of Green's function is the following. It is required to

FIG. 262.



find a distribution over a spherical surface which for all points on and within the surface will produce a potential equal to that produced by a single particle at an external point F. Let AEA' represent a central section of the spherical surface by the paper, and let the line from F through the centre O meet the circle in A, A'. Let G divide OA so that $OG : OA = OA : OF$, and denote OF, OG, OA, EF, GE by f, f', a, r, r'. Then if E be any point on the sphere, we have $f'/a = a/f$, so that the triangles OGE, OEF are similar. Hence $f'/r' = OE/EF = a/r$, and $1/r = f'/ar'$. Hence a quantity ma/f (that is mf'/a) at G will produce the same potential km/r at E as a quantity m at F would produce at the same point.

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If then, according to § 488, a distribution of amount ma/f be made over the surface so that the potential of the surface may be at each point that due to m at F, the potential at each point of the internal space will also be equal to that due to m at F.

For if this distribution were made with negative matter the potential at E would be zero, and so the potential at each point of the surface being zero the potential at every internal point would be zero. The negative distribution thus counteracts the potential due to m at F for all points on or in the sphere, and therefore when reversed gives at each such point the same potential as does m at F.

Let the negative distribution exist, so that the potential within the sphere is zero, the outward force just external to E is then $-4\pi k\sigma$, if σ be the surface density (sign included). But the force consists of two components, one towards F of amount km/r^2, and the other along GE of amount $km(a/f)/r'^2$. The former resolves into two components along EO and OF, which are proportional to these lines. These components are kma/r^3 and kmf/r^3 respectively. The latter resolves into two components along OE, GO, the amounts of which are respectively kma^2/fr'^3 and $kmaf'/fr'^3$. But $f'/r'^3 = a^3/f'^2r^3$ and $a/f' = f/a$, so that $f'/r'^3 = f^2/ar^3$, and the last written component becomes kmf/r^3. It is therefore equal and opposite to the component along OF of the force in the direction EF.

There remain therefore only the components along the radius, namely, kma/r^3 from E to O, and kma^2/fr'^3 from O to E. The latter, which may be written kmf^2/ar^3, is the greater, and therefore the resultant force is from O to E, and is of amount $km(f^2 - a^2)/ar^3$. Hence for the density of matter at E we have

$$\sigma = -\frac{m}{4\pi a}\frac{f^2 - a^2}{r^3}. \tag{58}$$

Hence a density σ given by the equation

$$\sigma = \frac{m}{4\pi a}\frac{f^2 - a^2}{r^3}, \tag{59}$$

will just produce at every point of the surface and at every external point the same potential as is produced by a particle of mass ma/f at G, and produces at every internal point the same potential as is produced by the particle m at P.

The two point-charges, $-ma/f$ at G and m at F, together keep the spherical surface at zero potential. A distribution of density $-m(f^2 - a^2)/4\pi ar^3$ on the surface, with m at F, does the same thing, and moreover produces the same potential at every external point as would be produced by $-ma/f$ at G, and at every internal point counteracts the potential due to m at F. The charge at G is called the electric image of the charge at F. The theory of electric images is due to Lord Kelvin, and in conjunction with his theory of electric inversion enables many problems to be solved

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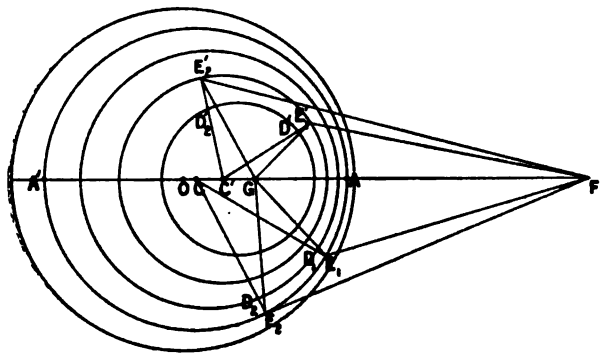
which could only be overcome with great difficulty by ordinary analysis. (See vol. iii. of the present work.)

The whole quantity of matter on the spherical surface, as may easily be verified by direct integration, is ma/f, so that we have here a distribution which attracts every external particle as if its whole mass were collected at the eccentric point Q. This is an example of what are called centrobaric distributions (see § 504 below).

*492. **Sphere Built up of Centrobaric Shells.**—Let now F and G be kept fixed, and successive spheres (Fig. 263) be described with respect to which F, G are inverse points. This process can be continued, the spheres becoming smaller and smaller as the centres approach G, until the whole space within the outer sphere has been

FIG. 263.



divided up into thin spherical shells, each consisting of the space between two successive surfaces. Now a distribution on any one of these spheres, the surface-density of which varies as the inverse cube of the distance of each element of the surface from F, has all the properties of the surface distribution just stated; that is, it produces at all points external to itself the same potential as would be produced by an equal quantity (m' say) of matter at G, and at all points internal to itself the same potential as would be produced by a quantity $m'f/a$ situated at F.

Hence a succession of such surface distributions made upon the spheres described as stated would be a centrobaric distribution of matter; that is, it would give the same field for all external points as would be given by the same total quantity of matter placed at G.

Further, the distance between two successive spheres, the thickness that is of a shell, varies as the square of the distance of the point considered from F. For let y be the distance of the centre C' of one of these spheres from C the centre of the outer sphere, on which are situated the points A, A', and x the distance from A of the nearest point of its circumference. The radius of the sphere is $a-x-y$, and we have $f'-y=(a-x-y)^2/(f-y)$ since F and G are

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fixed. Hence, if we take another sphere within this, the radius of which is $a-x-y-dx-dy$, we easily obtain from the relation just written $dx=(f-a+x)^2dy/2(f-y)(a-x-y)$. Also if θ be the angle which the radius $C'E'$ to any element E' of the surface makes with $C'F$, we have for the thickness τ of the shell at E' the value $dx+dy(1-\cos\theta)$. But from the triangle $C'E'F$, if $E'F=r$, we have $r^2=(f-a+x)^2+2(f-y)(a-x-y)(1-\cos\theta)$, and therefore

$$dx+dy(1-\cos\theta)=r^2dy/2(f-y)(a-x-y).$$

The thickness τ is therefore proportional to r^2 for different elements of any one shell.

Now let each of the spherical distributions, of surface density inversely as the cube of the distance from F, be distributed through the adjacent shell of space, so that the matter on each surface element takes volume density, the average value of which along the thickness there is ρ, given by $\sigma=\rho\tau$. The density ρ will then be for each shell inversely as the fifth power of the distance of each element from F.

The density ρ may be quite different for the different shells; what is essential is that its average value along the thickness of the shell at each element of any one shell should vary inversely as the fifth power of the distance of the element from F. Thus we obtain any number of spherical centrobaric distributions.

Of course a distribution the density of which at every point varies inversely as the fifth power of the distance of the point from F fulfils this condition.

Any uniform distribution on the spherical surface may be combined with the distribution on the surface found in § 491. If the amount of matter uniformly distributed be m', the external potential at every point will be that due to m' at the centre C and ma/f at the point G. If a/f be $1/4$ and $m'=m$ the equipotential lines and lines of force in a plane section of the field through the points C and G will, for points outside a circle of radius $a=\sqrt{CG.f}$, be precisely those shown in Fig. 257, where B and A represent C and G. Inside the sphere the potential due to m' is uniform, that due to ma/f placed on the surface according to the law of density given in (59) is the same as the potential due to m at F.

The density at any point is then

$$\sigma=\frac{1}{4\pi a}\left\{\frac{m'}{a}+m(f^2-a^2)\frac{1}{r^3}\right\}, \tag{60}$$

which at the points, A, A', respectively, becomes

$$\sigma=\frac{1}{4\pi a}\left\{\frac{m'}{a}+m\frac{f+a}{(f-a)^2}\right\},\quad \sigma=\frac{1}{4\pi a}\left\{\frac{m'}{a}+m\frac{f-a}{(f+a)^2}\right\}. \tag{61}$$

493. **Image Charge in a Plane.**—A plane is the particular case of a sphere in which the radius is infinite. The inverse point P' (Fig. 264)

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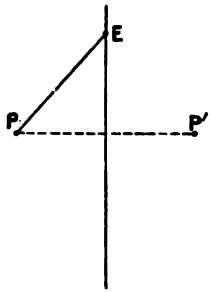
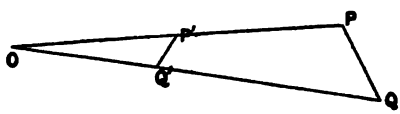
is then on the normal to the plane through the direct point P and as far behind the plane as P is from it in front. P' is in fact the optical image of P in the plane regarded as a plane mirror. The inverse charge corresponding to m at P is now m also. Thus a distribution, on a plane infinite in extent in every direction, the density of which at each element varies inversely as the cube of the distance of the element from a point on one side of the plane, produces at every point in the space on that side of the plane, the same potential as would be produced by a charge m at P', and at every point on the same side as P' a potential equal to that produced by m at P. The density on the surface is $h/2\pi r^3$ where h is the distance of P from the surface, and r the distance of the element E from P.

FIG. 264.



494. **Geometrical Inversion.**—The method of inversion is of great utility in deriving geometrical arrangements and their properties from others that are known. Take any point P of a geometrical arrangement whether of particles in space or of points in a diagram, and join it to a point O. From O draw a sphere of radius a, say. Then if $OP = r$ take another point P' on OP at a distance r' from O given by the relation $rr' = a^2$. O is called the *centre of inversion*, and a the *radius of inversion*. Now take any other point Q of the arrangement, and in the same way find along the radius of the sphere drawn in the direction OQ, the inverse point Q; and so on. A new arrangement is obtained which is called the inverse of the given one. Thus in Fig. 265 the points P', Q' are the inverses of

FIG. 265.



P, Q. The line $P'Q'$ is, however, *not* the inverse of PQ. As we shall see the inverse of a straight line is a circle passing through the centre of inversion.

It will be observed that the triangles OPQ, $OQ'P'$ are similar, and the pairs of angles OPQ, $OQ'P'$ and OQP, $OP'Q'$ are equal. If PQ be an element of a curve *so that OP, OQ are very close together*, $P'Q'$ is the inverse of the element PQ, and the two elements cut the radius vector OP at the same angles $P'PQ$, $PP'Q'$ or at the supplementary angles OPQ, $OP'Q'$. It follows that the inverses of two intersecting curves intersect at the same angle as do the direct curves, and that the same thing holds for two intersecting surfaces.

The inverse of a circle is another circle, and therefore that of a

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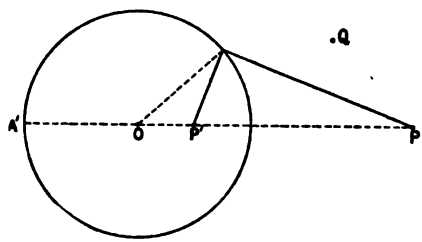
sphere is another sphere. The reader may easily verify as follows (constructing a figure for himself) the truth of the former proposition for the case at least in which the circle lies in a plane through O. For let PQ be the diameter of the given circle, and let R be any other point. Then PRQ is a right angle. Let P', R', Q' be the inverses of the points P, Q, R (the lines $P'Q'$, $Q'R'$, $R'P'$ are not the inverses of PQ, QR, RP) the angle $P'R'Q'$ is equal to $PRQ \pm POQ$ according as a line along OR' does or does not intersect PQ.

Now the intersection of a sphere by a plane is a circle. Hence the inverses of the circles of intersection of a sphere by planes through O are circles lying in the respective planes. Hence the inverse of a sphere is another sphere.

That the inverse of a straight line is a circle may be verified by the reader, and that the circle passes through the centre of inversion is evident from the fact that if the line be prolonged to infinity in both directions, the points at an infinite distance invert into the centre of inversion.

*495. **Inverse Distributions.**—The space outside the sphere of inversion inverts into the space within it, and the space within into

FIG. 266.



the space without. This is of great importance, as we can at once transform any solution of a gravitational problem which holds for the space within a sphere into one which holds for all space outside the sphere. The inversion of distributions of attracting or repelling matter and their fields was added by Lord Kelvin in one of his early papers to the theory of ordinary inversion, and used by him in some remarkable solutions of problems of electric distribution.

The charge ma/f of matter at P' is called the inverse of the charge m at P (Fig. 266). If instead of a simple point-charge m at P, there be a number of point-charges m_1 at P, m_2 at Q, ..., there will correspond to each of these a charge m_1a/f_1, m_2a/f_2, ... at the inverse points P', Q', ... in the sphere. Thus also if the direct distribution be a continuous linear, surface, or volume distribution, the inverse distribution will be a linear, surface, or volume distribution made up by inverting the elements of the direct distribution; and the line, surface, or volume occupied by the direct distribution inverts in the ordinary geometrical way into the line, surface, or volume occupied by the inverse.

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Let df be the dimension of an infinitesimal element of volume in the direction of the radius f from the centre of inversion, the corresponding dimension of the inverse element is df'. But $f' - df' = a^2/(f + df) = (1 - df/f)a^2/f$ so that $df' = df.a^2/f^2$. Also if $fd\theta$ be a dimension of the element in a direction at right angles to f, the corresponding dimension $f'd\theta = fd\theta.a^2/f^2$. Hence an element of volume which is the inverse of an element of volume $d\varpi (= f^2 d\theta^2 df)$ has the volume $d\varpi' = d\varpi.a^4/f^6 = d\varpi f'^6/a^6$.

Similarly an element of surface dS' which is the inverse of an element dS is $dS.a^4/f^4 = dS.f'^4/a^4$.

Again, if m be a direct element of matter occupying volume $d\varpi$, the inverse element is ma/f and occupies volume $d\varpi'$. But if ρ, ρ' be the direct and inverse volume densities, $\rho = m/d\varpi$, $\rho' = ma/fd\varpi'$, and therefore $\rho = \rho' a^5/f^5 = \rho' f'^5/a^5$. Similarly for surface densities

$$\sigma = \sigma' a^3/f^3 = \sigma' f'^3/a^3.$$

These are all shown in the table of ratios

$$\left.\begin{aligned} \frac{d\varpi'}{d\varpi} &= \frac{a^6}{f^6} = \frac{f'^6}{a^6} = \frac{f'^3}{f^3}, \quad \frac{dS'}{dS} = \frac{a^4}{f^4} = \frac{f'^4}{a^4} = \frac{f'^2}{f^2} \\ \frac{\rho'}{\rho} &= \frac{f^5}{a^5} = \frac{a^5}{f'^5}, \quad \frac{\sigma'}{\sigma} = \frac{f^3}{a^3} = \frac{a^3}{f'^3}. \end{aligned}\right\} \quad (62)$$

*496. **Inversion of Potentials.**—Let V be the potential at any point Q due to the direct charge m at P. Then $V = km/PQ$. The inverse of this potential is the potential V' produced at the inverse Q' of Q by the inverse point-charge ma/f (where $f = OP$) at P'. Thus $V' = kma/f.1/P'Q'$. But if r, r' denote OQ, OQ' we have, by Fig. 265,

$$\frac{V'}{V} = \frac{r}{a} = \frac{a}{r'}. \quad (63)$$

This ratio is independent of the position of P; it is the ratio of the potentials produced at the corresponding points Q', Q by any pair of inverse and direct distributions made up of point-charges.

Now let V be a potential constant throughout any given space S: V' is not constant unless r be constant, or r' be constant, that is unless the equipotential space is a sphere concentric with the sphere of inversion.

The surface of the inverse space may, however, be reduced to zero potential by placing at the centre of inversion a quantity of matter $-Va$. This will produce at every point at distance r' the potential $-Va/r'$ which is equal and opposite to V', the potential at the point due to the inverse distribution. With this modification the inverse surface becomes also an equipotential surface.

Thus if S, the surface of the direct sphere, be a surface on which matter is so distributed that the space within is at constant potential,

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the inverse surface distribution together with $-Va$ at O just maintains the inverse surface at constant potential.

Applications of these results are made in the Theory of Electricity. At present we have not space to pursue the matter further.*

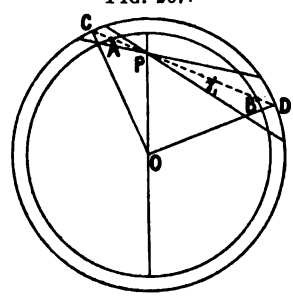
*497. **Internal Field of Thin Homœoid.**—The theorem stated in § 490 may be applied to a number of interesting particular cases of gravitational attraction.

First of all we consider the field due to matter of uniform density filling the space included between two similar and similarly situated ellipsoidal surfaces. Such a distribution of matter can be obtained by subjecting a spherical shell to *homogeneous strain*, that is altering the dimensions of the shell in the directions parallel to three diameters at right angles to one another in the ratio of three lines $a:1$, $b:1$, $c:1$. Such a strain results in the alteration of every element of volume of the shell, say a rectangular element with edges $\delta x, \delta y, \delta z$ in the direction of these diameters to a rectangular element of edges $a\delta x$, $b\delta y$, $c\delta z$, so that the volume is changed from $\delta x\delta y\delta z$ to $abc\delta x\delta y\delta z$.

Every element has its volume increased thus in the ratio of $abc:1$, and the shell if originally of uniform density remains so, though the density is altered.

Again draw a chord (Fig. 267), of the sphere intersecting the shell in four points, A, B on the internal surface, C, D on the external, and let that be divided in any ratio by a point P. The reader may easily verify that the ratio of distances $AP:PB$ remains unaltered after the strain, as does also $CP:PD$. Hence consider a cone of small solid angle $d\omega$ drawn from vertex P with CD as axis. It intercepts on the spherical shell two elements, the masses of which remain unchanged.

FIG. 267.



First suppose the shell of thickness exceedingly small in comparison with the distance PA or PB. Let OP be a line drawn through the centre of the shell, and τ be the thickness of the shell, r_1, r_2 the distances PA, PB, and θ the angle OCP. The area intercepted on the spherical surface is $r_1^2 d\omega/\cos\theta$ at A, and $r_2^2 d\omega/\cos\theta$ at B. The masses at A and B of the shell intercepted by the cone are proportional therefore to $r_1^2\tau d\omega/\cos\theta$, $r_2^2\tau d\omega/\cos\theta$, that is they have the ratio r_1^2/r_2^2. But their attractions at P are in opposite directions along the axis of the cone and have the ratio r_2^2/r_1^2. Hence the attractions are equal and opposite. In the same way other cones may be drawn from P as vertex so as to exhaust the spherical shell by equal and opposite elements, so that the field-intensity at any point is zero.

* See the Author's Treatise on Magnetism and Electricity (Macmillan & Co., 1898), where many worked-out cases are given.

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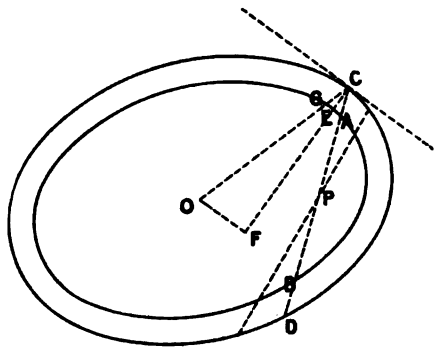
Clearly since the masses of the elements intercepted by the cones drawn from any point P are not altered by the strain, and the distances AP, PB remain in the same ratio as before, the attractions of the opposite elements of the shell intercepted by the cone are equal and opposite. There is therefore no field-intensity at any point within the deformed shell.

This result holds for any shell of finite thickness provided the density is uniform and the internal and external surfaces are similar and similarly situated ellipsoidal surfaces. For the shell can be

FIG. 268.



divided up into infinitely thin shells bounded as stated, for each of which the result holds.

Such a shell as that just considered has been called in Thomson and Tait's Treatise on Natural Philosophy (vol. i. Part II.) an *elliptic homœoid*, and we shall adopt this designation.

It is easy to show as follows that the thickness τ of an elliptic homœoid at any point is proportional to the length of the perpendicular let fall from the centre to the tangent plane drawn to the point.

When a body is homogeneously strained, parallel lines remain parallel though their directions are altered. Short arcs of the circumferences of the spherical shell in any plane through the centre and at the points where a radius intersects the surface are parallel lines. Hence short arcs at A, C in Fig. 268 (which represents a central section of a homœoid perpendicular to the tangent plane at C) are parallel. If CF be drawn parallel to the perpendicular from the centre O on the tangent at C, and OF be drawn parallel to the tangent and OC, FC intersect the inner surface in G, E, the two triangles COF, CGE are similar and $CG/CO = CE/CF$, that is $CE = CF.CG/CO$. But CG/CO is a constant ratio for all directions of CO, so that CE, which is the thickness τ of the shell at C, is proportional to CF.

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*498. **External Field of Thin Homœoid.**—Now considering the field due to a thin homœoid we see that the potential is constant at all external points close to the surface, and we have by § 489 for the force outwards at any point of it $dV/dn = -4\pi k\tau\rho$. Since the distance between one equipotential surface and another near to it is inversely as the field-intensity we see that this distance, τ' say, is inversely as τ. But this will give a surface confocal with the homœoid just obtained. To prove this we take as the equation of the thin homœoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

Let x, y, z be the co-ordinates of a point on this surface, $x+dx$, $y+dy$, $z+dz$ those of a point on a near external equipotential surface, and lying on the normal drawn to the surface at x, y, z. The direction cosines of the normal are proportional to x/a^2, y/b^2, z/c^2. The distance τ' from one surface to the other is therefore

$$\{xdx/a^2+ydy/b^2+zdz/c^2\}/\{x^2/a^4+y^2/b^4+z^2/c^4\}^{\frac{1}{2}}.$$

The denominator of this expression is $1/p$ where p is the perpendicular let fall from the centre on the tangent drawn to the homœoid at x, y, z. Now, by what has been proved in § 497, τ' is equal to $\frac{1}{2}f/p$ where f is a constant, and therefore, since the points x, y, z, $x+dx$, $y+dy$, $z+dz$ lie on the normal, and τ' is the distance between them, we have $dx=\tau' xp/a^2=\frac{1}{2}fx/a^2$, and similarly $dy=\frac{1}{2}fy/b^2$, $dz=\frac{1}{2}fz/c^2$. Therefore $2xdx=fx^2/a^2$, $2ydy=fy^2/b^2$, $2zdz=fz^2/c^2$, so that

$$\frac{x^2+2xdx}{a^2+f}+\frac{y^2+2ydy}{b^2+f}+\frac{z^2+2zdz}{c^2+f}=1\,;$$

or to quantities of the second order of smallness,

$$\frac{(x+dx)^2}{a^2+f}+\frac{(y+dy)^2}{b^2+f}+\frac{(z+dz)^2}{c^2+f}=1\,;$$

that is, the point $x+dx$, $y+dy$, $z+dz$ lies on an ellipsoidal surface confocal with the homœoid.

*499. **External Distribution Equivalent to Thin Homœoid.**—It thus follows from the theorem of the replacement, with identity of external field, of a distribution internal to a surface by a distribution over the surface, that a thin elliptic homœoid can be replaced by an external elliptic homœoid confocal with the former one and of the same mass. Thus the potential and force at an external point P due to an elliptic homœoid are the same as those at the same point due to another elliptic homœoid confocal with the former, of the same mass, and having P external, but infinitely close, to its surface.

We infer from the foregoing that the equipotential surfaces of a homœoid of uniform density are confocal ellipsoidal surfaces. Of

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this theorem, which seems to have been first given by Chasles, we shall give another proof in § 501.

The determination of potential and force at any point due to an infinitely thin homœoid is thus reduced to the determination of the potential and force at the surface of a confocal homœoid. It follows that any two thin confocal homœoids of equal mass produce at any point P external to both the same potential and field-intensity.

*500. **Homœoids of Finite Thickness. Maclaurin's Theorem.**—Further, if we have two homœoids of finite thickness different in the two cases, of uniform density and equal mass, which are confocal in the sense that the internal surfaces and the external surfaces form each a pair of confocal surfaces, the potential, and therefore the force due to each at any external point P, is the same. For each homœoid may be made up of the same number of thin homœoids, each of one set being of the same mass as the corresponding homœoid of the other set taken in the same order. Each homœoid of the one set will be confocal with the corresponding homœoid of the other set, as it is easy to see or prove if necessary. The theorem holds for each pair of corresponding homœoids, and therefore holds for the finite homœoids.

The internal surfaces may be made as small as is desired, and hence any two solid ellipsoids, each of uniform density and equal mass, the external surfaces of which are confocal, produce the same potential at all points external to both.

This is what is commonly called Maclaurin's theorem of the attraction of ellipsoids.* The proof here given of the theorem is a version of one due to Chasles.† Another proof will be given in § 503.

The process of dividing an ellipsoid up into infinitely thin homœoids seems to have been first used by Poisson in his *Mémoire sur l'attraction d'un ellipsoïde homogène.*‡ He there determines the field-intensity due to such a homœoid at an external point, and shows that it is directed along the axis of the cone which has its vertex at the point and envelops the homœoid. The following is a geometrical proof of the last result due to Steiner.§ Consider a point C (Fig. 269) on the homœoid; the plane determined by this element and the axis of the enveloping cone cuts the surface in an ellipse which is touched by the two generating lines of the cone which lie in this plane. Let A, B be the points of contact, then the angle APB is bisected by the axis of the cone. Let Q be the point in which AB is cut by the axis, and draw CQ and produce it to meet the

* Maclaurin seems to have given the theorem only for points on the prolongation of the axes. In his *Theoria Attractionis Corporum Sphæroidicorum* (*Werke*, Bd. v.) Gauss says: *Disquisitionem generalissimam de attractione sphæroidum non per revolutionem ortarum . . . jamiam inchoaverat Maclaurin, sed substiterat in attractione punctorum in aliquo trium axium positorum.* He then attributes the complete theorem to Laplace, whose proof is contained in the well-known passage of the *Mécanique Céleste*, vol. ii. liv. iii. chap. i.

† *Journal de l'Ecole Polytechnique* 25me *Cahier*. Paris, 1837.

‡ *Mem. de l'Institut de France*, vol. xiii. 1835.

§ *Crelle's Journal*, vol. xii.

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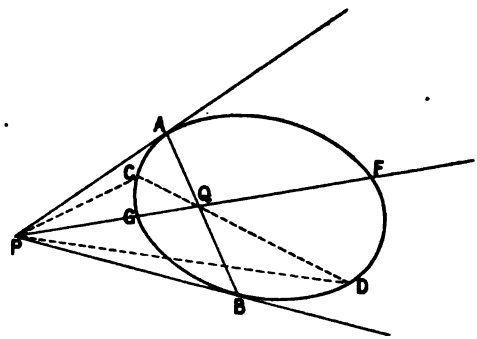
ellipse again in D. It can be proved that $\angle CPQ = \angle DPQ$. Now from Q as vertex take any small cone surrounding CD; this intercepts on the shell two elements which, as have been seen, give equal and opposite attractions on a particle at Q. These two elements have volumes in the ratio of the squares of their distances from Q, that is in the ratio QC^2/QD^2. But from the equality of angles stated $QC/QD = PC/PD$, and therefore *vol. of element at* C : *vol. of element at* $D = PC^2/PD^2$. Hence the elements attract a particle at

FIG. 269.



P equally, and the resultant attraction is along the bisector of the angle CPD, that is along PQ.

It follows that any plane through Q cuts the homœoid into two parts which have equal attractions on a particle at P.

Also since the axis of the enveloping cone is perpendicular to the ellipsoidal surface through P confocal with the homœoid we see that the equipotential surface at P coincides with the confocal surface, and thus we have another proof of Chasles' theorem with regard to the form of the equipotential surfaces.

These theorems have important applications in the theory of electricity. The density at any point of the equilibrium distribution on an ellipsoidal conductor alone in its own field is proportional to the thickness there of a thin homœoid coinciding with the surface of the conductor, and there is no field within the homœoid. The *potentials and fields* due to two confocal charged ellipsoidal conductors are the same at every point, the confocal surfaces external to a charged ellipsoid are its equipotential surfaces, and so on.

*501. **Chasles' Theorem of Two Confocal Homœoids.**—The following theorem is also given by Chasles for two thin confocal homœoids of equal mass. The potential produced at a point P of one homœoid by the other is equal to the potential produced at the corresponding point Q of the second homœoid by the first. [Corresponding points on two confocal ellipsoids are the points which would be transformed into one another if one homœoid passed into the other by a series of changes throughout which the surfaces remain confocal.]

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First we shall show that if P, P' be two points on one surface, Q, Q' their corresponding points on the other, the distances PQ', $P'Q$ are equal.

Let the equation of one surface be $x^2/a^2+y^2/b^2+z^2/c^2=1$, and of the other $x^2/(a^2+h^2)+y^2/(b^2+h^2)+z^2/(c^2+h^2)=1$, and let x, y, z, x', y', z' be the co-ordinates of the points P, P' on the first. The co-ordinates of the points Q, Q' on the second corresponding to P, P' are $x\sqrt{a^2+h^2}/a$, $y\sqrt{b^2+h^2}/b$, ..., $x'\sqrt{a^2+h^2}/a$, $y'\sqrt{b^2+h^2}/b$, ... Hence $PQ'^2=(x-x'\sqrt{a^2+h^2}/a)^2+\ldots$, $P'Q^2=(x'-x\sqrt{a^2+h^2}/a)^2+\ldots$, and therefore

$$PQ'^2-P'Q^2=h^2\{(x'^2-x^2)/a^2+(y'^2-y^2)/b^2+(z'^2-z^2)/c^2\}=0. \quad (64)$$

From this Chasles' theorem follows at once. For take any element of one homœoid at P' say. This is marked out by points which have corresponding points marking out a corresponding element of the other homœoid at Q', and these elements are of equal mass. Hence whatever the law of variation of attraction with distance may be, the potential at Q due to the first element situated at P' is equal to the potential at P due to the second element situated at Q', and similarly for other elements.

If the law of attraction be the inverse square of the distance we know that the potential at points within the outer homœoid produced by the matter of that shell is constant. Hence the potential produced at every point of the outer surface by the matter of the inner homœoid has the same value, that is the outer confocal surface is an equipotential surface of the inner shell, the result inferred in § 498.

*502. **Ivory's Theorem of the Attraction of Ellipsoids.**—If P, Q be corresponding points on the surfaces of two confocal ellipsoids A, B (of axes a, b, c, a', b', c' parallel to x, y, z) of uniform density, the component field-intensity parallel to one of the axes, that of x say, produced at Q by A, is to the component field-intensity in the same direction at P produced by B, in the ratio of bc to $b'c'$.

Consider a prism of A parallel to the axis of x and of cross-section $dydz$, and let EQ, FQ be the distance of its ends E, F from Q, r the distance of any element from the same point. If the law of attraction be $f(r)$ the attraction dX, parallel to x, of an element of length dx of the prism on a unit particle at Q is (Fig. 269) $\rho dydzdxf(r)\cos\theta=-\rho dydzf(r)dr$. Hence if $\int f(r)dr=-\psi(r)$ we have for that part of the field-intensity required which is given by the prism

$$dX=\rho dydz\{\psi(FQ)-\psi(EQ)\}. \quad (65)$$

Now of the other ellipsoid take a prism the ends of which E', F' correspond to E, F; by § 501, $E'P=EQ$, $F'P=FQ$, so that if the cross-section be made $dydzb'c'/bc$, we obtain for the attraction dX' parallel to x at P exerted by this prism

$$dX'=\rho\frac{b'c'}{bc}dydz\{\psi(F'P)-\psi(E'P)\}=\frac{b'c'}{bc}dX. \quad (66)$$

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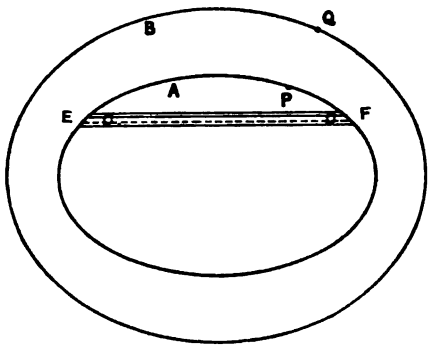
If prisms of cross-section $dydz$ be taken so as to exhaust the ellipsoid A, and corresponding prisms of B having cross-sections $dydzb'c'/bc$ be taken to correspond, B will be exactly exhausted also, and the whole component field-intensities X, X' will be in the same ratio as their parts, that is we shall have

$$\frac{X'}{X}=\frac{b'c'}{bc}. \tag{67}$$

This is Ivory's theorem. It holds, as was first pointed out by Poisson,

FIG. 270.



whatever be the law of attraction of the elements, provided it be according to a function of the distance.

*503. **Extension of Ivory's Theorem. Passage from Ivory's Theorem to Maclaurin's.**—From Ivory's theorem can be deduced another which possibly has been noticed before. Equation (67) can be written

$$\frac{X'}{X}=\frac{a'b'c'}{abc}\,\frac{a}{a'}$$

and, since the density is the same in both ellipsoids A, B, if M, M' be their respective masses this becomes

$$\frac{X'}{X}=\frac{M'}{M}\frac{a}{a'}. \tag{67'}$$

If now the density of one of the ellipsoids, say that of B, be altered, the attraction X' exerted by it in the direction of x on a unit particle at P will be altered in the same ratio as M' is. Thus (67′) will still hold. This forms an extension of Ivory's theorem.

In particular let M' be made equal to M, then

$$\frac{X'}{X}=\frac{a}{a'}; \tag{68}$$

that is, the x-component of the field intensity of B at P is to the

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x-component of the field intensity of A at Q, in the ratio of the semiaxis a of A to the corresponding semiaxis a' of B.

So far nothing depends on the law of distance followed by the attraction. Now let the law be that of the inverse square of the distance. It is proved (see *Absolute Measurements in Electricity and Magnetism*, vol. ii. Part I. p. 53) that the components of the field-intensity due to a homogeneous ellipsoid, at any point x, y, z on the surface or within it, are Ax, By, Cz where A, B, C are constants.* Hence, if X'' be the field-intensity produced at Q by the ellipsoid B,

$$\frac{X''}{X'}=\frac{x\sqrt{a^2+h^2}/a}{x}=\frac{a'}{a}.$$

But $X'=Xa/a'$, so that we obtain $X''=X$. The same thing holds for the other components, and thus as in § 500 we obtain again Maclaurin's theorem.

*504. **Law of Force for Gravitating Matter Deduced from Non-Existence of Field within Spherical Shell.**—The following important deduction from Ivory's theorem is due to Duhamel.† Two concentric spheres of the same density form a particular case of the two confocal ellipsoids just considered. The field-intensities X, X' of two such spheres at corresponding points, the extremities of two radii r', r drawn from the common centre in the same direction, are in the ratio r^2/r'^2. If the law of attraction be such that the mass of the shell of the larger sphere external to r have no field-intensity in its interior, the intensity X', due to the larger sphere at the surface of the smaller, reduces to that due to the matter within the smaller. Hence X, X' are the attractions of the smaller sphere on unit particles at the distances r', r from the centre respectively, and we have seen that they are inversely as the squares of these distances. Since the small sphere can be made as small as we please, and we can compare the field-intensity which it produces at its own surface with that which it produces at the surfaces of larger spheres of different radius, we see that the law of attraction which must hold is that of the inverse square of the distance.

Hence the law of the inverse square of the distance could be properly inferred from an experiment proving that there is no

* The value of A is $2\pi k\rho abc\int_0^\infty \frac{d(\phi^2)}{\{(a^2+\phi^2)^3(b^2+\phi^2)(c^2+\phi^2)\}^{\frac{1}{2}}}$, those of B and C are obtained from this by multiplying the element of the integral by $(a^2+\phi^2)/(b^2+\phi^2)$, $(a^2+\phi^2)/(c^2+\phi^2)$ respectively. The components of field-intensity at an external point x, y, z are given by exactly similar formulæ, in which, however, the lower limit of the integral is replaced by ϕ_1^2 where ϕ_1^2 is the positive or numerically least root of the cubic in ϕ^2 given by

$$\frac{x^2}{a^2+\phi^2}+\frac{y^2}{b^2+\phi^2}+\frac{z^2}{c^2+\phi^2}=1.$$

† *Cours de Mécanique, Première Partie*, § 151.

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gravitational field-intensity within a hollow spherical shell of uniform thickness and density. A proof depending on this principle was given by Cavendish for the law of electrical attraction and repulsion.

Another proof that the law of attraction is the inverse square of the distance if there be no force within a uniform spherical shell is given by Laplace in the *Mécanique Céleste*, Liv. II., No. 12. See also Maxwell, *Electricity and Magnetism*, vol. i., art. 74e.

505. **Centre of Gravity. Centrobaric Bodies.**—As has been pointed out by Lord Kelvin,* Green's theory of replacement of attracting matter leads to the determination of distributions which have in the strict sense of the term a *Centre of Gravity*. As already noticed, this term has a distinct meaning of its own, and is not to be confounded with the centroid or centre of mass. The following is a definition :—

Let a distribution of gravitating matter the configuration of the particles of which are fixed relatively to one another (a rigid body) be given, and let the forces between its particles and those of another rigid body reduce to a single force which always passes through a point fixed relatively to the first body, however the relative positions of the two bodies may be changed, that point is said to be the Centre of Gravity of the first body.

As we shall see, a large number of distributions of matter do not possess centres of gravity. A body which does possess one is called a *centrobaric* body.

Let a centre of gravity G be found to exist for a body B and the attractive forces between it and another body A. We may prove that the point G is the centre of gravity of B for the action between B and any other distribution of matter wholly external to the bounding surface of A.

First, since the resultant force passes through G for any one position of A, G is the centre of gravity of B for a distribution made up of a repetition of A in any number of different positions. Take such a position of A that a sphere, centre C say, can be described so as to completely enclose it without including any part of B. Now turn A round a diameter of the sphere as an axis so that each particle describes a circle: since the resultant attraction between it and B passes through G for every position, so also will the resultant attraction between B and a body formed by distributing each particle of A uniformly round the circle it has described.

Also since this holds for every axis through C round which A can be revolved, it holds for the attraction between B and the body formed by uniformly distributing each particle of A over the sphere of centre C on which it lies in any of the positions just considered. Thus it holds for the attraction between A and a body made up of uniform concentric shells.

But the force between any uniform shell and each particle of an

* *Proc. R.S.E.*, February 1864.

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external body is the same as it would be if the mass of the shell were concentrated at the centre. Hence the attraction between B and a particle equal in mass to B placed at C passes through G as well as C.

Hence also the resultant attraction between any particle at C and B passes through G. This holds for any position of G for which a sphere can be described as stated above; but it can be shown that C can be taken *anywhere* in the space external to B.

For take a position of A for which a sphere can be drawn to include A and exclude B. There will be a certain region within which the centres of such spheres can lie and outside which they cannot. Place A partly within partly without such a region. The attraction between B and a particle placed at any point within the region is through G, and therefore the direction of the resultant attraction between B and the part of A within the region passes through G, since the same thing is true of every particle. But since the whole attraction between A and B acts through G, the direction of the attraction between B and the remainder of A passes through G. Spheres can now be drawn from points within a new region of space so as to include the latter part of A and exclude B, and it can be proved as before that the action between B and a particle placed at any point within the new region passes through G. This latter region may be extended like the former one; and in this way it can be proved by successive steps that a particle situated at any point external to B is attracted in a line joining it with G.

Since the lines of force are straight lines passing through G, the equipotential surfaces described in space external to B are evidently spheres with their centres at G. It follows that G must be surrounded by matter, that is, that no path can be drawn to G from space external to B without passing through the matter of B. For if such a line could be drawn G would lie in free space continuous with that outside B, and in that space the lines of force would be straight lines through G. Thus, a closed surface could be described in free space about G the surface integral of normal force over which would not be zero, and which therefore must according to § 456 include matter, which contradicts the supposition as to the position of G. Hence a body such as an anchor ring, an open cylinder, an incomplete spherical shell, &c., cannot have a centre of gravity.

Since the equipotential surfaces are spheres with a common centre G, the body must attract external particles as if its whole mass were collected at G. For it is easy to show by considering different spherical surfaces that the field-intensity at external points falls off as the square of the distance increases.

For external points on spheres which are described from G as centre but which do not include the whole of B, the equipotential surfaces are the spherical surfaces passing through those points. This is a possible case when we have a sphere as at § 492, the density of which varies as the inverse fifth power of the distance from an external point F. The Centre of Gravity G is then the inverse point of F relatively to the sphere.

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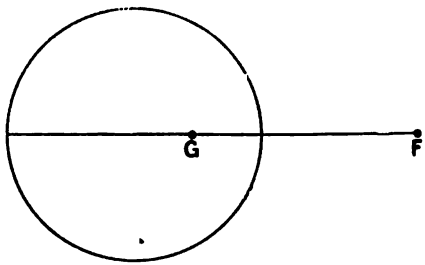
506. **Formation of Centrobaric Distributions.**—It is possible to distribute any given quantity M' of matter over any closed surface, completely enclosing a given mass M of matter so as to produce a system centrobaric as regards the field external to the surface, and having any specified internal point as centre of gravity.

For (1) distribute over the surface a quantity of matter in such a way as to produce at the given surface a potential equal and opposite to the potential V produced by M at each point of the surface. The

FIG. 271.



potential at each point of the surface and without it is thus reduced to zero, and the quantity of matter on the surface is $-M$.

Now (2) distribute matter over the surface so as to produce at any element E a potential kM/GE. The potential at each element E has this value, and the external field is precisely that which would be produced by M at G. Therefore the system is centrobaric, and the quantity of matter on the surface zero, and the whole quantity within is the same as before. Now add any quantity of matter over the surface so as to produce at each element E a potential kM'/GE.

The potential at each element of the surface is now $k(M+M')GE$, and at each external point is the same as that produced by $M+M'$ at G. If we do not contemplate negative matter at any part of the surface, we must, if the field-intensity is outward instead of inward across the surface at any element, diminish M, so that M' may be made sufficient for the required purpose.

We have an important but less general problem when M is zero. Particular cases of this are a spherical shell of matter, the surface density of which varies as the inverse cube of the distance from an external point F, and a spherical volume distribution the density of which varies as the inverse fifth power of the distance from F. The shell attracts external particles as if its whole mass were collected at G, the inverse point of F with respect to the sphere. Hence, if any internal point is to be chosen as G, the density must be made to vary inversely as the cube (or the fifth power according to the case) of the distance from the corresponding inverse point F outside.

It is clear that the centre of gravity of a centrobaric body coincides with its centre of inertia. For the body is centrobaric for *all*

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external extracting bodies. It is so for a body so distant that the forces on the different particles of the centrobaric distribution are infinitely nearly parallel; the forces on the particles are proportional to the masses, and the centre of these forces (§ 174) is the centre of gravity.

507. **Centrobaric Distributions are Kinetically Symmetrical.**—It may be noticed here that a centrobaric body has the same moment of inertia about every axis through the centre of gravity, that is to say, all such axes are principal axes, and the momental ellipsoid for that point is a sphere. The reader may endeavour to supply a proof of this theorem, or have recourse to Thomson and Tait's *Natural Philosophy*, vol. i., Part II., where further information regarding centrobaric bodies will be found.

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CHAPTER XII.

ASTRONOMICAL DYNAMICS.*

508. **Orbital Motion of a Particle under a Central Force. Specification of the Central Force.**—We shall take first the case of a particle which moves under the influence of a force directed towards a fixed point, and varying according to a single-valued function of the distance of the particle from that point. Afterwards we shall consider shortly the problem of the relative motion of two bodies which act on one another with equal and opposite forces, and illustrate the result by means of the motions of the planets about the sun. It will not be possible to do more than add a few general considerations regarding the problems of greater complexity that exist in the motion of the planets and give references to further sources of information.

The line drawn from the fixed point to the particle is called the radius-vector, its length is denoted by r. If, then, F be the central force, we have $F=f(r)$. Unless it is otherwise stated, it will be assumed that the force on the particle is an attraction, and that F is its numerical value. Also the positive sense of the radius-vector will be taken as outwards from the centre: hence, when the mass-acceleration in this sense is calculated, we shall have to equate it to $-F$. Until it becomes necessary to apply the results of the theory to the motions of the bodies of the solar system, the particle will be taken as of unit mass. To modify the equations at any stage, however, it is only necessary to multiply the accelerations by the mass m of the moving particle.

509. **The Motion is in One Plane.**—The particle is supposed given in motion at a certain instant with velocity v_0 at a point at which the radius-vector has length r_0. The distance traversed by the particle in an infinitesimal interval of time, dt, including or immediately succeeding the instant is, of course, v_0dt, and the acceleration is in the plane determined by the centre of force and the element of path v_0dt. The velocity generated in dt therefore lies in that plane, so that in the next element of time the motion lies in the same plane as before, and so on.

510. **Accelerations along and at Right Angles to the Radius-Vector. Law of Uniform Description of Areas.**—These are, of course, accelerations along directions fixed in space and coinciding at

* This chapter may be omitted on a first reading and by beginners.

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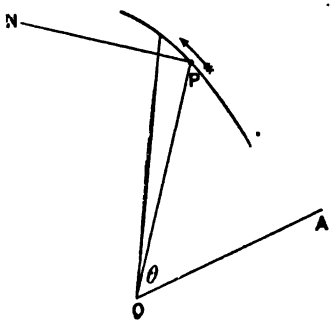
the instant with the radius-vector and the perpendicular to it. We shall suppose that the particle P is moving, as shown in Fig. 272, about the centre of force O, and that the accelerations are taken in the directions OP, PN, where OP is the position of the radius-vector, making an angle θ with the fixed line OA in the plane of motion, and PN is at right angles to OP, as shown in the diagram. The velocity relatively to the moving direction OP is $\dot{r}$. But this direction is turning round in the direction indicated by the arrow with angular velocity $\dot{\theta}$. Hence the velocity parallel to a fixed line coinciding with OP of a point on PN distant q from P is $\dot{r} - \dot{\theta}q$. If q is fixed in value the velocity of the same point relatively to the moving axis PN is zero, and its velocity $\dot{q} + \dot{\theta}r$ along a fixed axis coinciding with the instantaneous position of PN reduces to $\dot{\theta}r$. The velocities of P relatively to the fixed axes coinciding with OP, PN are therefore $\dot{r}$, $\dot{\theta}r$.

FIG. 272.



The accelerations are therefore, by (76), § 271, $\ddot{r} - r\dot{\theta}^2$ along OP, and $r\ddot{\theta} + 2\dot{r}\dot{\theta}$ along PN. The former must be equal to the force from O to P, and the latter must be zero.

We thus get

$$\ddot{r} - r\dot{\theta}^2 = -F, \quad r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0. \tag{1}$$

The latter equation may be written

$$\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = 0, \tag{2}$$

and has the following obvious interpretation. The area swept over by the radius-vector, while the angle θ is increased by the element $d\theta$, is $\frac{1}{2}r^2 d\theta$, so that $r^2\dot{\theta}$ is twice the rate of description of area by the radius-vector. Equation (2) is therefore equivalent to the statement

$$r^2\dot{\theta} = h \tag{3}$$

where h is a constant; that is, that the rate of description of area by the radius-vector is unvarying. This is, in other words, the theorem that the moment of momentum of the particle about the centre of force O is constant, a theorem which is obvious from the fact that the direction of the rate of change of momentum of the particle at each instant passes through O, and therefore has no moment about O.

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511. **Differential Equation of the Orbit.**—If $1/u$ be put for r, we have, since r and θ vary together,

$$\dot{r} = -\frac{\dot{u}}{u^2} = -\frac{\dot{\theta}}{u^2}\frac{du}{d\theta} = -h\frac{du}{d\theta}, \tag{4}$$

by (3). Hence also

$$\ddot{r} = -h\dot{\theta}\frac{d^2u}{d\theta^2} = -h^2u^2\frac{d^2u}{d\theta^2}. \tag{5}$$

Also $r\dot{\theta}^2 = h^2/r^3 = h^2u^3$, and this with (5) converts the first of (1) into

$$\frac{d^2u}{d\theta^2} + u = \frac{F}{h^2u^2}, \tag{6}$$

which is the differential polar equation of the orbit.

512. **Equation of Orbit when Force is Function of Distance. Law of Inverse Square.**—When F is given as a function of the distance, $f(u)$, (6) can be used to find the orbit; or conversely, if the orbit is given, (6) can be used to find the law of force. As an example of the former problem take the case of a force varying inversely as the square of the distance. Here $F = \mu u^2$, and we have

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2}, \tag{7}$$

of which the complete solution is

$$u - \frac{\mu}{h^2} = A\cos(\theta - \alpha),$$

where θ and α are constants. It is clear that when $\theta - \alpha = 0$, $u = \mu/h^2 + A$, or $r = 1/(\mu/h^2 + A)$, and that when $\theta - \alpha = \pi$, $u = \mu/h^2 - A$ or $r = 1/(\mu/h^2 - A)$. Calling the first of these values of r $a(1-e)$, and the second $a(1+e)$ we obtain $A = e/a(1-e^2)$, and $\mu/h^2 = 1/a(1-e^2)$.

Hence the solution becomes

$$r = \frac{a(1-e^2)}{1 + e\cos(\theta - \alpha)}, \tag{8}$$

which is the equation of a conic section, of latus-rectum $2a(1-e^2)$, major axis $2a$, and eccentricity e. It is a hyperbola if $e > 1$, and an ellipse if $e < 1$.

513. **The Orbit is Concave or Convex towards the Centre of Force according as the Force is an Attraction or a Repulsion.**—This is true whatever the law of force may be. The particle at any instant is moving along the tangent to the path. According as the force is an attraction or a repulsion, the velocity generated in an

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interval of time dt following the instant is towards or from the centre of curvature. Thus, in the former case the direction of motion changes from P to Q, as shown in Fig. 273, and in the latter from P' to Q', as in Fig. 274.

The same result is easily obtained analytically. It is clear from Fig. 275, in which PQ denote the positions of the particle at the beginning and end of an interval of time dt, that if PQ be denoted by ds, and QS be drawn perpendicular to OP, we have $QS = rd\theta$,

FIG. 273.

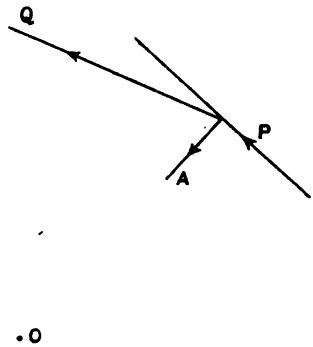
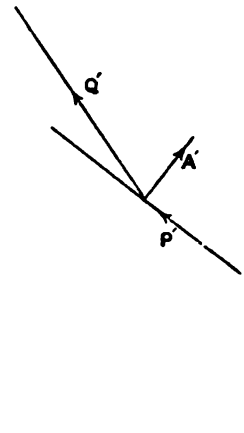


FIG. 274.



$SP = -dr$. Hence $ds^2 = r^2d\theta^2 + dr^2$. Also if OMP be a right angle $\sin SPQ = p/r = rd\theta/ds$. Thus $ds^2 = r^4d\theta^2/p^2$. Hence

$$\frac{1}{p^2} = \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 + \frac{1}{r^2} = \left(\frac{du}{d\theta}\right)^2 + u^2, \tag{9}$$

and we obtain

$$\frac{1}{p^3}\frac{dp}{dr} = u^2\left(\frac{d^2u}{d\theta^2} + u\right) = \frac{F}{h^2}, \tag{10}$$

by (6) since $du/d\theta . d\theta/dr = -u^2$.

It is clear that for concavity towards O, p must increase or diminish with r, that is, dp/dr must be positive. Hence $d^2u/d\theta^2 + u$ must also be positive. The reverse is the case for convexity. Thus, according as F is positive or negative, that is, is an attraction or a repulsion, the curve is concave or convex towards the centre of force.

514. **Law of Inverse Square. Criterion of Orbit. Velocity from Infinity.**—Calculating the value of v^2, the square of the velocity when the radius-vector is of length r and makes an angle θ with a line fixed in the plane of the orbit, we obtain from (9) (since $pv = h$) and (8)

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$$v^2 = h^2\left\{\left(\frac{du}{d\theta}\right)^2 + u^2\right\} = \mu\left(\frac{2}{r} - \frac{1}{a}\right). \tag{11}$$

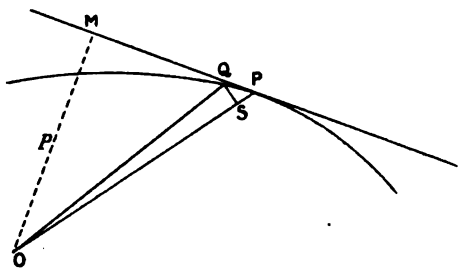
But the velocity which would be acquired by a unit particle in falling under the influence of a centreward force μ/x^2 from infinity to a distance r from the centre of force is given by

$$V^2 = 2\mu\int_r^\infty \frac{dx}{x^2} = \frac{2\mu}{r}. \tag{12}$$

Hence

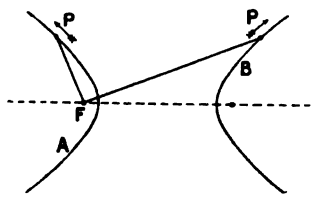
$$V^2 - v^2 = \frac{\mu}{a} = \frac{h^2}{a^2(1-e^2)}. \tag{13}$$

FIG. 275.



Thus, if $v^2 > V^2$, then $e > 1$, and the path is a hyperbola; and if $v^2 < V$, then $e < 1$, and the path is an ellipse. The path is therefore an ellipse or a hyperbola with the centre of force as a focus according as

FIG. 276.



the velocity v is less or greater than the velocity V, or, as it is generally expressed, is less or greater than the velocity from infinity. When the orbit is a hyperbola under an attractive force, the particle moves in that branch of the hyperbola within which lies the centre of force.

On the other hand, when the force is repulsive and varies according to the inverse square of the distance, the orbit is a hyperbola with the centre of force as a focus, but the particle moves in the branch which is convex to the centre of force. The two cases are shown in Fig. 276. F is the centre of force, and the particle moves

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in the branch A or the branch B, according as the force is attractive or repulsive.

If the law of force be μ/r^n, where $n>1$, the velocity from infinity is given by

$$\tfrac{1}{2}V^2=\int_r^\infty \frac{\mu}{r^n}dr=\frac{\mu}{n-1}\frac{1}{r^{n-1}}. \tag{14}$$

On the other hand, if the law of force be μr^m, where $m>-1$, the velocity from infinity is infinite. But in this case the velocity acquired in moving from a point at distance r to the centre of force is finite. Using V in this sense we obtain

$$\tfrac{1}{2}V^2=\mu\int_0^r r^m dr=\frac{\mu}{m+1}r^{m+1}. \tag{15}$$

515. **Energy of Orbital Motion of Particle.**—We can now assign the energy of the motion. The potential energy of the moving particle in any position, the distance of which from the centre of force is r, is the work which must be done against the attraction $f(r)$ in carrying the particle to that position from a chosen standard position at distance c, say. If $f(r)=\mu r^{m'}$, where m' may be positive or negative, the potential energy E is given by

$$E=\int_c^r f(r)dr=\mu\int_c^r r^{m'}dr=\frac{\mu}{m'+1}(r^{m'+1}-c^{m'+1}). \tag{16}$$

Hence, if at a distance r the velocity be v, the principle of conservation of energy gives

$$\tfrac{1}{2}v^2+E=\tfrac{1}{2}v^2+\frac{\mu}{m'+1}(r^{m'+1}-c^{m'+1})=K, \tag{17}$$

where K denotes the whole energy of the particle.

If $m'<-1=-n$, say, this last equation becomes

$$\tfrac{1}{2}v^2+E=\tfrac{1}{2}v^2-\frac{\mu}{n-1}(r^{1-n}-c^{1-n})=K. \tag{18}$$

Taking in this case $c=\infty$ for the standard position of zero potential energy, this last equation becomes

$$\tfrac{1}{2}v^2-\frac{\mu}{n-1}r^{1-n}=K,$$

or

$$\tfrac{1}{2}v^2-\tfrac{1}{2}V^2=K, \tag{19}$$

where V is the velocity from infinity.

On the other hand, when $m'>-1$, E becomes infinite when $a=\infty$,

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and we choose as the standard position $r=0$. Hence in this case $m'+1$ is always positive, and

$$E=\frac{\mu}{m'+1}r^{m'+1}=\tfrac{1}{2}V^2.$$

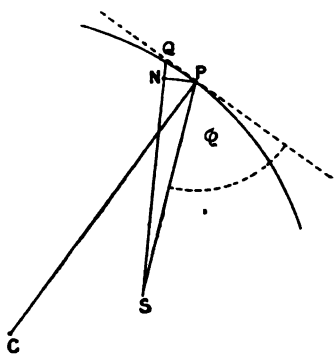
The energy equation is therefore

$$\tfrac{1}{2}v^2+\frac{\mu}{m'+1}r^{m'+1}=\tfrac{1}{2}v^2+\tfrac{1}{2}V^2=K. \qquad (20)$$

The energy equation (18) gives, of course, if v_0 be the velocity at distance r_0 in the orbit,

$$\tfrac{1}{2}v^2-\tfrac{1}{2}v_0^2=\frac{\mu}{n-1}(r^{1-n}-r_0^{1-n}). \qquad (21)$$

FIG. 277.



If the motion be in an ellipse under a force varying as the inverse square of the distance and v_0 be the velocity at the shortest distance, that is, at the distance $a(1-e)$ (§ 512) from the focus which is the centre of force, the equation of energy is

$$\tfrac{1}{2}v^2-\tfrac{1}{2}v_0^2=\mu\left(\frac{1}{r}-\frac{1}{a(1-e)}\right). \qquad (22)$$

But since the particle, when at distance $a(1-e)$, is moving perpendicularly to the major axis, $v_0a(1-e)=h$, or $v_0^2=h^2/a^2(1-e)^2=\mu(1+e)/a(1-e)$, since $\mu=h^2/a(1-e^2)$. Thus (22) becomes

$$\tfrac{1}{2}v^2=\mu\left(\frac{1}{r}-\frac{1}{2a}\right), \qquad (22')$$

the equation (11) already obtained above.

516. **Tangential and Normal Resolution. Chord of Curvature.**—Let the velocity v at the point P be inclined to the radius-vector at an angle ϕ, as shown in Fig. 277. The component of central force along the tangent is $F\cos\phi$, in the direction opposing the motion, the other rectangular component $F\sin\phi$ is inwards towards the centre of curvature. Hence, if R be the length of the radius of curvature at P,

$$v\frac{dv}{ds}=-F\cos\phi, \quad \frac{v^2}{R}=F\sin\phi. \qquad (23)$$

By Fig. 277 $\cos\phi=dr/ds$, and therefore the first of (23) becomes $vdv=-Fdr$, so that we obtain by integration

$$\tfrac{1}{2}v^2=-\int Fdr+C,$$

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which is another form of the energy equation. If $F=f(r)$ this is

$$\tfrac{1}{2}v^2 = -\int f(r)dr + C,$$

or

$$\tfrac{1}{2}v^2 - \tfrac{1}{2}v_0^2 = F(r_0) - F(r), \tag{24}$$

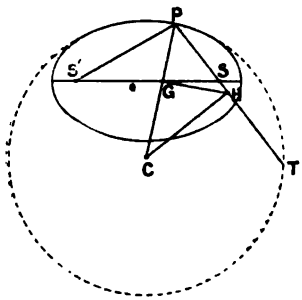
if $F(r)=\int f(r)dr$, and r_0 is the value of the distance at the point from which the integration is regarded as begun.

For example, if $f(r)=\mu/r^2$, and r_0 is $a(1-e)$, that is, if the orbit is an ellipse round a centre of force in one of the foci,

$$\tfrac{1}{2}v^2 - \tfrac{1}{2}v_0^2 = \frac{\mu}{r} - \frac{\mu}{a(1-e)}. \tag{25}$$

This is equation (22), which was obtained in § 515 by practically the same process.

FIG. 278.



The second equation of (23) can be written in the form

$$v^2 = \tfrac{1}{2}F.2R\sin\phi. \tag{26}$$

But $2R\sin\phi$ is the length of what is called the "chord of curvature" through the centre of force, that is, that chord of the circle described from the centre of curvature with radius R which coincides in direction with PO. [Fig. 278 shows the centre of curvature C of the orbit (an ellipse) at the point P, S is the centre of force, and PST the chord of curvature.] By comparison with the equation $v^2=2gh$, we see that v^2 is the velocity which the particle would acquire in moving from rest under a constant acceleration F in the direction of motion, through a distance equal to one fourth of the chord of curvature.

Thus, in the particular case of a circular orbit of radius a about a centre of force at the centre and under a force μ/a^2 the velocity is that which would be acquired by the unit particle in moving under constant acceleration μ/a^2 in the direction of motion through a distance $\frac{1}{2}a$. The velocity proper to an infinitesimal satellite revolving round the earth in a circle of radius R about the earth's centre is therefore $\sqrt{gR}$ where g is the acceleration due to gravity at distance R. The period of revolution of the satellite would be $2\pi R/\sqrt{gR}=2\pi\sqrt{R/g}$, the period of oscillation of a pendulum oscillating under gravity g, and of length equal to the earth's radius.

517. **Apsides and Apsidal Distances.**—In an elliptic orbit described round a centre of force in the focus, there are only two points at which the direction of motion is perpendicular to the radius-vector. Such a point is called an *apse*, and its distance from the focus is called an *apsidal distance*.

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The condition for an apse is that $du/d\theta = 0$, and by using this condition in the general differential equation of a central orbit it can be proved that while the number of apsides depends on the nature of the orbit, there cannot be more than two apsidal distances if the force varies as a single-valued function of the distance. This, however, can be seen without analysis from the following considerations. Let the velocity of the particle at the apse be reversed in direction without change of numerical value, the portion of the orbit just before described will be retraced, since the acceleration is the same as before at each point, but inclined to the direction of motion at an angle which is the supplement of its former inclination to the direction of motion at the same point. Hence, whatever rate of increase or diminution of velocity took place in any element of the path before, there will be an equal diminution or increase in the same element in the backward motion. But the part of the orbit described in the backward motion must be that which would be given by the part beyond the apse, if the diagram were turned round through an angle of 180°, the radius-vector to the apse as an axis. The parts of the orbit on the two sides of an apse therefore lie symmetrically about the radius-vector to the apse, and the apsidal distance first on one side is equal to that first on the other side of the apse.

518. **Determination of Orbit.**—If the plane of the orbit and the centre of force, and the velocity and direction of motion be given, the orbit can be determined. The case of greatest physical importance is the elliptic orbit described by a particle under a force varying inversely as the square of the distance.

The centre of force is the focus S, Fig. 278, of the ellipse. The plane of the orbit being given, and the centre of force, if the velocity v at a point P, distant r from the centre of force be known in magnitude and direction, the orbit can be completely determined. For the perpendicular from the centre of force on the direction of motion at P is known in length and position. If its length be p, we have $vp = h$; hence h is known.

For an ellipse, as has already been seen, v^2 must be less than $2\mu/r$, the square of the velocity from infinity at P. This is known since μ is supposed known. By §§ 512, 514, $a(1-e^2) = h^2/\mu$, and $v^2 = \mu(2/r - 1/a)$. Thus

$$a = \frac{\mu r}{2\mu - v^2 r}, \quad e^2 = \frac{\mu a - h^2}{\mu a}, \tag{27}$$

and therefore the length of the semi-axis minor is by Fig. 277

$$b = a\sqrt{1-e^2} = h\sqrt{\frac{a}{\mu}} = \sqrt{\frac{a}{\mu}}\, vr \sin\phi. \tag{28}$$

The orbit is finally determined as follows. The direction of motion being known, a normal to the ellipse at P can be drawn. This drawn inward makes an angle $\pi/2 - \phi$ with the radius-vector; a line drawn inward from P making an equal angle with SP will pass

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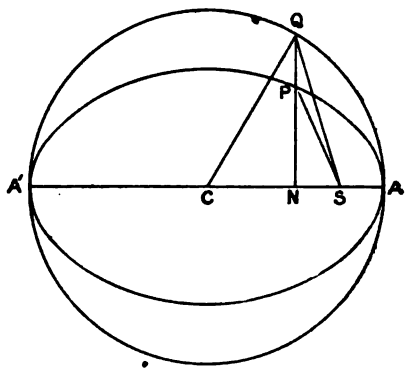
through the second focus S', and since $SP + PS' = 2a$, the second focus is determined.

For this determination there have been required the plane of the orbit, the position of the centre of force, a point in the orbit, and the direction and magnitude of the velocity at that point.

519. **Elements of an Orbit.**—In the case of a planetary orbit the position of the plane of the orbit is determined by its inclination to the plane of the earth's orbit (or *ecliptic*), and the position of the line of intersection of these two planes (which passes, of course, through the sun's centre), or *line of nodes* as it is called. The nodes are the two points where the planet in its motion passes through the plane of the ecliptic. To an eye placed in space away beyond the north pole of the earth, and looking down on the ecliptic, the planet will appear to pass from the under or southern side of the ecliptic to the upper or northern side at one node, and to pass from the upper to the under side at the other node. The former is therefore called the *ascending node*, the latter the *descending node*.

FIG. 279.



The position of the line of nodes is defined by the angle the line drawn from the sun's centre to the ascending node makes with that drawn to the position of the earth in its orbit at the vernal equinox. This is called the "heliocentric longitude of the ascending node."

The position of the major axis and the centre of the orbit are given by the heliocentric longitude of the perihelion (or point at which the planet is nearest the sun) and the perihelion distance $a(1 - e)$.

This determination of the orbit involves a knowledge of six quantities; (1) the semi-axis major a, (2) the eccentricity e, (3) the inclination ψ of the plane of the orbit to the ecliptic; (4) the longitude α of the extremity of the major axis nearest to the centre of force (the "longitude of the perihelion"); (5) the longitude β of the ascending node; and (6) the longitude ϖ of the planet at the zero of reckoning of time. This last is called the *epoch*, and enables the position of the planet in its orbit at any instant to be stated.

520. **Anomalies:—True, Eccentric, Mean. Relations connecting them.**—The angle turned through by the radius-vector as the particle moves from its position A, Fig. 279, of shortest distance from the centre of force S is called the *true anomaly*. It is thus the angle ASP. If a circle be described on the major axis AA of the orbit as diameter, and an ordinate NP be drawn at right angles to

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the major axis to the particle P, and produced to meet the circle at Q, the angle ACQ is called the *eccentric anomaly*.

The mean angular velocity n with which the radius-vector turns round is h/ab, since πab is the area swept over in one revolution, and h is twice the rate of description of area. Or, since $b = a\sqrt{1-e^2}$ and $h = \sqrt{\mu a(1-e^2)}$

$$n = \sqrt{\frac{\mu}{a^3}}. \tag{29}$$

The quantity nt, if t be time of describing the arc AP of the orbit, is called the *mean anomaly*. The *period*, or time in which the radius-vector makes one turn, is thus $2\pi\sqrt{a^3/\mu}$.

Referring to Fig. 278, and calling the eccentric anomaly ϕ and the true anomaly θ, we can express (1) nt, (2) θ, in terms of ϕ. Clearly area $ASP = (b/a)$ area $ASQ = (b/a)$(area ACQ − area SCQ) $= \frac{1}{2}ab(\phi - e\sin\phi)$. But

$$\text{area } ASP = \frac{nt}{\pi}\text{area } APA' = nt \times \tfrac{1}{2}ab.$$

Hence

$$nt = \phi - e\sin\phi. \tag{30}$$

Also

$$NS = CS - CN = ae - a\cos\phi = a(e - \cos\phi).$$

But also

$$NS = -SP\cos\theta = a(e\cos\phi - 1)\cos\theta.$$

Hence

$$\cos\theta = \frac{e - \cos\phi}{e\cos\phi - 1} \quad \text{or} \quad \frac{1-\cos\theta}{1+\cos\theta} = \frac{1+e}{1-e}\,\frac{1-\cos\phi}{1+\cos\phi},$$

that is

$$\tan\tfrac{1}{2}\theta = \sqrt{\frac{1+e}{1-e}}\tan\tfrac{1}{2}\phi. \tag{31}$$

Also as the reader may verify

$$\sin\phi = \sqrt{1-e^2}\,\frac{\sin\theta}{1+e\cos\theta}. \tag{32}$$

From (31) we have $\tan\frac{1}{2}\phi = \tan\frac{1}{2}\theta\sqrt{(1-e)/1+e}$ and, therefore, by (32) we can write (30) in the form

$$nt = 2\tan^{-1}\left\{\sqrt{\frac{1-e}{1+e}}\tan\tfrac{1}{2}\theta\right\} - e\sqrt{1-e^2}\,\frac{\sin\theta}{1+e\cos\theta}, \tag{33}$$

which enables the time taken to describe any arc of the elliptic orbit to be calculated.

A problem of great importance in physical astronomy is the calculation of r and θ for any known value of t. This is known as Kepler's problem, and several solutions by which the calculation can be made by approximation have been given; but for these the reader must

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consult treatises on Physical Astronomy or special works on the Dynamics of a Particle.

521. **Problem of Two Bodies.**—So far we have supposed the particle to move round a fixed centre; in other words, the satellite has been regarded as having a mass infinitesimal in comparison with that of the primary attracting body. But in the actual case of planetary motion this condition is only approximately fulfilled. The mass of no planet can be regarded as incomparably smaller than that of the sun, and therefore there is acceleration of the sun towards the planet which is to the acceleration of the planet towards the sun in the ratio of the mass of the planet to the mass of the sun. We have thus the so-called *problem of two bodies* if we consider the motion of any single planet. Of course the problem of the exact motion of a planet is infinitely more complicated than this, inasmuch as both the planet and the sun are attracted by all the other bodies of the solar system. The exact solution of the problem of the motion of any one body can only be approximated to by laborious calculations: a solution in finite terms, even for the comparatively simple case of three mutually attracting bodies, is, except in certain particular configurations of the bodies, beyond the powers of analysis.

As has been pointed out more than once already, the motion of a system must, in order that the law of action and reaction may apply, be referred to a system of axes fixed by bodies which are dynamically independent of the moving system. Now, if we assume this law for the attraction between a planet and a satellite, it follows at once from § 156 that the motion of the centre of inertia of the two bodies is unaffected by their mutual action. The reader may as an exercise write out the detailed proof for this case; but the general considerations stated include all cases that can arise.

522. **Orbit relative to Sun. Correction of Kepler's Third Law.**—The planet and the sun thus each describe an orbit in the plane of motion about the centroid of the two bodies; but what is observed is the relative motion of the planet about the sun regarded as having its centre fixed in position. To find this motion from the other we have to apply to both planet and sun an acceleration equal and opposite to that of the sun towards the planet. Let M be the mass of the sun and m that of the planet; then if kMm/r^2 be the mutual force between the two bodies, kM/r^2 is the acceleration of the planet towards their centroid, and km/r^2 that of the sun in the opposite direction. The relative acceleration of the planet is thus $k(M+m)/r^2$ inwards along the radius-vector, and in order to find the relative orbit we must use this acceleration.

The period, as we saw in § 520, is $2\pi\sqrt{a^3/\mu}$ for a satellite moving round a fixed centre. In the present case we have $\mu = k(M+m)$, and so for the period we obtain

$$T = 2\pi\sqrt{\frac{a^3}{k(M+m)}}. \tag{34}$$

Now consider another planet of mass m_1, and period T_1, and let

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the major axis of the orbit be a_1, and assume (see § 529) that the acceleration of the planet towards the centroid of the sun and planet is kM/r_1^2, where k is the same constant as before. The acceleration in this case is $k(M+m_1)/r_1^2$. Here

$$T_1 = 2\pi\sqrt{\frac{a_1^3}{k(M+m_1)}},$$

so that

$$\frac{T^2}{T_1^2} = \frac{a^3}{a_1^3}\,\frac{M+m_1}{M+m}. \tag{35}$$

It was observed by Kepler that $T^2/T_1^2 = a^3/a_1^3$, and this latter relation is generally stated as Kepler's third law of the planetary motions. (See § 526.) Equation (35), however, gives a correction of the law applicable to the case in which the ratio m/M is not insensible. The following table * illustrates the confirmation of the connection by observation. The mean distances a are in terms of the mean distance of the earth from the sun as unit, and the times T are in terms of the orbital revolution of the earth as unit.

Planet.	a	T	a^3	T^2	$a^3 - T^2$
Mercury.	·3871	·2408	·0580	·0580	– ·0000003
Venus .	·7233	·6512	·3785	·3785	– ·0000002
Earth .	1	1	1	1	
Mars . .	1·5237	1·8808	3·5375	3·5375	– ·00001
Jupiter .	5·2028	11·8618	140·701	140·701	+ ·131
Saturn .	9·5388	29·4560	867·914	867·658	+ ·256
Uranus .	19·1824	84·0123	7058·44	7058·07	+ ·37
Neptune.	30·037	164·616	27098·4	27098·4	+ ·16

It will be noticed from the last column that in the case of the planets which have masses less than that of the earth the difference $a^3 - T^2$, which is made zero for the earth, is negative, and positive for the planets of greater mass, and has its greatest value for Jupiter, the largest and most massive of all the planets. This is clearly in accordance with (35).

523. **Determination of Mass of Planet from Period of Satellite.**—The problem of determining the mass of a planet is very easily solved if the planet possesses a satellite. Let m' denote the mass of a satellite revolving round a planet of mass m in period T: we have $T = 2\pi\sqrt{a'^3/k(m+m')}$, for experience has shown that the mutual force between the planet and satellite is kmm'/r'^2, where k is the same multiplier as has been used for the attraction between the sun and a

* Maxwell's *Matter and Motion*, p. 115.

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planet. If T be the period of the planet and the disturbing action of the satellite on the planet's motion be neglected, $T = \pi\sqrt{a^3/k(M+m)}$. Hence

$$\frac{T'^2}{T^2} = \frac{a'^3}{a^3}\frac{M+m}{m+m'}. \tag{36}$$

Neglecting m' we get from this

$$m = \frac{T^2 a'^3}{T'^2 a^3 - T^2 a'^3} M. \tag{37}$$

Thus the masses of planets such as Mars, Jupiter, and Saturn have been determined. The motion of each satellite where more than one exist—for example, in the system of Jupiter, which has five satellites—is affected by the presence of the others; but allowing for this the periods are in accordance with Kepler's third law. The observed distances enable the accelerations to be found. A comparison of these with the acceleration of the planet towards the sun shows that the same constant k applies in both cases, as has been stated.

524. **Kepler's Laws. Law of Description of Areas.** — The motion of a planet round the sun was inferred by John Kepler, astronomer and mathematician to the Emperor of Bohemia, from an examination of observations, mainly of the planet Mars, made by his predecessor in office, Tycho Brahé. His results are contained in his *Astronomia Nova*, published in 1609, and though the dynamical ideas contained in that work are faulty, it contains also what was perhaps the first statement of the doctrine of gravitational attraction as a *mutual* force between two portions of matter. Kepler's conclusions as to the planetary motions were the result of an unavailing effort to reconcile the observed places and their times of occurrence with the hypothesis of motion in a circular orbit with uniform angular velocity about an eccentric point. The centre of the orbit was supposed to be midway between the sun's position and this point. Kepler noticed that at the greatest or the least distance from the sun the velocities of the earth in its orbit were inversely proportional to these distances, that is, that the rates of description of area by the radius-vector were equal at these points. He concluded, it appears, that this relation of the distances would hold at every point, and that, therefore, the times of describing equal small arcs would be proportional to the distance of each from the sun, and that therefore the sum of the distances of the equal small arcs, into which any finite small arc was divided, would vary as the time taken to describe the arc. The calculation of this sum proved difficult, and, therefore, for a rough verification the area included between two radii-vectores and the finite arc was used instead. It was found that this area varied accurately as the time; an entirely different conclusion, it is to be observed, from that which Kepler set out to verify.

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525. **Law of the Elliptic Orbit.**—Thus was established for the earth the law of equable description of areas by the radius-vector to the sun. On endeavouring to test whether this law held for Mars, still adhering to the idea of a circular orbit about an eccentric point, he found a discrepancy which could only be explained by supposing that the orbit was an ellipse with its major axis along the line of apsides and the sun in a focus of the ellipse. This is what is commonly called the "First Law of Kepler."

526. **Third Law of Kepler. Relation between Periods and Mean Distances.**—It was not until much later that Kepler arrived at his third law, viz., that the squares of the periodic times of the different planets are proportional to the cubes of their mean distances from the sun (the cubes of the lengths of the major semi-axes of their orbits). This law he obtained, not from any theory of the matter, but by trying whether various relations which occurred to him agreed with the observed times and distances.

We shall refer to these laws as first, second or third, according to the order of their discovery, and show how Newton was led to his gravitational theory by their means.

527. **Newton's Deductions from Kepler's Laws.**—From the first law, that of the description of areas, Newton inferred that the force, if any, on the planet was directed along the radius-vector; for from the law of equable description of areas *by itself* it is impossible to say whether a force does or does not act on the planet. For example, the radius-vector, drawn from any point whatever to a particle moving with uniform velocity in a straight line, describes equal areas in equal times, and there is no force on the particle.

But if the radius-vector describes equal areas in equal times there can be no force having moment round the fixed point through which the radius-vector always passes. For twice the rate of description of area is the moment of momentum of the motion about the fixed point, and this is, by supposition, constant. Hence, the force, if any, on the particle must be directed along the line through the fixed point. This was Newton's dynamical deduction from the first law of Kepler.

528. **Deduction of Law of Force from the Law of the Elliptic Orbit.**—From Kepler's law of the elliptic orbit Newton deduced that a force acts towards the sun, and varies (as the planet changes its position in its orbit) inversely as the square of the planet's distance from the sun's centre, where, according to the law of attraction of spherical masses, the total mass of the sun may be supposed concentrated. The deduction of this result will afford an example of Sir W. Rowan Hamilton's method of the hodograph.

Let APA', Fig. 280, represent the orbit of the planet, and a tangent to the orbit at the position P of the planet meet a circle described on the major axis as diameter in the points R, R'. Then, as stated in § 78, the lines SR, $S'R'$ drawn from the foci are perpendicular to the tangent and $ST = S'R'$. Let the sun's centre be supposed to be at rest (§ 522 above), and to be situated at S: SP is

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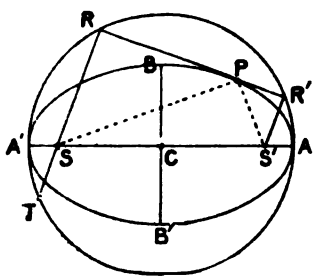
the radius-vector. In consequence of the uniformity of description of areas by SP, $v.RS=h$. But since $RS.ST=b^2$, $v.RS=vb^2/ST$. Hence $v=ST.h/b^2$, or the velocity of the planet is proportional to ST. Thus the circle ARA' represents the hodograph of the planet's motion. The hodographic origin is S, and the line ST represents the velocity turned through an angle of 90°. The motion of the point T, which accompanies the motion of P, represents the acceleration of P, on the same scale as that on which ST represents the velocity. The direction of motion of T is along the tangent to the circle at T, and is towards A if P is moving towards R; hence as the hodograph is turned through 90°, so that ST represents a velocity from P to R, the acceleration of P is at right angles to the tangent at T, that is along CT, that is along PS. [That TC, which, since TRR' is a right angle, must if produced pass through R', is parallel to SP, may be seen geometrically from the fact that since the tangent at P makes equal angles with PS and PS', $RS : ST = RP : PR'$.]

FIG. 280.



But since TC remains parallel to SP, and SP turns with angular velocity $\dot{\theta}$, the speed of T in the circle is $a\dot{\theta}$. This represents the acceleration on the same scale as that on which ST represents the velocity, and therefore the acceleration is $a\dot{\theta}h/b^2$. If the mass of the planet is unity, this is the force upon it from P towards S. But $h/r^2=\dot{\theta}$, $(SP=r)$, so that the force F is ah^2/b^2r^2. Hence the force varies inversely as the square of the distance SP.

The acceleration is thus $4\pi^2a^3/T^2r^2$. If the mass of the planet be m the force is $4m\pi^2a^3/T^2r^2$. Now the force of attraction between the sun and the planet is kMm/r^2, and the acceleration produced by it is less than that just calculated in the ratio of $M/(M+m)$. Hence we have $kM/r^2=4\pi^2/T^2.Ma^3/(M+m)r^2$, so that

$$T=2\pi\sqrt{\frac{a^3}{k(M+m)}}, \tag{38}$$

as was found before.

529. **Dynamical Interpretation of Kepler's Third Law.** — Newton's dynamical interpretation of Kepler's third law was that the force of attraction towards the sun on the different planets was proportional to their masses, and inversely proportional to the squares of their distances from the sun.

Let us suppose k to be the gravitation constant for one planet of mass m, period T, and mean distance a, and k' that for a

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planet for which the corresponding quantities are m', T', a'. Then by (38)

$$\frac{T^2}{T'^2} = \frac{a^3}{a'^3}\frac{M+m'}{M+m}\frac{k'}{k}. \tag{39}$$

But Kepler found that this ratio was a^3/a'^3. We know that it is very approximately $a^3(M+m')/a'^3(M+m)$. Hence $k=k'$; that is, the forces per unit of mass on the different planets are inversely proportional to the squares of their distances. This result for solar gravitation may be compared with that of Newton's pendulum experiment for terrestrial gravity, which shows that the forces on different bodies at the earth's surface are directly proportional to their masses, or that the forces per unit mass are the same.

530. **Newton's Discovery of Law of Universal Gravitation.**—Newton was in possession of the theory of elliptic motion a considerable time before he was in a position to announce his discovery of universal gravity. This he arrived at by showing that the acceleration of the moon towards the earth was to the acceleration of a falling body at the earth's surface in the inverse ratio of the square of the distance of the moon to the square of the earth's radius. This, on the assumption, afterwards justified, that the earth attracts external bodies as if its whole mass were collected at its centre, afforded a proof that the two accelerations were due to the same centreward attraction exerted by the earth on external bodies.

It is stated that Newton's first comparison of the two accelerations gave a serious discrepance. The moon's mean distance was then tolerably well known in terms of the earth's radius, but this latter distance was underestimated through an erroneous impression that a degree of longitude at the equator corresponded to a distance of sixty miles on the earth's surface. At a meeting of the Royal Society on January 11, 1682, Newton learned that a new measurement of an arc of the meridian of Paris had been made by Picard, and had given 69·7 miles for the length referred to. Newton repeated the calculation and found that the discrepance had disappeared. About three years later he succeeded in proving that a spherical body of density symmetrical about the centre attracted an external particle as if the whole mass of the body were collected at the centre. The last obstacle to the enunciation of his great principle of gravitational attraction was thus removed, and he stated the law which has been already illustrated above in §§ 195...202, and by some of the discussions in chap. v., that every particle of matter attracts every other particle with a force proportional to the product of the masses, and inversely proportional to the square of the distance between them. There is thus only one constant k which gives in absolute units of force by the formula kmm'/r^2 the attraction between two particles of masses m, m' at a distance r apart. We shall show in the next chapter how this constant has been determined.

The comparison of accelerations referred to above is approximately as follows: The moon's mean distance is 237,000 miles, the

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period of revolution of the moon is 27·321 days. Hence, the acceleration of the moon towards the earth is, in feet per second per second,

$$=\left(\frac{2\pi}{27\cdot321\times86400}\right)^2 237000\times5280=\cdot008866.$$

Hence for the acceleration due to gravity at the earth's surface we should have by the law of inverse square of the distance, taking the earth's radius as 3960 miles,

$$g=\frac{237000^2}{3960^2}\cdot008866=31\cdot76,$$

in feet per second per second, which agrees fairly with the actual value.

531. **Orbital Motion of a Cluster of Meteorites.**—The question of the motion of a cluster of particles is important in connection with the natural history of comets and swarms of meteors. A particle in such a cluster is under the attraction of the sun and the attraction exerted upon it by the particles of the swarm. If the assemblage of particles is spherical in shape and of uniform distribution of matter, its attraction on a particle at any point is directed towards the centre of the sphere.*

Referring to Fig. 129, let C be the place of the sun supposed fixed, A be the centre of the swarm, and B an individual particle of unit mass; the force towards the centre is kM/AB^2. But $m=\frac{4}{3}\pi\rho r^3$, where ρ is the (average) mass per unit volume, and $r=AB$. The force is therefore $\frac{4}{3}k\pi\rho r$, or μr if μ be put for $\frac{4}{3}\pi k\rho$. Then, by §§ 272, 273 (the notation being as stated there, except that r_1 is used instead of a to denote the distance of the centre A of the cluster from C, the sun), the distance r_1 and the angular velocity n being both supposed variable, we obtain, since the distance of the particle from the sun along the line of centres of the sun and the nebula is r_1+x,

$$\left.\begin{aligned}\ddot{r}_1+\ddot{x}-2n\dot{y}-y\dot{n}-n^2(r_1+x)&=X\\ \ddot{y}+2n(\dot{r}_1+\dot{x})+(r_1+x)\dot{n}-n^2y&=Y\end{aligned}\right\}\quad(40)$$

To these, since all the particles do not lie in the plane of motion of the centre, we must add a third equation. Let z be taken at right angles to the plane of motion, then we have approximately

$$\ddot{z}=-kM\frac{z}{r_1^3}-\mu z.$$

But

$$X=-kM(r_1+x)/\{(r_1+x)^2+y^2\}^{\frac{3}{2}}-\mu x=-kM/(r_1+x)^2-\mu x,$$

approximately, and $Y=-kMy/(r_1+x)^3-\mu y$. Hence the equations are (40) above, with these values of X, Y used in them.

* The assumption of sphericity is somewhat violent, and can be only very roughly in accordance with fact in any actual case. It seems on this account unnecessary to correct the result given in (45) below for the effect of eccentricity of the orbit of the swarm.

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Now for the motion of the central particle we have by (40)

$$\ddot{r}_1 - n^2 r_1 = -\frac{kM}{r_1^2}, \quad 2n\dot{r}_1 + r_1\dot{n} = 0. \tag{41}$$

These equations used in (40), with the values of X, Y stated, give approximately

$$\left.\begin{aligned} \ddot{x} - 2n\dot{y} - y\dot{n} - n^2 x &= 2kM\frac{x}{r_1^3} - \mu x \\ \ddot{y} + 2n\dot{x} + x\dot{n} - n^2 y &= -kM\frac{y}{r_1^3} - \mu y. \end{aligned}\right\} \tag{42}$$

If r_1 be constant $\dot{n}$ will be constant also, since then the centroid of the cluster will move round the sun in a circle. The attraction towards the sun at the centre is $n^2 r_1$ per unit of mass, and this must be kM/r_1^2. Hence $n^2 = kM/r_1^3$. Thus we obtain for the equations of motion of the particle whose co-ordinates relatively to the centre of the cluster are x, y, z,

$$\left.\begin{aligned} \ddot{x} - 2n\dot{y} - (3n^2 - \mu)x &= 0 \\ \ddot{y} + 2n\dot{x} + \mu y \qquad &= 0 \\ \ddot{z} \qquad + (n^2 + \mu)z &= 0. \end{aligned}\right\} \tag{43}$$

532. **Condition that a Swarm of Meteorites may keep together under the Sun's Attraction.**—If we suppose now the values of x and y to oscillate about certain constant values, we must write

$$x = a\cos(mt + \epsilon), \quad y = b\sin(mt + \epsilon),$$

that is suppose the relative orbit of a particle to be an ellipse of semi-axes a, b.

These values of x, y substituted in (43) give the condition

$$(m^2 - \mu)(m^2 + 3n^2 - \mu) - 4m^2 n^2 = 0, \tag{44}$$

and the roots of this quadratic in m^2 must be real and positive if the oscillation in the values of x and y is to be stable. Now if $m^2 = \pm\infty$, the expression on the left is positive. If $m^2 = \mu$, or $m^2 = \mu - 3n^2$, it is negative. But if $m^2 = 0$ it becomes $\mu(\mu - 3n^2)$, which is positive if $\mu > 3n^2$. Hence the graph of the expression on the left of (44) will cross the axis along which values of m^2 are measured twice on the positive side of the origin if $\mu > 3n^2$. The condition required is therefore that just stated.*

Now, replacing μ by $\frac{4}{3}\pi k\rho$ and n^2 by kM/r_1^3, we obtain the inequality

$$\tfrac{4}{3}\pi\rho r_1^3 > 3M. \tag{45}$$

Hence, in order that the swarm of small particles may keep together

* It is to be observed that the two conditions, obtained by substituting the values of x, y in the first two equations of (43), show that the relative orbit cannot be a circle unless $3n^2$ be numerically small in comparison with $\mu - m^2$. The swarm therefore cannot be a *spherical* cluster revolving like a rigid body about an axis at right angles to its orbit unless this be the case.

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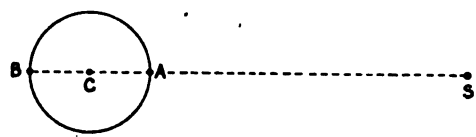
it is necessary that the average density be greater than that of a spherical distribution of matter of radius equal to the sun's distance and of three times the sun's mass. This conclusion is due to M. Charlier,* whose investigation is somewhat more elaborate than that given above, though the results are confined to the special case of a circular orbit.

533. **Elementary Discussion of Stability of Meteor Swarm.**—A full investigation, from the equations of motion is necessary if the problem is to be pushed further to the case of a swarm of meteorites moving in an elliptic orbit; but for the reasons indicated in the footnote on p. 500, this is unnecessary. So far as the circular orbit is concerned, the main result can be obtained by simple considerations. Consider what is practically the case supposed above—a spherical swarm of radius a moving round the sun, and turning as a whole about an axis perpendicular to the orbit in the period of revolution, so that it turns the same face always towards the sun. A particle, of unit mass, say, at the centre, C, Fig. 281, at distance SC $(=r_1)$

FIG. 281.



from the sun, is in relative equilibrium under the sun's attraction and the so-called centrifugal force. That is, we have for that particle $kM/r_1^2 - n^2 r_1 = 0$.

Again, a particle on the outside of the swarm at the point nearest the sun is at a distance $r_1 - a$, and under attraction $kM/(r_1 - a)^2$. Hence there is a preponderance of attraction over the acceleration $n^2(r_1 - a)$ towards S. This excess is

$$\frac{kM}{(r_1 - a)^2} - n^2(r_1 - a) = kM\left\{\frac{1}{(r_1 - a)^2} - \frac{1}{r_1^2} + \frac{a}{r_1^3}\right\}$$
$$= 3kM\frac{a}{r_1^3}$$

nearly. This must be balanced by the attraction towards the centre C exerted by the swarm if the particle is not to leave the swarm. Hence we must have $\frac{4}{3}\pi\rho k a^3/a^2 > 3kMa/r_1^3$ or,

$$\tfrac{4}{3}\pi\rho r_1^3 > 3M,$$

as before. The same result would be obtained for a particle at B. In that case the attraction of the sun $kM/(r_1 + a)^2$ would be insufficient to supply the acceleration $n^2(r_1 + a)$ towards the sun. The condition that this should be supplied by the attraction of the swarm is that $\frac{4}{3}\pi\rho r_1^3$ should be at least equal to $3M$.

* *Bulletin de l Académie de St. Pétersbourg*, t. xxxii. No. 2.

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This result holds, of course, for all particles within the swarm on the line SC, for any such particle experiences no force on the whole from the spherical layer outside it.

It is to be observed that a particle at A or B (or on the line SC) is in greater danger of leaving the swarm from the causes just explained, than a particle elsewhere on the spherical surface.

534. **Condition that a Satellite may be retained by a Planet or a Meteor Swarm.**—If we consider an isolated particle outside the spherical swarm at a distance r from the centre the equations wil require modification. Thus, putting μ for km, we find instead of (43

$$\left.\begin{aligned} \ddot{x} - 2n\dot{y} - \left(3n^2 - \frac{\mu}{r^3}\right)x = 0 \\ \ddot{y} + 2n\dot{x} + \frac{\mu}{r^3}y = 0 \\ \ddot{z} + n^2z + \frac{\mu}{r^3}z = 0. \end{aligned}\right\} \quad (46)$$

Multiplying these by $\dot{x}$, $\dot{y}$, $\dot{z}$ respectively, adding and integrating we obtain

$$v^2 - 3n^2x^2 + n^2z^2 - \frac{2\mu}{r} + C = 0, \quad (47)$$

where $v^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2$, and C is a constant. This is the equation of relative kinetic energy for the case in question.

The meaning of the constant C can be assigned as follows. The equation of energy in an elliptic orbit is $\frac{1}{2}v^2 - \mu(1/r - 1/2a) = 0$, where a denotes the length of the semi-axis major. Writing then (47) in the form

$$\tfrac{1}{2}v^2 - \mu\left(\frac{1}{r} - \frac{1}{2a}\right) - \frac{3}{2}n^2x^2 + \frac{1}{2}n^2z^2 = 0, \quad (48)$$

so that, $C = \mu/a$, we see that μ/C is the semi-axis major of the elliptic orbit that the particle would describe about the centre of the swarm if n were zero, that is, if the sun's action were entirely negligible.

We can show that this equation assigns an upper limit to the distance to which the particle can move, in the plane of motion of A, from the centre of the swarm. For when v^2 has a given value the particle must lie on the surface given by placing that value in (48). Hence, since v^2 is positive, the particle cannot pass across the surface for which $v^2 = 0$, that is, the surface of which

$$3n^2x^2 - n^2z^2 + \frac{2\mu}{r} - C = 0 \quad (49)$$

is the equation.

By putting $z = 0$ in this we obtain the equation of the curve in which the surface intersects the plane of x, y, namely

$$3n^2x^2 + \frac{2\mu}{r} - C = 0,$$

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or if $r\cos\theta$, $r\sin\theta$ be put for x, y,

$$3n^2\cos^2\theta.r^3 - Cr + 2\mu = 0. \tag{50}$$

If this cubic equation in r have at least one finite positive root for every value of $\cos\theta$, the trace of the surface on the plane of x, y will have a closed branch round the centre. Now it is clear by Descartes' rule of signs that it cannot have more than two positive roots or than one negative root. The roots are all real if

$$\cos^2\theta < \frac{C^3}{81n^2\mu^2}.^* \tag{51}$$

Hence there are two positive roots if this inequality is satisfied, and the particle, if within the closed curve, cannot pass beyond it in the plane of the orbit.

The greatest value of $\cos\theta$ is 1. Hence, if the inequality is satisfied when $\theta = 0$ or $\theta = \pi$, it is *a fortiori* satisfied for all other values. We have therefore only to examine whether the inequality

$$\frac{C^3}{81n^2\mu^2} > 1 \tag{51'}$$

is satisfied for any problem for which data are given. We shall suppose that this is the case.

It is clear that when $\cos\theta = 0$ the equation has one finite root $r = 2\mu/C$, and two infinite roots, one of which is positive the other negative. When $\theta = 0$, as can easily be shown, a root corresponding to this lies between $2\mu/C$ and $3\mu/C$, and for values of θ ranging from 0 to 90° its value gradually diminishes to $2\mu/C$ at $\theta = 90°$.

Besides the closed curve there are two infinite branches given by the other roots. These branches are asymptotic with the two lines represented by $3n^2x^2 = C$. For points between the closed curve and the infinite branches, the value of the expression on the left of (50) is negative, and therefore by (48) v^2 is negative. Hence the velocity is imaginary. Inside the closed curve or outside the infinite branches v^2 is positive. Hence the particle must in the one case remain inside the closed curve, or in the other case remain outside the infinite branches.

The closed curve or oval, and the infinite branches are shown in diagram a of Fig. 282. AA', BB' are the lines represented by $3n^2x^2 = C$. The oval has its greatest diameter in the direction of the axis of x, that is xx in the diagram.

In diagrams b/c the trace of the surface is shown for the respective cases $C^3 = 81\mu^2n^2$, $C^3 < 81n^2\mu^2$. The oval and the infinite branch in b meet on the axis of x, and it appears as if at these points the particle might escape from the oval. The condition, however, $C^3 = 81n^2\mu^2$, which gives equality of the two positive roots, shows that at the points of meeting the acceleration of the particle, as well as its velocity, is zero.

* See any Treatise on the Theory of Equations.

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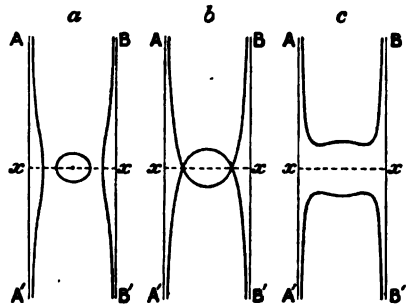
In the remaining case, illustrated by diagram *c*, there are no real roots for $\cos\theta = 1$, and there is no closed curve surrounding the origin. There is therefore no upper or lower limit given for the distance of the particle from the origin, the centre of the swarm.

We can apply this theory to the moon regarded as a particle of infinitesimal mass, and supposed to move in the plane of the earth's orbit as specified in § 274. It is easy to verify that the inequality is satisfied, and that therefore the moon lies within the closed curve. If we suppose that $C = \mu/a$ where a is the radius of the present orbit of the moon, we obtain as a superior limit a value between $2a$ and $3a$.

FIG. 282.



Retaining terms in y in (40) above we should obtain by a similar process a more accurate determination of the limiting surface. Thus Mr. G. W. Hill* has shown that the limiting surface consists of three sheets, one enclosing the earth, the other enclosing the sun, and a third surrounding the other two, but unclosed in the direction of z. The value of v^2 is positive only within either closed sheet or outside the third, and therefore v is imaginary everywhere else. A particle, therefore, revolving under the attractions of the earth and moon cannot cross the limiting surface if within either of the closed sheets. Mr. Hill's result for the maximum distance for the first of these sheets from the earth's centre is 110 radii of the earth; the moon's average distance is about sixty times the earth's radius.† The moon is therefore within the surface enclosing the earth, and cannot escape from it.

535. **Planetary Perturbations.**—The subject of the perturbations of the planetary motions has been touched upon in the sketch of elementary lunar theory given in §§ 274–276 above, where certain effects of the solar attraction on the orbit is shown in the relative orbit of Fig. 283. The whole subject of *Celestial Mechanics* has undergone improvement at the hands of Hill, Gylden, Poincaré, and others, and recourse must be had to the memoirs and works of these authors for full information. The calculation of the disturbances of the lunar and planetary orbits by the action of the sun in the former case, and of other planets in the latter, and the formation of tables of the places of the moon and planets must, however, still in practice proceed by the older processes of successive approximation. These

* *American Journal of Mathematics*, vol. i. 1870.

† The reader will find further information on the subject of the preceding sections in Routh's *Dynamics of a Particle*, or Tisserand's *Mécanique Céleste*, tome iv.

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will be found explained in Laplace's *Mécanique Céleste*, or in the more modern work of Tisserand, which has the same title.

536. **Newton's Revolving Orbit.**—One result given by Newton is interesting—that of a planet under a disturbing force, which is proportional to the inverse cube of the distance, and acts along the radius-vector. Let the law of force be $f(u)$, and the disturbing force be μu^3, then the equation of motion is

$$\frac{d^2u}{d\theta^2}+\left(1-\frac{\mu}{h^2}\right)u=\frac{f(u)}{h^2u^2}. \tag{52}$$

Hence, if, instead of θ, we write $\theta'=\theta\sqrt{1-\mu/h^2}$, and

$$h'=h\sqrt{1-\mu/h^2},$$

the equation reduces to

$$\frac{d^2u}{d\theta'^2}+u=\frac{f(u)}{h'^2u^2}, \tag{53}$$

which is of the usual form. The motion is therefore precisely that of the particle in the orbit under the force $f(u)$, with the difference that the angular velocity is $\dot{\theta}'=c\dot{\theta}$, where $c=\sqrt{1-\mu/h^2}$. Hence, the orbit is the undisturbed orbit supposed turning at each instant round the centre of force with angular velocity $(c-1)\dot{\theta}$.

537. **Disturbing Action of Small Tangential Force on Motion of Planet. Tidal Action on Moon's Motion.**—Consider a particle moving round a primary in a very gradual spiral path tending outwards in consequence of a small tangential force in the direction of motion. The action of this force taken by itself is to increase the orbital velocity of the particle, but, as the central force is inadequate to give a centreward acceleration corresponding to an increased velocity, the particle moves out to a greater distance, and the velocity is diminished by the action of the component of central force resisting the outward motion. The motion may be approximated to by supposing the particle subjected to a series of tangential impulses of small amount, and to pass out after each to the new orbit corresponding to the new velocity; but we shall suppose the tangential force to act, and the adjustment to the new orbit to take place, continuously.

If v be the velocity and a the radius of the path, the kinetic energy is $\frac{1}{2}v^2$, and the potential energy $-\mu/a$. Hence the energy equation is by (22′)

$$\tfrac{1}{2}v^2-\frac{\mu}{a}=-\frac{\mu}{2a}. \tag{54}$$

If now the tangential force act for any interval, it will do an amount of work w, say, on the particle and the total energy will now be $-\mu/2a+w$. Hence, if v' be the velocity and a' the distance of the particle in its new path from the centre of force, the energy equation is now

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$$\tfrac{1}{2}v'^2 - \frac{\mu}{a'} = -\frac{\mu}{2a} + w. \tag{55}$$

But for the relative equilibrium we must have $v'^2 = \mu/a'$, so that the equation just written becomes

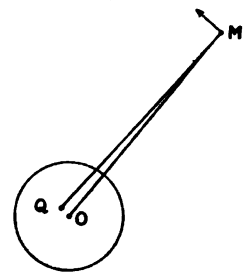
$$-\tfrac{1}{2}v'^2 = -\frac{\mu}{2a} + w.$$

Adding this to (54) we obtain

$$\tfrac{1}{2}v^2 - \tfrac{1}{2}v'^2 = w; \tag{56}$$

that is, the kinetic energy is now less than before by an amount equal to the increase of energy w produced by the tangential force. The tangential component of the central force against the motion must, therefore, have been double the tangential force disturbing the motion.

FIG. 283.



The gain of potential energy, however, has been that involved in the change from $-\mu/a$ to $-\mu/a'$, or $-\mu/a' - (-\mu/a) = \mu/a - \mu/a'$. This by (54) and (55), is

$$\tfrac{1}{2}v^2 - \tfrac{1}{2}v'^2 + w = 2w, \tag{57}$$

that is, the gain of potential energy is twice the loss of kinetic energy, or there is a gain w of energy on the whole.

Such a tangential action is exerted on the moon in consequence of the fact that high water (see chap. xii.) is not directly under the moon, but in advance of the radius-vector drawn from the centre of the earth to the moon. Hence the resultant action is not through the centre O of the earth, Fig. 283, but in the line QM. This gives a disturbing tangential force in the direction of the moon's motion of the kind discussed above. The tendency is to cause the moon to move further off in the first instance; but the whole subject will be more fully explained in chap. xiv. in connection with the subject of the Tides.

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CHAPTER XIII.

GRAVITATIONAL ATTRACTION: THE EARTH'S MEAN DENSITY.

538. **Determination of the Constant of Gravitation and of the Earth's Mean Density.***—There is good reason for believing that the attraction of each of the planets is very approximately the same as if the whole mass of the planet were collected at its centre of figure. There are, however, in some cases, probably in all, deviations from sphericity; all have, no doubt, more or less oblateness of figure, produced originally by the axial rotation. To this are due, in the case of the earth, the phenomena of precession and nutation; but the attraction on an external particle is practically the same as if the mass of the earth were collected at its centre. It is a result of observation that the earth is nearly spherical in figure, and rotates about a diameter, round which the deviation from sphericity is very nearly symmetrical. Hence, from the facts that, allowing for the effect of rotation (§ 281), the acceleration of a particle is nearly constant in amount at all points on the same level, and is directed very approximately towards the earth's centre, we conclude that the matter composing the earth is distributed with a close approach to symmetry about the centre; that is, that the earth may be regarded as composed of a series of concentric spherical shells each of uniform density.

Also, by the third law of Kepler, which in its most general form (§ 522) is found fulfilled, not only by the planets and the sun, but by minor systems such as Jupiter and his satellites, we are led to the conclusion that the same constant of gravitation, k, applies to the sun and all planetary bodies; and it was shown, as we have seen by Newton, that it is the same formula of attraction kmm'/r^2 that gives the force between the earth and the moon, and the acceleration of a falling body at the earth's surface. Hence we have the problem to determine the constant k; in other words, to determine the attraction between two particles, each of unit mass, placed at unit distance apart. This is a problem, of course, belonging to universal gravitation, and could be solved at once from the observed acceleration of

* A valuable account of this subject is to be found in Professor Poynting's Adams Prize Essay, *The Mean Density of the Earth.* London: Charles Griffin & Co., 1894.

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any planet or satellite if the mass of the central attracting body were known in terms of the dynamical unit of mass. But by such observations we can only compare the masses of the heavenly bodies with one another, or with the unknown mass of the earth.

539. **Elementary Theory.**—If, however, the attraction of a sphere of matter of known density, say a ball of lead, on a particle of unit mass at a measured distance from the centre of the sphere be determined, the value of k can be found, and also the mean density of the earth. For let M be the mass of the sphere, r the distance of the attracted particle from the centre, and F the force observed in absolute units. Then $F = kM/r^2$, or

$$k = \frac{r^2}{M}F. \tag{1}$$

The value of k can then be used to express g, the known value of gravity at the place of experiment, in terms of the earth's mass and dimensions. By § 281, if G denote the total gravitational attraction at the earth's surface, R the earth's mean radius, n the earth's angular velocity of rotation, and λ the latitude of the place,

$$g = G - n^2R\cos^2\lambda = G\left(1 - \frac{n^2R}{G}\cos^2\lambda\right).$$

But $G = kV\rho/R^2$, where V is the volume of the earth and ρ its mean density, and $n^2R/G = \frac{1}{289}$; so that

$$g = \frac{kV\rho}{R^2}\left(1 - \frac{1}{289}\cos^2\lambda\right). \tag{2}$$

But this equation rests on the hypothesis that the earth is spherical, which is not accurately true, as in fact the earth has, according to the geodetic results of Colonel A. R. Clarke, R.E., the following dimensions*—

mean equatorial semi-diameter	a	= 20926202 feet
polar semi-diameter	c	= 20854895 ,,

so that it has an ellipticity $e[=(a-c)/a] = 1/293{\cdot}465$.

To make the necessary correction† the multiplier of $kV\rho/R^2$ in (2) must be diminished by $\frac{1}{2}m - \frac{1}{3}e - (\frac{3}{2}m - e)\sin^2\lambda$, where m denotes $1/289$. Hence, including a correction-term $-2h/R$ for height h above the mean level (and taking h in centimetres), we have

$$g = \frac{kV\rho}{R^2}\{1 - \tfrac{3}{2}m + \tfrac{1}{3}e + (\tfrac{5}{2}m - e)\sin^2\lambda - 3 \times 10^{-9}h\}. \tag{3}$$

The value of V in cubic centimetres is $1{\cdot}0832 \times 10^{27}$ and of R in centimetres is $6{\cdot}37 \times 10^8$.

* *Geodesy*, Oxford, 1880, p. 319.

† Thomson and Tait, *Nat. Phil.*, vol. i. part ii. § 797.

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Now let g have been determined for the place of experiment either by direct pendulum observations, or from the formula

$$g = 980{\cdot}6056 - 2{\cdot}572\cos 2\lambda - {\cdot}000003h,^* \qquad (4)$$

which agrees with (3) and expresses in centimetre-second units very approximately the results of gravity determinations made at different parts of the world. It is only necessary to insert in (3) the values of k, V, and R and calculate that of ρ. It remains, therefore, only to discuss the determination of F, by means of which k is found. In the statement of results below we shall use the symbol Δ, according to usage, for the earth's mean density.

The value of ρ may also be found, without using the value of k, as follows. If $E = V\rho$ we have $G = kE/R^2 = \frac{4}{3}\pi k\rho R$. Hence, $G/F = \frac{4}{3}\pi\rho Rr^2/M$, and

$$\rho = \frac{3}{4\pi}\frac{G}{F}\frac{M}{Rr^2},$$

from which ρ may be calculated from the value of M, ascertained by weighing the attracting sphere, and the observed value of F.

540. **Cavendish's Experiment.**—The mutual attraction of two balls of lead was measured by the Hon. Henry Cavendish † by a method which he states had been proposed by the Rev. John Michell, who had gone so far as to construct an apparatus for the purpose. This apparatus was improved and in great measure reconstructed by Cavendish, but the principle and general arrangement remained the same. At the extremities of a light horizontal rod suspended by a torsion wire were placed two equal balls of lead. There being practically no force on these spheres except the downward force of gravity on each, the rod would, if undisturbed by air-currents, take up the position of rest for which the wire was without torsion. Then two large equal balls of lead were to be placed, one near one of the suspended spheres, the other at an equal distance from the other, but on the opposite side of the suspension rod. Pulls would thus be exerted on the balls in opposite directions, and a couple applied on the suspended system. This would be deflected to a new position of equilibrium in which the deflecting couple would be balanced by the couple of torsion due to the twist given to the suspension wire. The deflection of the suspended system would give the angle of torsion, and from a separate determination of the torsional rigidity of the wire the couple could be obtained, and therefore the pull in absolute units between each pair of balls.

541. **Details of Apparatus.**—Figs. 284, 285 show, in elevation and plan, the arrangement of the apparatus as made by Cavendish. The suspension-rod h, h, with balls b, b and suspension-wire w, was

* The coefficient of the second term is given in Everett's *Units* as 2·5028, but this has been obtained by using a lower value, 1/282, of the ellipticity.

† Experiments to Determine the Density of the Earth, *Phil. Trans.*, vol. lxxxviii. 1798.

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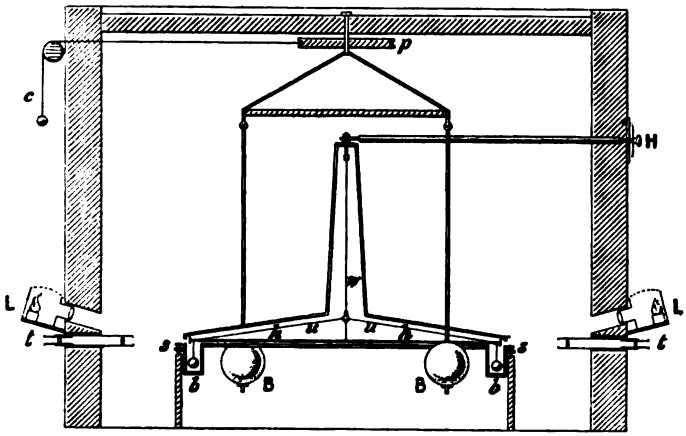
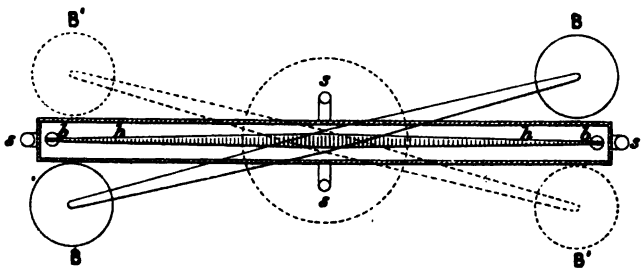
placed within an inner case which partially prevented disturbances from currents of air, and the whole apparatus was enclosed in a large outer chamber which remained constantly closed. The rod was of deal and six feet in length, and was stiffened by a wire u, u stretched

FIG. 284.



from its ends over a rigid vertical strut at the centre. To the uppermost point of the light girder thus constructed the suspension-wire was attached; the other end was held by a torsion-head resting on the top of the inner case. This head could be reached and operated

FIG. 285.



by a rod H from outside the chamber. Four levelling screws shown in the plan enabled the whole arrangement to be set up in proper position.

The large masses were mounted on a frame turning round a pivot in the roof of the chamber by which they could be brought first to the position, say that shown in full lines in the diagram, then to that shown in the dotted lines; or, if required, the line joining the balls could be placed at right angles to h, h, so as to exert no

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couple on the suspended system. The turning-frame was operated from outside the chamber by a cord *c* passed over a horizontal pulley *p*.

Two vertical copper rods carried the spheres at their lower ends, and these rods were arrested on either side of the torsion-chamber by stops attached to the outer wall, so as to place the large spheres in position as close as was possible to the small balls without disturbing the inner case.

The position of the rod *h*, *h* was observed on scales opposite its extremities by means of telescopes *t*, *t* placed in the walls of the chamber. These scales were of ivory divided to twentieths of an inch, and the rod carried a vernier of five divisions at each end. The necessary illumination of scales was provided by the lamps *L* placed opposite apertures in the wall above the telescopes.

The suspended balls *b*, *b*, as well as the attracted spheres *B*, *B*, were of lead. The former were two inches in diameter, the latter twelve inches. The torsion-wire was of silvered copper, and was 39¼ inches in length. At first the wire was of such thickness that the time of a torsional oscillation (half-period) was fifteen minutes, but this was soon replaced by a thicker wire, with which the time of oscillation was about seven minutes. When the turning-frame was against the stops on either side of the torsion-case, the centres of the large balls were 8·85 inches from the central vertical plane of the case, and on a level with the centres of the balls *b*, *b*.

542. **Method of Experimenting.**—It was not found possible to observe the positions of static equilibrium; what was done was first to free the suspension wire as far as possible from torsion, with the line joining the large balls perpendicular to *h*, *h*. The large balls were then turned into position on one side of the frame, which set the rod *h*, *h* into vibration about the equilibrium position. Three consecutive turning positions, or positions of rest, of the vibrating rod *h h* were observed on the scale, after which the frame was carried round to the position on the other side, and three consecutive turning positions of *h*, *h* again observed. From these the equilibrium position on either side could be deduced, and the angle between these was taken as the angular deflection produced by carrying the balls *B*, *B* from one configuration to the other.

The time of vibration was carefully determined by a series of observations made for the purpose, and served, with the calculated moment of inertia of the suspended system, to enable the torsional couple applied by the wire *w* to be calculated. These observations were made for both positions of the attracting spheres *B*, *B*. It was found that for one the time of oscillation (half-period) was 426 seconds, for the other 427 seconds.

543. **Corrections Applied.**—In reducing the results various corrections had to be made, with regard to which it is impossible here to go into details. Reference may be made to the original paper, or to Professor Poynting's treatise. For one thing, the attracting masses *B*, *B* were not quite opposite *b*, *b*, as the distance

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between them was a little too small. Then there was attraction on each ball b due to the distant large sphere, which diminished the effect of that due to the near ball; there was attraction exerted on the rod h, h, attraction on the suspended system exerted by the copper rods, and a couple also applied by the case in consequence of the deviation from symmetry produced by the deflection. This last Cavendish found to be less than $\frac{1}{10}$ per cent. of that due to the attraction of the masses.

544. **Theory of the Experiment.**—It had been found by previous experiments, carried out by Coulomb, that the couple required to keep a wire twisted through any angle was directly proportional to the twist. Thus, if the lower end of the wire w was turned through an angle θ relatively to the upper end, a couple $c\theta/l$ would be required to maintain the twist (measured by θ/l), where l was the length of the wire and c a constant, called the "torsional rigidity of the wire" (see § 638). We shall denote c/l by μ: the couple is then $\mu\theta$. Now $2a$ denoting the distance between the lines of action of the pulls between the pairs of adjacent balls, the couple due to these forces is represented by $2kMma/r^2$, where M is the mass of a large ball, m that of a small one, and r is the distance between their centres.

If now r' be the distance between two remote balls (a large and a small), and $2a'$ be the distance between the (parallel) lines of action of the attractions of the components of each pair of such balls, we have an opposite couple of amount $2kMma'/r'^2$. Hence

$$\mu\theta = 2kMm\left(\frac{a}{r^2} - \frac{a}{r'^2}\right). \tag{5}$$

It is theoretically more correct to use for M and m not the true masses of the balls as determined by reducing to weighing in vacuum, but these masses diminished in each case by the mass of air displaced by the ball. For, suppose two opposite balls to be removed and the spaces vacated filled with air, there will be attraction between these two portions of air which will, if there is equilibrium of the whole mass of air, be balanced by the action of the surrounding air. This action will be exerted on the immersed balls, and there will be left unbalanced only the attraction between the *additional* masses now contained in the spaces.

The point is not, however, of practical importance, but it should be noted, as in one determination at least the weights of the balls have been "corrected" to the weight in vacuum.

For the vibration experiments we have the equation

$$T = 2\pi\sqrt{\frac{I}{\mu}} \tag{6}$$

where I is the moment of inertia of the vibrating system, and T its period. For $\mu\theta$ is the restoring couple, and, therefore, the angular

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acceleration towards the equilibrium position is $\mu\theta/I$, so that $\mu\theta/I\theta = 4\pi^2/T^3$, by the theory of simple harmonic motion.

A correction had to be made in the application of this equation for the fact that the couple of attraction did not remain constant during the oscillations by which the period was determined.

545. **Result of the Experiments.**—These equations enable the value of F, the attraction exerted by a large sphere on unit mass of a small one, to be determined, and hence k to be calculated as already explained. By means of the result can be found the value of Δ, the earth's mean density. Twenty-nine results obtained by Cavendish gave for Δ a mean value of 5·448 with a probable error of $\pm$ ·033.

546. **Effects of Inequalities of Temperature.**—Some trouble was experienced through the difficulty, owing to the great size of the apparatus, of maintaining the temperature uniform throughout the inner case. If the large balls were left for a time in proximity to the small ones the effect seemed to alter, sometimes gradually increasing, at others gradually diminishing. This was traced to difference between the temperature of the case and that of the large balls, so that the distribution of temperature changed when the masses were moved, and air-currents were set up, producing spurious attractions or repulsions, and in course of time deflections which very slowly died away as the apparatus was left to itself.

Thus, when to test this effect the large balls were heated with a lamp and then brought near the case, the outside of the case rose in half an hour $1\frac{1}{2}°$ F., and the small spheres were found deflected through 14 scale-divisions instead of 3, as at first. Cooling the large spheres with ice to 6° or 7° F. below the temperature of the case, and bringing them near the sides, produced a reverse effect, a diminution in one hour of $2\frac{1}{2}$ divisions. When the spheres were heated there would be generated an ascending current of air on the sides of the small balls b, b adjacent to B, B, and a descending current on the other side, with an excess of pressure on the latter side above that on the former; and the reverse would be the case in the experiments made with the balls cooled.

Professor Poynting, in this connection, calls attention to some important observations and a suggestion made by Sir William Crookes. In the course of his experiments on the actions on bodies in high vacua, Crookes noticed that, when a large mass was brought near a delicately suspended light ball in air of ordinary density, "if the mass is colder than the ball, it repels the ball, and when the mass is hotter than the ball, it attracts the ball," and that opposite effects were produced when the small ball was in a vacuum. He also observed that there was an intermediate pressure at which differences exerted little or no interfering action, and remarked that "by experimenting at this critical pressure, it would seem that such an action as was obtained by Cavendish, Reich, and Baily should be rendered evident."

547. **Repetitions of Cavendish's Experiment. Reich's Experiments.**—Determinations in one or two cases not yet finally com-

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pleted have been made by this method, by several experimenters, among others by Reich,* Baily,† Cornu and Baille,‡ and Boys.§ In Reich's experiments the arrangements were very similar to those of Cavendish. But the attracting balls were used one at a time. Being suspended by brass wires from wheels running on rails along the ceiling of the room, the ball could be run out from the middle position, or position of zero couple, to the position opposite the ball hung at either of the ends of the torsion-rod. Four successive extreme positions of the torsion-rod in its swing were observed, and the readings for these being a, b, c, d, $\frac{1}{8}(a+3b+3c+d)$ was taken as the reading for the mean centre of the range of vibration. The mass was then at once shifted to the middle position, and four readings taken also for that position. The last, d, of the former set was taken as the first of the new set, and this, as was pointed out by Cornu and Baille, is not free from objection, as the transfer of the attracting ball could not be regarded as instantaneous. The ball was then run out to the attracting position used before, and the observations repeated. Similar operations were performed with the other attracting ball in the corresponding positions on the other side of the torsion-rod.

The result obtained on a final computation of the mean density was 5·49.

Reich repeated his research with an arrangement in which only a single attracting sphere was used, mounted on a turn-table, by which the sphere could be placed on either side of a small ball hung from one end of the torsion-rod. The table turned about a tube in which the ball hung, so that the attraction of the turn-table on the suspended ball was the same in all positions, and the correction for the attraction of the brass wires was done away with. Reich also tested for possible error due to magnetic action. A diamagnetic attracted sphere of bismuth gave as the mean of 10 determinations 5·5266, and an attracted sphere of iron as the mean of 12 determinations 5·6887. He attributed the difference to a diamagnetic repulsion between the lead mass and the bismuth.

548. **Baily's Experiments.**—Francis Baily's experiments were very extensive, and for a long time were regarded as the most reliable of all that had been published. A short account of these experiments and of the criticism directed against them in later years will be instructive to the physical student.

The torsion-case was attached to the ceiling of the room, was lined inside with metallic-foil, wrapped round outside with flannel, and finally fitted with an outside casing which was gilded to prevent, as far as possible, heating by radiation from outside. The attracting balls were spheres of lead 380 lbs. in weight, turned as nearly true

* *Versuche über die mittlere Dichtigkeit der Erde*, Freiberg, 1838, and later, *Neue Versuche*, &c., Pogg. Ann. 85 (1852), *Phil. Mag.*, 5 (1853).

† *Mem. Roy. Ast. Soc.* 14, (1842).

‡ *Comptes Rendus*, 76 (1873) and 86 (1878).

§ *Proc. R. S.*, 46 (1889), and *Nature*, vol. l. 1894.

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as possible, placed on a turn-table mounted on an upright pivot resting on the floor of the room. The turn-table was operated by ropes from one corner of the room, from which the observations of the positions of the torsion-rod were made by means of a telescope.

Baily used balls of different materials and different sizes and various forms of suspension, and periods varying from 200 to 1000 seconds. The distance of an attracting sphere and the opposite attracted ball was about 11 inches, and the centres of the attracting spheres were 80 inches apart.

In beginning a set of observations a sufficient oscillation was got up by turning the attracting balls first to one side, then to the other of the torsion-rod, and so on. When this was done the end of the swing was noted, and the attracting balls were moved round to the opposite position. Baily assumed, as did Reich, that this change might be regarded as instantaneously effected, so that the reading taken just before might be regarded as the first of a set of four which was completed by taking three consecutive elongations in the new position. These were then combined as before to give the centre of swing by calculating $\frac{1}{8}(a+3b+3c+d)$. Let this reading be denoted by r_1.

The attracting balls were then carried over to the opposite position, which was always such as to aid, by the attraction, the motion of the suspended system from the position marked by the reading d, and four turning positions taken as before, of which the first was the last on the side turned from. These gave a reading for the centre of swing on the other side of zero. Let this reading be r_2, and suppose the positions to have been such that $r_2 > r_1$. The attracting masses were then carried back to the first position, and the observations repeated, giving a reading r_3, say, for the centre. The deflection was then taken as $\frac{1}{2}\{r_2-(r_1+r_3)/2\}$.

The object of this procedure was to eliminate the effect of change of position of the centre of swing, which was almost continually going on, usually in the same direction. Supposing the shift to have been at a uniform rate, the correction was accurately made in this way.

The period of vibration was obtained by observing, during each set of observations of deflection, the instants of transit of two consecutive divisions of the scale which the first observations showed had the centre of swing between them, and from these calculating the period of the centre of swing. The time of vibration calculated by the formula $\frac{1}{4}(t_1+2t_2+t_3)$ for three times of vibration obtained from three successive sets of observations was taken as free from effect of drift of the centre.

The distances between the centre of each attracting sphere and an assumed central mark was measured by means of a microscope for each of the two positions of the attracting sphere, and the mean of these was taken as the distance between the centres of the attracting and attracted balls.

Several objections have been taken to Baily's mode of experimenting. In the first place, it was pointed out by Cornu and Baille

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that the assumption that the interval occupied in reversing the turn-table might be neglected led to serious error. Since the balls were quickly whirled round, just after the last elongation of a group of four had been read, into the position to aid by their attraction the next half swing, it was assumed that this reading, d, say, might be taken as the first of the group of four readings for the new position. But the attraction for the time occupied in putting the balls into the new position being wanting, the vibrating system passed through a neutral position corresponding to a smaller deflection than d. Thus d was taken so as to give too small a deflection from the turning-point to the following reading, and gave a middle position too close to the former position of the attracting ball, and this occurred on both sides. Thus the deflection came out a little too small in each case, and gave too large a value of the earth's mean density.

Baily observed, but could not explain, the fact that the mean density obtained diminished as the mass of the attracted balls was increased. This may have been due to the error just referred to. Increase of mass of the attracted balls would increase the moment of inertia of the suspended system. Thus, the period would be increased, and the loss of attraction during the time required to change the position of the attracting masses would be of less importance.

An examination of Baily's experiments by Cornu and Baille justified their suspicion that the retention of the last vibration of each set as the first of the next set led to serious error. Baily's final result was $5{\cdot}6747 \pm {\cdot}0038$. But ten of his results taken at random by Cornu and Baille give a density of 5·731, which was reduced by rejection of the first vibration to 5·615. The final result reduced in the same ratio would become 5·55.

There are other anomalies in Baily's results which indicate that there were disturbances which were not sufficiently allowed for or guarded against. It has been pointed out by Prof. W. M. Hicks that the results show that the mean density falls almost uniformly as the temperature at which the experiments were made rises. Also Baily seems not to have realised that to approach the apparatus so closely as to read the distance of the balls from the assumed middle position by means of a microscope would introduce serious temperature disturbances unless proper precautions were taken.

As has been pointed out in § 544 above, the masses of the balls to be taken should have been the masses determined by *weighing in air*. The weights were, however, corrected for the air displaced, an error of principle, though not one really affecting the results in any appreciable degree.

549. **Experiments of Cornu and Baille.**—This investigation, which does not seem to have been yet finally concluded, has been carried on by means of a torsion-balance designed to avoid disturbances to which apparatus on the Cavendish model were subject. The attracting masses are spheres of mercury weighing 12 kilogrammes, contained in cast-iron shells fixed in position. The change of position of the masses is effected by transference of the mercury

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by tubes, which completely prevents the vibrational disturbance attending the use of a turn-table, and contributes to the preservation of uniformity of temperature. The attracted balls are of copper, 109 grammes each in weight, and are carried by a light tube of aluminium only 50 centimetres long. The whole apparatus is thus very much reduced in size, an advantage both in regard to sensitiveness and to avoidance of temperature disturbances. Electric registration of the vibrations is employed to give a complete record of the movement with time marks on the same sheet.

In 1872, 1873 two mean results were obtained for ρ. viz. 5·56 and 5·50, of which the first was regarded as the more trustworthy, as a slight bending of the rod was discovered, which probably accounted for the lower value of the second set of experiments. In 1878 the balance had been improved, and behaved so well that the time vibration remained for a year almost exactly at 408 seconds. The mean result obtained with the improved appliances was again 5·56 for the mean density of the earth.

550. **Apparatus and Experiments of Boys.**—With regard to diminution of length of beam made by Cornu and Baille, it may be noticed that, if the other dimensions be kept the same, the moment of inertia is diminished in the square of the ratio of the diminution of length. Thus, if the length be diminished in the ratio of n to 1, the moment of inertia is altered in the ratio of n^2 to 1. If, then, the period be kept the same, we must have μ/I (§ 544) the same as before, that, is μ must be changed also in the ratio n^2 to 1. Thus for a given deflection θ the couple to be applied by the attraction is equal to the former amount $\times n^2$; but at the same distance as before, the couple is equal to the former couple $\times n$. The sensibility is, therefore, since $n < 1$, increased in the ratio of 1 to n. The beam may, therefore, be diminished in length, with accompanying diminution of μ and increase of sensibility, so long as the fibre remains strong enough to support the total suspended mass.

Increase of length of the suspension wire or fibre without diminution of cross-section, it is to be remembered, gives also a proportionate reduction in the value of μ; so that it is possible, by lengthening the fibre, to maintain its strength unaltered.

Continual diminution of length of beam has, it will be seen, the disadvantage of increasing the opposing couple due to the attraction of each large ball on the remote suspended ball, and this may become serious when the length is made very small.

If every dimension of the apparatus is diminished while the suspension wire is changed so that the period is kept constant, the attraction on the suspended ball becomes changed in the ratio n^6/n^2, or $n^4/1$. The couple is, therefore, changed in the ratio n^5 to 1. But the moment of inertia is altered in the ratio $n^3 \times n^2$, that is n^5 to 1. Thus for the same period the value of μ must be changed in the ratio n^5 to 1. The torsional rigidity is thus changed in the same ratio as the couple at the diminished distance apart of the balls, and, therefore, the same angular deflection of the beam will

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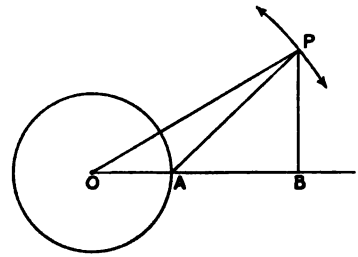
be produced with the masses in this position. The sensibility is, therefore, unaltered.

These results of alteration of dimensions were pointed out by Mr. C. V. Boys in the paper already referred to (*loc. cit.* § 547), and he applied them to the construction of an apparatus of very small dimensions, in which his important invention of quartz fibres of great thinness and strength is utilised to give a suspension fibre of sufficiently small torsional rigidity.

FIG. 286.



It was also shown by Boys that, by placing the centres of the attracting and attracted masses at positions A, P on a line AP inclined at a certain angle PAB, less than 90°, to the vertical plane through the centres of the attracted masses, the moment of the attracting forces on the balance (the opposing couple due to the distant masses being neglected) can be made a maximum for a given distance c of the centre of the attracting mass from the axis O and a fixed position of A. That the maximum exists can be seen without calculation, and the reader may easily find the exact position. If OP, OA, AB (Fig. 286), be denoted by c, a, x, and M, m denote the attracting and attracted masses, the attraction, which is along AP, is

$$KMm/\{x^2+c^2-(a+x)^2\},$$

the moment of this about O is this expression multiplied by $c\sin OPA$, that is by

$$a\sqrt{c^2-(a+x)^2}/\sqrt{x^2+c^2-(a+x)^2},$$

and therefore the square of the moment is proportional to

$$\{c^2-(a+x)^2\}/\{x^2+c^2-(a+x)^2\}^3.$$

This can be shown to have a maximum for constant a and c when

$$ax^2+(3a^2+c^2)x-2a(c^2-a^2)=0$$

or if θ denote $\angle POB$, when

$$ca\cos^2\theta+(c^2+a^2)\cos\theta-3ca=0.$$

To obviate the opposing couple due to the distant masses, Boys proposed to place the attracted masses on different levels, and therefore in the apparatus constructed by him and described in his first paper (*loc. cit.* § 547) he adopted the form of torsion-balance indicated in Fig. 287.

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Instead of two small balls, two small cylinders of lead only 11·3 millimetres long and 3 millimetres in diameter, fixed by light arms of brass on the two sides of a central tapering stem of glass tube, are used as attracted masses. The centres are on different levels 50·8 millimetres apart, and the distance apart of their axis is 13 millimetres. Attached to the upper end of the glass stem is a quartz fibre secured at the upper end to a torsion-head. A mirror carried by the glass tube enables the deflections of the balance to be read in the ordinary way by a reflected ray from a lamp, thrown on a scale about 16½ metres from the mirror. A plate-glass window in the tube protecting the torsion-balance allowed passage of the light.

FIG. 287.

The balance is thus so small that it is wholly enclosed in a narrow brass tube (not shown in Fig. 287), which supports the torsion head, and serves to protect the apparatus from currents of air, and to preserve its temperature constant.

The attracting masses are two cylinders of lead 50·8 millimetres in diameter and length, the centres of which are at the levels of the small cylinders. These are fastened by screws to the inside of an outer brass cylinder coaxial with the suspended system, with their axes in a diametral plane of the tube. The cylinder is supported on a levelled base plate (which also carries the tube supporting the torsion-balance), and can be turned round its own axis through any angle so as to bring the attracting masses to act on the small cylinders in the positions to exert the maximum deflecting couple in either of the two directions about the axis of the balance.

Mr. Boys states (*loc. cit.*) that the result of his preliminary experiments with this apparatus has been "complete and satisfactory. As a lecture experiment the attraction between small masses can be easily and certainly shown, even though the resolved force causing motion is, as in the present instance, no more than the one two-hundred-thousandth of a dyne (less than one ten-millionth of the weight of a grain), and this is possible with the comparatively short period of 80 seconds." He goes on to say: "It is my intention, if I can obtain a proper place in which to make the observation, to prepare an apparatus specially suitable for absolute determinations. The scale will have to be increased, so that the dimensions may be determined to a ten-thousandth part at least. Both pairs of masses should, I think, be suspended by fibres or by wires, so that the distance of their centres from the axis may be accurately measured, and so that in the case of the little masses the moment of inertia of the beam, mirror, &c., may be found by alternately measuring the period with and without the masses attracted. The unbalanced

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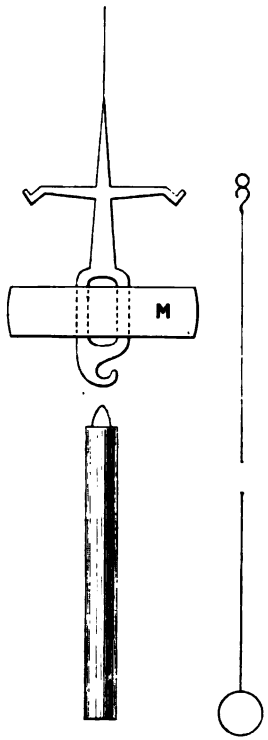
attractions between the beam, &c., and the large masses, and between the little masses and anything unsymmetrical about the support of the large masses, will probably be more accurately determined experimentally by observing the deflections when the large and small masses are in turn removed, than by calculation."

FIG. 288.



Mr. Boys has carried out the investigation, with such an apparatus as that here indicated, in the Clarendon Laboratory at Oxford. Experiments were made with two pairs of attracting balls of lead, one pair $4\frac{1}{4}$ inches in diameter, the other pair $2\frac{1}{4}$ inches in diameter. For the attracted masses there were used in different experiments a pair of gold balls ·25 inch in diameter, another pair ·2 inch in diameter, and a pair of gold cylinders ·25 inch in diameter and length. All these masses were constructed with very great care to avoid want of homogeneity: they were made to solidify upward to prevent the formation of cavities, the lead balls were subjected after solidification to hydraulic pressure, and the gold balls were beaten in steel dies. The lead balls were hung by phosphor-bronze wires from supports standing on a lid above the case, and were counterpoised by weights so that the lid with the attracting masses could be carried round from one attracting position to the other, by a mechanism arranged for the purpose, with but little friction.

The attracted masses were hung by quartz fibres ·2 inch apart, attached by hooks to the ends of the arms of a cross of gilded copper. This cross, with its attached mirror M for enabling the deflection to be measured by a reflected ray of light, formed the principal part of the balance, and is shown in the adjoining diagram (Fig. 288). The mirror was the strip left after two equal segments had been cut off from the two sides of a circular mirror of perfect construction, and had across its back two microscopic vertical grooves which received the fibres hanging from the arm above.

To a hook below the centre of the mirror a cylinder of silver of weight just equal to the combined weight of two gold balls could be attached. This enabled the observations of the period of oscillation of the apparatus to be made with these balls, and with the cylinder without change of stretching force, and therefore the unknown moment of inertia of the cross and attached mirror to be eliminated.

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The details of the apparatus and of the mode of observing are very important and are full of instruction to the experimentalist, and the reader should carefully study these in Mr. Boys' paper, Phil. Trans. R.S. 1895, and *Nature*, vol. l. 1894. Though the forces to be measured were extremely small, only one five-millionth of the gravity of a grain in some of the experiments, there can be no doubt whatever of the great accuracy of the work. The values obtained were $k = 6{\cdot}6576 \times 10^{-8}$ in C.G.S. units, and mean density of the earth $= 5{\cdot}527$ times that of water.

Another very careful determination lately completed by the torsion-balance method deserves notice.* To reduce the disturbance from air currents as nearly as possible to zero, Dr. C. Braun, at Mariaschein, in Bohemia, has designed, and to a considerable extent made with his own hands, a torsion-balance enclosed in a tolerably high vacuum (about $\frac{1}{200}$ of an atmosphere). His apparatus is much larger than that of Boys, the torsion-beam being 9 inches long, the attracting masses each 5 kilogrammes or 9 kilogrammes, and the attracted masses 54 grammes. The suspension wire was of metal, but a repetition of the experiments is in progress with this replaced by a fibre of quartz. The result obtained for Δ is $5{\cdot}527 \pm {\cdot}0014$, almost exactly that of Boys.

551. **Determinations by the Ordinary Balance.** — Instead of a torsion-balance an ordinary chemical balance may be used to measure the pulls exerted by the attracting masses. Imagine a small ball hung from one end of the beam of a delicate balance, and equilibrated by weights in a scale-pan on the other end of the beam, then a large ball to be brought under it so that the centres of the two balls shall be at a known vertical distance. The balance will be disturbed, and a weight w will have to be added to the weights in the scale-pan to restore equilibrium. Let the masses of the large and small balls be M, m respectively, and the distance between their centres be r. The pull between the balls is kMm/r^2, and this is equal to wg. Hence we have

$$k = \frac{wg}{Mm} r^2,$$

and for the earth's mean density the value obtained by substituting this value of k in (3). Or, neglecting the corrections used in (3), and putting R for the distance of the earth's centre from the attracted ball, and E for the mass of the earth, we obtain $Er^2/MR^2 = m/w$, so that

$$E = \frac{m}{w} \frac{R^2}{r^2} M.$$

This method has been employed by von Jolly† and by Professors

* *Denksch. d. Math. Cl. d. Akad. d. Wiss. Wien.* 64, 1896, also *Naturwiss. Rundschau*, 1897.

† *Wied. Ann.* xiv. 1881.

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König, Richarz, and Krigar-Menzel* in Germany, and in an elaborate and important research in this country by Professor Poynting. It is not possible to more than mention here the first of these investigations: but the reader will find a critical account of it in the treatise of Professor Poynting already referred to.

The research of Professors Richarz and Krigar-Menzel (for Professor König was connected with it only in the preliminary stages) is of considerable importance and value from the systematic care with which every part of it was carried out. The experiments were carried out in an earth-protected casemate of the fortress of Spandau, near Berlin, where it was certain that there was little daily variation of temperature and great freedom from disturbance due to railways or vehicles. In brief, the method was this: From the ends of the beam of a sensitive balance were hung two wires which carried each two scale-pans about 2 metres apart, but so that each pair of scale-pans (composed of one on each wire), the upper pair and the lower, were on one level. A kilogramme weight was then placed in the upper scale-pan on one wire, and another kilogramme in the lower scale-pan on the other wire, and the position of the beam observed. Then each kilogramme was carried over to the adjoining scale on the same level, and the position of the beam again observed. The deflection gave clearly four times the difference of weights due to the difference of levels of the masses in the pans.

Then a great pile of lead, about 2 metres cube, was built up of lead bars, so that the wires attached to the balance passed along vertical perforations in the pile, and the experiments described above were repeated. In consequence of the pile the weight in the lower scale-pan now experienced, in addition to the former forces, a pull upwards, and the weight in the upper pan a pull downwards. With an allowance for the difference of weights already observed, the deflection gave four times the pull applied to each weight by the lead pile, and from this the constant of gravitation and the earth's mean density could be obtained. The result was $k = (6{\cdot}685 \pm {\cdot}011)10^{-8}$ in C.G.S. units, and $\Delta = 5{\cdot}505 \pm {\cdot}009$.

552. **Poynting's Experiments.**—In Professor Poynting's experiment the balance, which was a large bullion balance with a four-foot beam, was employed with a turn-table by which a large attracting ball could be placed first underneath one of two equal balls suspended at the same level from the extremities of the beam of a sensitive balance, then underneath the other ball. To counterpoise the turn-table, a mass half as great as the large attracting one was placed at double the distance from the axis on the other side, and its attraction on the suspended weights was allowed for. To eliminate the pull on the beam, &c., the experiments were made for two different levels of the suspended masses.

The balance deflections were magnified by means of a mirror hung by a bifilar, one thread of which was attached to a fixed point,

* *Wied. Ann.* xxiv. 1885; *Nature*, xxxi. (1884–85).

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the other to the end of a small bracket carried by the pointer of the balance. Thus, when the balance was deflected, the upper end of one thread was moved with the pointer, and the mirror was carried round into a new plane. The sensibility of this arrangement is obviously inversely proportional to the distance between the threads, and therefore can be made very great. Vanes carried below the mirror, and, immersed in oil in a dash-pot, prevented undue oscillation. The deflections were read by a telescope from a room above that in which the balance was placed.

For the details of measurement, the various precautions observed, and the reduction of the observations, the reader must refer to Professor Poynting's essay. His final results are

$$k = 6{\cdot}6984 \times 10^{-8}, \ \Delta = 5{\cdot}4934.$$

553. **Pendulum Methods.**—Newton was, at one time at least, of opinion that the attraction between terrestrial bodies was too small to be measured; indeed, he expressed the view that a hemispherical mountain, 3 miles in radius, would not draw the plumb-line two minutes of angle out of the vertical (*De Mundi Systemate*, Horsley's ed. vol. iii. § 22). This estimate was erroneous, but it shows that the possibility of measuring the attraction between ordinary bodies had occurred to Newton. The methods of determining the mean density of the earth by pendulum deflections produced by a mountain, or by comparison of the periods of a pendulum at the top and bottom of the shaft of a deep mine, though interesting in many respects, are not capable of giving results comparable in point of accuracy with those obtained by either form of balance. In the first case, difficulty of exactly contouring a mountain to find its volume and figure, and the uncertainty as to its mass that must always remain, however well the geological formation and disposition of strata have been made out, and in the second, the uncertainty as the arrangement of the strata in the more distant neighbourhood of the shaft, prevent these methods from being likely to yield results as trustworthy as those to be obtained by direct experiment with masses of accurately known constitution.

An estimate, however, of the value of Δ, which, though rough, showed clearly that Δ was far greater than the density of the crust of the earth, was made 160 years ago by Bouguer, who made observations in Peru, by vibrating a pendulum (1) near the sea-level; (2) at Quito on a tableland about 9375 feet above sea-level, and on the summit of a mountain about 6000 feet higher.

554. **Alterations of Level of Mercury Surface produced by Tidal Waters.**—Boscovich and after him Cavendish* conceived the idea of observing the deflection of the plumb-line produced by tidal rise of the water in an arm of the sea, and it was suggested by Robison in his *Mechanical Philosophy*, 1804, that the tides in the Bay of Fundy might produce a sensible deflection of a plummet in the

* Wilson's *Life of Cavendish*, p. 473.

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neighbourhood, and so enable the earth's mean density to be determined.* In some ways it certainly seems more promising to endeavour to observe the alterations of level of a mercury surface opposite a long, nearly straight channel which is filled and emptied to a considerable depth by the tide, than to make observations on the attraction of a mountain, the mass of which can never be obtained with any accuracy from surface specimens. The mass of water could be determined with an accuracy which could not be approached in the case of the mountain or the mine; but it seems probable that there would be great difficulties due to a warping of the ground, which seismometric observations show goes on to such a surprising extent.

Thus, M. D'Abbadie found on the shore of the Bay of Biscay, by comparing a fixed mark with its image formed by reflection from the surface of mercury, that a deflection of the surface in one direction was produced by rise of the tide and a reverse deflection by its fall. In about two-thirds of an extended series of observations the surface was inclined, towards the sea at high water and from it at low water, as compared with its position when the sea was at the mean level; in a certain proportion the level showed no change, and in an equal proportion the change was opposite to that expected in Peru.

Lord Kelvin has made an estimate of the effect of a horizontal slab of water 50 miles long, 50 miles broad, and 10 feet deep, on a pendulum suspended opposite the middle of one of the edges of the slab at a distance of 300 feet, and found that the deflection would be about $\frac{1}{20}$ of a second of angle. The reader may verify this by the formula indicated in § 461 above. This slab of water represents roughly a tidal change which takes place at St. Alban's Head. It is clear that, for the observation of such a small deflection as this, some optical method, or perhaps the bifilar mirror arrangement referred to above, would have to be adopted.

555. **The Schiehallien Experiment.**—The experiment on the mountain Schiehallien, which was begun in 1774 by Maskelyne, deserves notice. The mountain is a ridge in Perthshire ranging nearly east and west, and consists of strata which are nearly vertical. The apparent difference of latitude between two stations on the same meridian (but not on the same level), one on the north, the other on the south side of the mountain, was determined by astronomical observations, in which, of course, the apparent levels at the two places were made use of. This was compared with the true difference of latitude, obtained from the positions of the stations. The difference of latitude, 49·94″, was found to be increased by 11·6″ by the attraction of the mountain.

A careful contour map of the mountain was afterwards made by Hutton, who estimated the distribution of attracting matter, and so deduced from Maskelyne's observations the value of the earth's mean density. His first estimate was 4·5, but the results of a survey made for him later by Playfair caused him to adopt the value 4·95.

* Thomson and Tait, *Nat. Phil.* Part ii. § 818.

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556. **The Harton Pit Experiment.**—The most notable of the pendulum experiments that have been made is, no doubt, that which Airy carried out at the Harton coal-pit in Durhamshire, and which gave finally the value 6·565. The elementary theory of this method may be here noticed.

Let R be the radius of the earth considered as a sphere without rotation and of symmetrical distribution of matter (mean density ρ) round the centre. Its mass is $\frac{4}{3}\pi\rho R^3$, and its attraction on a particle at the surface is $\frac{4}{3}\pi k\rho R$. The attraction of the earth on a particle at depth h below the surface is, if ρ' be the density of the spherical shell of thickness h, $4\pi k(\frac{1}{3}\rho R^3 - \rho' R^2 h)/(R-h)^2$. Hence, if g be the value of gravity at the surface, and g' the value at the depth h, we have

$$\frac{g'}{g} = \frac{\rho R^2 - 3\rho' Rh}{\rho(R-h)^2} = 1 + 2\frac{h}{R} - 3\frac{\rho'}{\rho}\frac{h}{R},$$

since h is small compared with R.

This theory is not strictly applicable, inasmuch as the shell of thickness h has not necessarily the average density which the strata have at the place of experiment. The formula of computation used is, however, that just obtained with this latter density as the value of ρ'.

For the details of this experiment as well as of many meritorious pendulum and survey investigations, which are not mentioned here, the reader may refer to Professor Poynting's essay, which also contains a valuable bibliography of work on the subject of the earth's mean density.

557. **Question of Attraction of Crystals and Gravitational Permeability.**—Some other interesting questions regarding gravitational attraction have lately been investigated. The fact that no outstanding discrepance between weighings, or results depending on weighings made under different circumstances, had been noticed shows that probably crystals experience the same attracting force in whatever way their axes are oriented with respect to the earth, and that the attraction between two bodies is not appreciably affected by the interposition of other matter between them. It is well known that when a small ball of quartz is placed in an electric field, and is moved by electric force, the action depends on the direction of the axis of the crystals in the field. Further, if between two oppositely electrified conducting balls in air a slab, say of glass, is interposed, the attraction between them is increased; the lines of force in the field, which, when the medium is unlimited and uniform, are distributed in a certain way round the balls, assume a new distribution when the glass is introduced, are crowded, in fact, more into the space between the balls, and there is an enhanced attraction of one on the other. The facts and theories of this action will be explained under Electricity; but nothing of a similar kind has been observed in a gravitational field of force.

Experiments on the pull between a lead ball and a sphere of calc-spar placed with its axis in different directions relatively to the lead

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ball were made in 1895 by Dr. Mackenzie (*Phys. Rev.* 2, 1895) with a torsion balance, but without any positive results. Professor Poynting and Mr. P. L. Gray have also experimented on the attraction between two quartz spheres, not directly on the pull between them in different positions, but to discover, what it seems probable should exist if the attraction between the spheres with axes parallel is not the same as when the axes are crossed, a couple resisting the turning from a position of greater force to that of smaller. For the position of greater force is that of smaller potential energy of the attraction. The experiments are described in the *Trans. R. S.* 192, A, 1899, and in *Nature*, August 23, 1900. The method employed was that of bringing to bear on a torsion pendulum what, if the effect existed, would be a succession of impulses in the free period of the pendulum, and therefore producing by accumulation of effects an oscillation of the pendulum through a sensible range.

It may be remarked here that a theory of gravitation was put forward many years ago by Le Sage of Geneva to the effect that throughout space there are flying equally in all directions innumerable small particles, and that ordinary matter is in a high degree, but not completely, permeable to these particles. A small proportion of these particles falling on any body is stopped, and this proportion must depend on the structure of the body. A single body in space would experience no force in any direction, but two bodies, as the Sun and Jupiter, would, from the shielding action of one on the other, experience force along the line joining their centres, and if the balance of particles stopped is sufficiently small, the force would be inversely proportional to the square of the distance between the bodies, and directly proportional to the product of the masses.

But, according to this theory, the weight of a crystal should depend on its orientation relatively to the earth, and this, though previously shown by experience to be probably not the case, is negatived by Messrs. Poynting and Gray's experiments.

Messrs. Austin and Thwing have described, in the *Physical Review*, 5, 1897, experiments on the permeability of different kinds of matter by lines of gravitational force. Screens of lead, zinc, mercury, alcohol, and glycerine were tried between the attracted balls of a torsion balance and the attracting masses without appreciably affecting the deflections.

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CHAPTER XIV.

THE TIDES.

558. **Equilibrium Theory of the Tides.**—The earth itself is not unyielding, and the waters of the oceans and seas upon it are mobile. The gravitational attraction of the sun and moon on equal portions of matter at different parts of the earth's surface differ in magnitude and direction. Hence there are produced distortions of the solid earth, and the much more marked alterations of sea-level and tidal currents which we are in the habit of referring to as the tides. We cannot here attempt to traverse the very extensive field of tidal theory and phenomena, but it is desirable to show how tidal forces arise, and to calculate their amounts, and to give some slight idea of their effects. It will be possible for the reader thus to gain some idea of the so-called equilibrium theory of the tides, and the manner in which the conclusions of tidal theory are modified by the rotation of the earth and the inertia and friction of the moving matter.

If the rotation of the earth were stopped and the sun, earth, and moon were held fixed in the positions they occupy at any moment, a configuration of the bodies would supervene, in which the disturbing stresses would be equilibrated by forces set up by changes of relative position of the matter composing the bodies. Considering only the change of configuration of the waters on the earth's surface, an equilibrium tide would be produced which would be maintained unchanged so long as the relative positions of the three bodies were not altered.

Suppose, now, the earth to turn about its axis, and the moon and sun to take the successive positions relative to the earth which they take in actual fact, but to do so so slowly that the state of static equilibrium proper to rest in each configuration is set up; the tides will be of the amounts at each instant contemplated in the equilibrium theory. This theory, however, although it gives much information regarding the tides as they actually occur, has its results profoundly modified by the rapidity of the earth's rotation, and the so-called dynamical (or, more properly, kinetic) theory of the tides becomes necessary for the explanation of the phenomena.

We shall calculate the tidal forces due to the action of the moon and sun, and then consider the equilibrium form of the surface of the water on the earth. We shall suppose, for simplicity, first that the whole earth is covered with water, and then endeavour to explain

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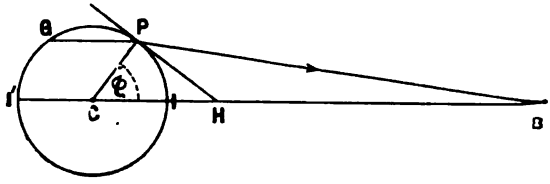
how the conclusions obtained on this supposition must be modified take account of the existence of continents and islands and their irations.

. Calculation of Tide-producing Forces. Horizontal and l Components.—Let C, Fig. 289, represent the centre of h, I, P, I' points on its surface, B the position of the moon's

FIG. 289.



centre, and, in accordance with a notation usual in this subject, denote the distance CB by D and the mass of the moon by M. The pull exerted by the moon's mass on a particle of unit mass at C is kM/D^2. If the force exerted by the moon on every unit particle in the earth were the same as this in magnitude and direction, every particle would have an acceleration of precisely this amount, and there would be no distortion of the earth, or change of relative position of the mobile matter on its surface. Further, it is clear that, if the earth were made up of a congeries of particles of equal mass, and there were applied to each of these particles any force whatever, P, the same in magnitude and direction for every particle, no change of relative position of the parts of the earth would ensue. The result would only be to superimpose on any formerly existing accelerations exactly the same acceleration of all the parts. We therefore suppose every particle acted on by a force of amount per unit mass kM/D^2, in the direction opposed to CB. This annuls the acceleration of the particles at or near C; and therefore the resultant of this force and the attraction of the moon is the force producing change of the relative configuration.

The force on a unit particle at P exerted by the moon acts along PB, and is of amount $kM/BP^2 = kM/(D^2 + a^2 - 2Da\cos\phi)$ if a be the distance CP. Besides this we have to suppose a force kM/D^2 to act along PG parallel to BC. The resultant has a horizontal component, that is, a component along PH, where H is the intersection of BC with a tangent at P to the circle of intersection of the plane BPC with the earth, and a vertical component, that is, a component along the radius CP. We can evaluate these components by resolving in these two directions the two forces kM/BP^2, kM/D^2, which have just been specified as acting at P. The component of kM/BP^2 along PH is $kM\cos BPH/BP^2$. But since

$$\angle HPC = \pi/2, \cos BPH = \sin BPC = D\sin\phi/BP.$$

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Hence, $kM\cos BPH/BP^2 = kMD\sin\phi/(D^2+a^2-2Da\cos\phi)^{\frac{3}{2}}$. Also the component of the other force along PH is $-kM\sin\phi/D^2$. Hence the total horizontal force from P to H is

$$\begin{aligned} kM\{D/(D^2+a^2-2Da\cos\phi)^{\frac{3}{2}} - 1/D^2\}\sin\phi \\ = kM\sin\phi\{1/(1+a^2/D^2-2a/D.\cos\phi)^{\frac{3}{2}} - 1\}/D^2 \\ = 3kMa\sin\phi\cos\phi/D^3, \end{aligned}$$

since a is small in comparison with D. Calling this F, we have the equation of horizontal tide-producing force

$$F = 3kM\frac{a}{D^3}\sin\phi\cos\phi. \tag{1}$$

The vertical component, R, say, taken in the direction from C to P, is $kM\{-\cos BPC/BP^2 - \cos\phi/D^2\}$. But $-\cos BPC = \sin BPH$, and $\sin BPH/\sin CHP = \sin BPH/\cos\phi = BH/BP = (D - a/\cos\phi)/BP$, so that $\sin BPH = (D\cos\phi - a)/BP$. Also approximately

$$1/BP^3 = (1 + 3a\cos\phi/D)/D^3.$$

Hence

$$R = kM\left\{\frac{D\cos\phi - a}{BP^3} - \frac{\cos\phi}{D^2}\right\} = \frac{kMa}{D^3}(3\cos^2\phi - 1). \tag{2}$$

560. **Variation of Tidal Forces over Earth's Surface.**—The horizontal force F has thus the same value for every point on any one circle surrounding the line II' as axis. Also it is directed at each point of these circles along the great circle passing through that point and II', as in Fig. 290, being of equal (according to the degree of approximation here adopted) numerical amount, but of opposite sign for supplementary values of ϕ. As a matter of fact, the tidal forces under the moon at I are about 5 per cent. greater than those in the opposite direction at the diametrically remote point I', since the ratio of the difference to either force is $3a/r$ very nearly. The force has zero values at II', at the circle which contains C, and its maximum value when $\sin\phi\cos\phi$ (that is, when $\sin 2\phi$) is a maximum. But $\sin 2\phi$ has its greatest value when $2\phi = \frac{1}{2}\pi$, so that F is a maximum when $\phi = \pi/4$. The maximum value is $\frac{3}{2}kMa/D^3$. The lines AB, $A'B'$ in Fig. 290 represent circles drawn round the earth at right angles to the line II', and so placed that a radius drawn from the centre C to any point in AB makes an angle $\pi/4$ with CI, and a radius drawn from C to any point in $A'B'$ makes the same angle with CI'. The horizontal force is greatest on these circles, and is directed at each point, as shown by the arrow-heads, along the great circles passing through II'. The line DE represents a circle dividing the spherical surface into two halves. The horizontal force is zero at all points on DE. At all points on the right of DE the force is towards I; at all points on the left of DE the force is towards I'.

The vertical force R is outwards along CP for values of ϕ for

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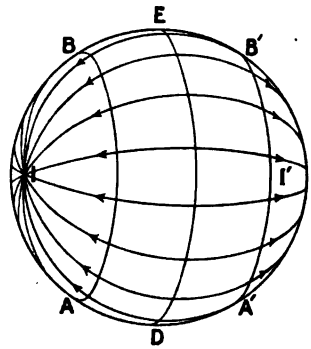
which $3\cos^2\phi > 1$, and in the direction PC when $3\cos^2\phi < 1$. The first inequality holds for values of ϕ between $0°$ and $54°44'8'$ and between $125°\ 15'\ 52''$ and $180°$; and the second for values of ϕ between $54°44'8''$ and $125°15'52''$. Thus R is outwards at both I and I', and at each of these points has its maximum value $2kMa/D^3$. The maximum inward value of R exists at all points for which $\phi = \pi/2$, and it is half the maximum positive value. The maximum value of F is thus three-fourths of that of R.

FIG. 290.



We shall see that the elevation of the water above a certain chosen mean level is equal to half that fraction of the earth's radius which the vertical tidal force is of gravity.

561. **Moon and Anti-Moon. Lunar Semi-diurnal Tides.**—If the mass of the moon were diminished from M to $\frac{1}{2}M$ the forces would all be halved, and, from the statements just made and illustrated by Fig. 290, it will be seen that the full values of the tidal forces would be restored, if another mass $\frac{1}{2}M$ were placed at a point B' on the line BC, such that $BC = CB'$. The tide-producing forces are thus what they would be if, instead of the moon at B, there existed a "moon and anti moon," each of mass $\frac{1}{2}M$ at B and B'. Thus we have the result often deemed paradoxical by persons who have not sufficiently considered the matter, that high water is produced under the moon and on the side of the earth remote from the moon, at the same instant, and that at any one place two high waters occur in each lunar day, that is, in the time (24 hours 50 minutes nearly) of one rotation of the earth relatively to the moon. These are called lunar *semi-diurnal* tides. It is a matter of common observation that these two tides, speaking generally, are not of the same height; there is, in fact, a difference between them which can be accounted for by supposing a *diurnal* tide, that is, a tide of period one lunar day superimposed on the semi-diurnal tides.

562. **Solar Tides. Spring Tides and Neap-Tides.**—Precisely similar effects are produced by the sun's attraction, and the sun may, in the way just explained for the moon, be replaced by sun and anti-sun. The reader will now perceive in a general manner how the height of the tides varies throughout the month. There are two solar semi-diurnal tides in 24 hours, and thus in time of occurrence the solar tides gradually overtake and get ahead of the lunar tides, which occur twice in 24 hours 50 minutes. At new moon, when the sun and moon are in the same part of the heavens as M and S, Fig. 291, and at full moon, when they are in opposite parts, as M and $A.S$, the solar and lunar tides conspire, and we have what

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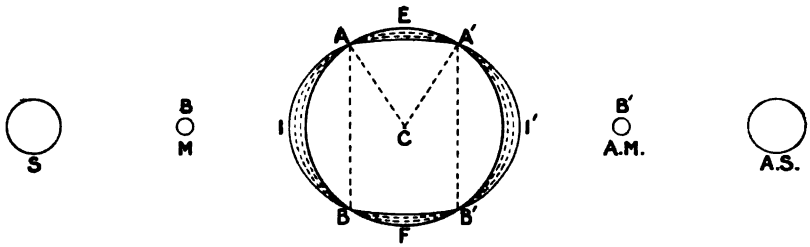
are called *spring-tides* occurring twice in a lunar month. Between every successive pair of spring-tides there is a time when the sun and moon are in *quadratures*, as it is called, that is, are (as seen from the earth) at an angular distance of 90°. Then the solar low water coincides with the lunar high water, and as the latter is greater the solar tide is subtracted from the lunar. Thus we have twice a month

FIG. 291.



what are called *neap-tides*, which are of a comparatively small range from low to high water.

Fig. 291, which as regards dimensions and distance of the bodies is not drawn to scale, illustrates these effects, of course with enormous exaggeration. The dotted spheroid close to the spherical surface BAE shows the form of the surface under the solar tide, regarded as produced by S and $A.S.$, the other dotted spheroid is the surface under the lunar tide (rather more than twice the solar tide) which may be regarded as produced by M and $A.M.$ The spheroid represented by the full-line $IAA'I'$ shows the surface under the two tides existing simultaneously. As will be seen presently, the surface of elevation, which consists of the two caps AIB, $A'I'B'$, is separated from that of depression by two circles (represented by AB, $A'B'$) at right angles to II', and so situated that the angles ICA, $I'C'A'$ are each equal to 54° 44′ nearly. The two tides thus conspire, and the elevations and depressions at any place are simply added together.

The diagram may be modified for the case of neap-tides by supposing the line joining the moon and anti-moon turned through a right angle about C, in the plane of the diagram, and to carry with it the lunar tide. The resultant tide will then be found by adding the tides where they conspire, and taking the balance of elevation or depression where they are opposite. The result in each case will be the deviation from the mean level.

563. **Magnitude of Tide-producing Forces.**—It is interesting to compare the magnitudes of the tide-producing forces of the sun and moon with forces of familiar amounts. The attraction of the earth on a body of unit mass at its surface is kE/a^2 where E is the earth's mass. The ratio to this of the maximum horizontal tide-producing force is $\frac{3}{2}Ma^3/ED^3$. Now M/E is about 1/83 and a/D about 1/60.

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Hence the ratio is roughly 1/12,000,000. The total force on a portion of matter is thus extremely minute compared with the gravity of the matter. The effect of the vertical tidal force when positive is to diminish gravity and when negative to increase gravity. The maximum positive value of the vertical force being $\frac{4}{3}$ of the maximum horizontal force, the greatest amount by which gravity is diminished by the tidal force is only 1/9,000,000 of its actual value, and the greatest increase of gravity from the same cause is 1/18,000,000.

The mass of the sun is about 332,000 times the earth's mass, or nearly 27,000,000 the mass of the moon, and the sun's distance is about 386 times the moon's distance. Hence, as the cube of 386 is $57{\cdot}5 \times 10^6$, the solar tide-producing force is rather less than half that due to the moon.

564. **Height of the Equilibrium Tide.**—In finding the height of the water above the mean sea-level we shall suppose the undisturbed surface to be a sphere of a certain radius a to be specified presently, and consider only the deviation from sphericity produced by the purely tidal forces acting on a stratum of water covering a solid unyielding nucleus. Of course there is a deviation from sphericity produced by the rotation of the earth about its axis, but this is a deviation which remains the same at any one place, and the ordinary tidal rise and fall will be approximately obtained without taking it into account. The process applied will proceed on the assumption that the chosen mean radius a is small in comparison with the distance, D, say, between the centre of the earth and the centre of the moon, so that we may neglect a^3/D^3, &c., in comparison with a/D or a^2/D^2.

Also, in the approximative process, it is to be noticed that the change in the field of force due to the displacement of the waters themselves is neglected, that is, the potential at any point of the surface, due to the matter of the earth, is taken to be the same as if the whole quantity of matter were collected at the centre C. In a strict discussion the potential of the surface stratum in its new configuration would have to be calculated.*

Supposing, then, the tide-producing forces to be due to moon and anti-moon (each of mass $\frac{1}{2}M$) at opposite sides of the earth, as explained in § 561, let V be the potential at any point on the surface of the earth, the distance of which from C is r, and angular distance from the line joining the centres of the earth and moon is ϕ, and let E denote the mass of the earth. We have

$$-V = kE/r + \tfrac{1}{2}kM\{1/(D^2+r^2-2Dr\cos\phi)^{\frac{1}{2}} + 1/(D^2+r^2+2Dr\cos\phi)^{\frac{1}{2}}\}$$

Expanding and retaining terms of the order r^2/D^2, since those of order r/D disappear, we have for the equation of the equipotential surface with which the surface of the water coincides

* See Thomson and Tait, *Nat. Phil.* vol. i. part ii., § 815.

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$$\frac{E}{r}+\frac{M}{D}\left\{1+\tfrac{1}{2}\frac{r^2}{D^2}(3\cos^2\phi-1)\right\}=\text{const.} \tag{3}$$

If now we denote by a the value of r for which $\cos^2\phi=1/3$, the constant value of the left-hand side is $E/a+M/D$. Taking, then, a as the distance of the surface of mean level from the centre, we obtain

$$E\left(\frac{1}{r}-\frac{1}{a}\right)+\tfrac{1}{2}\frac{Mr^2}{D^3}(3\cos^2\phi-1)=0.$$

If we write $r=a(1+u)$, u will be numerically small compared with unity, and therefore, neglecting the terms in u^2, we obtain instead of the last equation

$$u=\tfrac{1}{2}\frac{Ma^3}{ED^3}(3\cos^2\phi-1). \tag{4}$$

It will be observed that the quantity on the right is by (2) half the ratio of the vertical tidal-force to the value of gravity. The height above the mean level defined above is thus au. The value of u vanishes on the circles defined by $\phi=\cos^{-1}\sqrt{1/3}$ and already referred to in § 560. The maximum rise is at I and I' and the lowest fall at points on the circle CD. In the former case $au=Ma^4/ED^3$ and in the latter $\frac{1}{2}Ma^4/ED^3$. The total range from highest to lowest is thus $\frac{3}{2}Ma^4/ED^3$.

565. **Succession of Changes of Level of Sea in Lunar Day.**—In Figs. 291, 292, I represents the point in which a line joining the centres of the earth and the tide-producing body, the moon, let us say, intersects the earth's surface. The same line, produced backwards from the centre, intersects the surface in a second point I'. I, I' are the points of maximum high water. Circles drawn round the earth, which have this line as axis, are circles of equal tidal height above the sea-level. Thus, for example, AB, $A'B'$ are circles of zero change of level, CD is the circle of greatest depression. At all points on the surface between I and the circle AB, and between the circle $A'B'$ and I', the water is above, and at all points between AB and $A'B'$ it is below, the mean level.

Now consider the earth turning about its axis through N, while the tidal distortion of the surface is held fixed relatively to the moon. An observer at a fixed point on the earth's surface is carried round in space along the small circle E, G, H. When he has come to E the moon is on his meridian, and he observes that the tidal height is then a maximum at his station. As he moves on towards F he finds the tidal elevation diminishing, until at F he finds it zero. As he passes from F he observes a depression setting in which reaches a maximum when he has arrived at the position G. That depression diminishes again as his station moves on in space towards H, where the water is found to have resumed the mean sea-level. As the observer is carried on with the earth from H, the sea-level

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rises and reaches a maximum when he arrives at the meridian of I, I', at a point K (not visible in the figure) between N and I'. As the observer passes from K the tidal elevation diminishes and reaches zero when he again crosses the circle $A'B'$. Then a depression again sets in and continues until the circle AB is reached.

The maximum at K, it is to be noticed, is less than the maximum at G, inasmuch as the point K is farther from I' than G is from I. This difference is due to the diurnal tide represented in the second line of (6).

566. **Tidal Rise and Fall in Terms of Co-ordinates of Moon and of Place of Observation.**—It remains to express this value of the

FIG. 292.

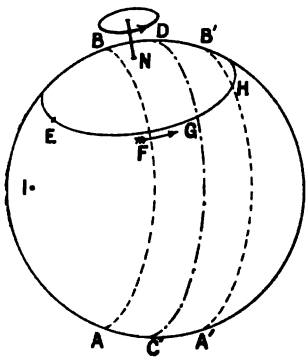
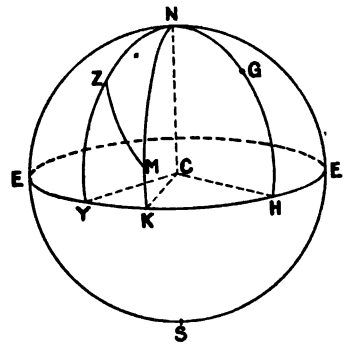


FIG. 293.



tidal rise or tidal above mean level in terms of the co-ordinates of the place of observation and those of the moon. The position of the place on the earth's surface is specified by its latitude and longitude, and that of the moon by the angle through which the earth has turned about its axis since the moon was on the meridian of Greenwich, or the moon's hour-angle, as it is called, and the angle which the line drawn from C to the moon makes with the plane of the equator, that is, the moon's declination. Let the longitude (west) and latitude (north), HY, YZ in Fig. 293, be denoted by l and λ, and the hour-angle HK, and declination MK, by ψ and δ. We have to express $\cos\phi$ in terms of these co-ordinates.

It is clear that ϕ is the moon's true zenith-distance, that is, the angle which the vertical at the place makes with the line drawn from C to the moon's centre. Let, then, Fig. 293 represent the celestial sphere, N, S the poles, EHE the equator, M, G, and Z the points in which radii drawn from C to the moon, to Greenwich, and to P intersect the celestial sphere. Draw great circles from N through G, M, and Z, meeting the equator in H, K, and Y, and join MZ by an arc of a great circle. MZ represents the moon's zenith distance ϕ.

Now consider the spherical triangle NMZ. The angle at N is

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measured by KY, that is, $HY-HK$ or $l-\psi$. Also NM, NZ, the sides of this triangle, are $\pi/2-\delta$, $\pi/2-\lambda$ respectively. The fundamental elementary formula of spherical trigonometry gives

$$\cos MZ = \cos NM \cos NZ + \sin NM \sin NZ \cos KY;$$

or

$$\cos\phi = \sin\lambda \sin\delta + \cos\lambda \cos\delta \cos(l-\psi).$$

Hence

$$\begin{aligned} 3\cos^2\phi - 1 &= 3\{\sin^2\lambda \sin^2\delta + \cos^2\lambda \cos^2\delta \cos^2(l-\psi) \\ &\qquad + 2\cos\lambda \cos\delta \sin\lambda \sin\delta \cos(l-\psi)\} - 1 \\ &= \tfrac{3}{2}\cos^2\lambda \cos^2\delta \cos 2(l-\psi) + 6\cos\lambda \sin\lambda \cos\delta \sin\delta \cos(l-\psi) \\ &\qquad + \tfrac{1}{2}(3\sin^2\delta - 1)(3\sin^2\lambda - 1). \qquad (5) \end{aligned}$$

Thus, for the total rise or fall au we have, putting $\mathfrak{h}$ for $\frac{3}{2}Ma^4/ED^3$,

$$\begin{aligned} au &= \tfrac{1}{2}\mathfrak{h}\cos^2\lambda \cos^2\delta(\cos 2l \cos 2\psi + \sin 2l \sin 2\psi) \\ &\qquad + \tfrac{1}{2}\mathfrak{h}\sin 2\lambda \sin 2\delta(\cos l \cos\psi + \sin l \sin\psi) \\ &\qquad + \tfrac{1}{12}\mathfrak{h}(1 - 3\cos^2\delta)(3\sin^2\lambda - 1). \qquad (6) \end{aligned}$$

A precisely similar expression holds for the solar equilibrium tide. The data respecting the moon and the earth are replaced by the corresponding data relating to the sun and the earth, and the tide evaluated and added to the lunar tide for the same instant to give the total elevation or depression of the water-level.

The first line of the expression on the right represents the semi-diurnal tide, the second line the diurnal tide, and the third line is called the *declinational* tide.

If dS denote an element of the earth's surface, then, as can easily be proved, $\int dS$ taken over the surface is equal to zero—that is, the volume removed from the region of depression is equal to the volume above the chosen mean level in the region of tidal elevation. This, however, is for an ocean covering the whole earth. If the earth is only partially covered, to the expression in (6) for au must be added a quantity au', fulfilling the condition that $\int a(u+u')dS$, taken over the whole water surface, is zero.

567. **Discussion of Terms in Formula for Tidal Rise and Fall.**—The semi-diurnal tide for any place changes from zero to a maximum and back again to zero, then to a maximum negative value and back again to zero, while the hour angle ψ changes by 180°. Save for the varying amplitude, the change is a simple harmonic variation running through all its phases twice in the interval between two successive passages of the moon in the same direction across the meridian. The amplitude varies as the square of the cosine of the declination of the tide-producing body, and also as the square of the latitude of the place, being thus zero at the poles and a maximum at the equator. The declination varies, in the case of the sun, from zero to the obliquity of the ecliptic, $\omega(=23^\circ\ 24'\ 10'')$ north or south,

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and in the case of the moon through a varying range, on each side of zero, which can never be greater than $\omega+i$ or less than $\omega-i$, where i is the inclination of the moon's orbit to the plane of the equator. Hence, with change of the value of the declination, the amplitude of the tidal rise and fall varies from a maximum to a minimum value without vanishing, being greatest when the declination is zero.

The declination runs through a whole range of values in one revolution, being negative in one half and positive in the other half of the body's period. It is a matter of the most ordinary observation that, if the moon be markedly low in the heavens, say at the end of the first quarter, it is correspondingly high at the end of the third quarter.

The two high-waters and two low-waters which are produced every 24 lunar hours, or every 24 solar hours, are of different heights in consequence of the tide represented by the second line, which goes through all its values as ψ varies from 0 to 360°. This component is called the diurnal tide. It varies as the sine of twice the declination, and therefore vanishes when the tide-producing body is on the equator. This is the case for the sun only when it is at the equinoxes, and for the moon when at one or the other of the nodes of its orbit round the earth. The amplitude is therefore positive from one equinox (or node) to the next, and is negative in the remaining half of the tropical period of the body; that is, the period from one equinox (or node) to the next but one.

Also the diurnal tide vanishes both at the equator and at the poles on account of the factor $\sin 2\lambda$ which occurs in the amplitude.

The third line represents the declinational tide; that is, in the case of the moon, the fortnightly tide; in the case of the sun, the semi-annual tide. It does not vanish because of the factor $1-3\cos 2\delta$, for δ never reaches the value for which $\cos 2\delta = 1/3$, but because of the factor $3\sin^2\lambda - 1$ it is zero for points in latitude 35° 15′ 52″, or, say, 35° 16′. When the body is on the celestial equator the term has a maximum positive value for a given latitude less than 35° 16′, and a maximum negative value for a given latitude greater than 35° 16′; for $1-3\cos 2\delta$ has then the value -2, while $3\sin^2\lambda - 1$ has the value -1 at the equator and 2 at the poles, and vanishes for $\lambda = \sin^{-1} 1/\sqrt{3}$.

Since the quantity, so far as it depends on the declination, is independent of whether the declination is north or south, and does not involve ψ, the declinational tide may be regarded as produced by two circles of matter, composed, in the case of the lunar tides, of the moon and anti-moon, each uniformly distributed round the circle parallel to the equator on which it lies. These circles are therefore at equal distances on the two sides of the equator, and as the declination varies from an extreme value to zero they come closer and closer together, until they coalesce into one circle on the plane of the equator. A similar statement is true for the semi-annual tide.

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568. **Tides of Long Period.**—The moon's orbit turns completely round in a period of 19 years, so that the limits of the declination go through a complete series of changes, from $\omega + i$ to $\omega - i$ and back again, in that time. Hence, all three tides vary on this account, and in a 19 years' period.

Also the distance D of the body goes through a complete series of variations in the period of revolution, on account of the ellipticity of the orbit. This gives the monthly and annual elliptic tides, as they are called.

569. **Rise and Fall is greatly Modified by Existence of Continents.**—It is to be most carefully noticed that, even as equilibrium tides, every one of the tides here discussed is very greatly modified by the fact that the earth is only partially covered by water. The reader will find a very full discussion of the question in Thomson and Tait's *Natural Philosophy*, or in Professor G. H. Darwin's article on the Tides in the *Encyclopædia Britannica* (19th ed.); but it may be stated that, when the theory is corrected (which can only be imperfectly done) by taking account of the distribution of land and water, there is no longer necessarily high water or low water for any place when it lies in the plane through the earth's centre and the tide-producing body; but the time for either semi-diurnal or diurnal tide may be either before or after the instant when this is the case.

The semi-diurnal tide does not vanish at the poles, as the expression above states. The diurnal tide is zero at neither the equator nor the poles. Again, the lunar fortnightly and solar semi-annual tides vanishes by the formula in latitude $\lambda = \sin^{-1} 1/\sqrt{3}$, and this is not the case in the corrected theory. It can be shown that the corrected value of the latitude of evanescent declinational tide is $\sin^{-1} \sqrt{(1+E)/3}$ where E is the mean value of $3\sin^2\lambda - 1$ for the part of the earth covered by water. Professor G. H. Darwin has found for $\sin^{-1} \sqrt{(1+E)/3}$ the value 34° 40′ or 34° 57′, according as a certain supposititious antarctic continent is taken into account or not. Thus the value 35° 16′ is but little departed from.

570. **Effect of Yielding of the Internal Parts of the Earth.**—It has been supposed that the part of the earth internal to the stratum of water is unyielding, and that the tidal elevation and depression are entirely due to relative displacement of the water. If the internal part of the earth yielded as freely as if it were liquid, if it were composed, for example, of liquid separated from the outside stratum by a flexible covering, the distortion into the spheroidal form would be almost entirely made up of displacement of the liquid within the envelope, and there would be but little due to elevation and depression of the surface waters relatively to the envelope. The surface liquid would behave very much as a thin covering of oil on the surface of the sea would behave; there would be almost no change in its thickness.

The tidal elevation and depression is undoubtedly less than it

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would be if there were no yielding of the internal and apparently solid matter; how much less is not certain. The question has been discussed by Lord Kelvin, who has concluded that the yielding of the earth, if it were made up of material as rigid as glass, would diminish the oceanic tides to $\frac{2}{3}$ of those which would be produced if the earth were absolutely unyielding. If the globe were as rigid as steel, the tides would be $\frac{2}{3}$ of those of the ocean on an unyielding earth. Lord Kelvin believes that the oceanic tide is certainly greater than half the tide on an unyielding earth, and perhaps rather less than the tide that would be produced if the internal matter were as rigid as steel, and concludes that the rigidity is certainly greater than that of glass, and possibly less than that of steel.

Lord Kelvin thus supposes that the earth is solid internally, and from the fact that the ellipticity of its surface corresponds nearly to the present rate of rotation, he is of opinion that it must have solidified comparatively recently, when the rate of rotation was approximately what it is at present. In this way strong evidence is obtained, which is reinforced by other considerations, that the earth is not composed of a great central fluid part and an outer and partial covering of water separated by a solid crust. The yielding of a solid crust, even several miles in thickness, to the oscillations of the confined fluid would be complete, and changes of the depth of the sea would be very slight.

571. **Results of Equilibrium Theory not borne out by Actual Tides. Question of Fortnightly Tides.**—The equilibrium theory is, as a whole, in no way a true theory of the tides. The conditions necessary for its being true are not even approximately fulfilled: the earth turns with such rapidity on its axis that no static adjustment to the form of a tidal spheroid stationary with respect to the tide-producing body is possible, and what is produced is a system of *forced vibrations* of the water covering the earth, modified by friction, and by the configuration of the continents and islands by which the surface sheet of water is bounded. The equilibrium theory is, however, valuable for the general view of the matter which it gives, and the analysis into component tides which it suggests: moreover, the lunar fortnightly and the solar semi-annual tides which it gives are more nearly representative of the tides which actually occur. It was remarked by Laplace that friction would enable the equilibrium form to be assumed by these latter tides, but, as has been pointed out by Professor G. H. Darwin, the period is not long enough to enable this adjustment to take place, at any rate for the fortnightly tides. For (see § 574) any motion of the water retarded by friction proportional to the velocity would be diminished in a time, t, in the ratio of the factor $e^{-kt/2}$ to unity, and it would depend on the value of t required to give $kt/2 = 1$, whether the friction was effective in this way or not. The smaller this value of t—that is, of $2/k$—that is, the larger k, the more effective is the friction; and t would have to be small in comparison with the half period of the tide—that is, in comparison with a week, to justify Laplace's remark.

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Were Laplace's view correct, it would be possible, by comparing the observed amount of the fortnightly tide with the theoretical amount, to form an estimate of the tidal yielding of the solid earth; but though the observed amount falls short of the theoretical, no such estimate can legitimately be made for the reason just stated. The amount of the semi-annual tide given by observation is too uncertain to afford any basis for an estimate of this kind.

572. **Digression on Free and Forced Oscillations.**—The term *forced vibration* has a certain technical significance. It means a vibration taking place under applied periodic forces which are independent of the vibrating system. Thus, if a pendulum be deflected and then left to itself, it makes what are called *free oscillations*. But if a periodic force in a period different from that of the motion be kept applied to it from outside—that is, apart from the periodic force due to its displacement relatively to the earth—the pendulum is compelled to oscillate in the period of the force. For example, consider a pendulum performing oscillations through a very small range, and suppose that the periodic force begins to act and continues to do so. The motion under the free vibration is given by the equation

$$x = a\cos(nt - \epsilon), \tag{7}$$

where x is the displacement from the middle position at time t, $2\pi/n$ is the period, ϵ the epoch, and a the amplitude (§ 43) of the motion. The differential equation is

$$\ddot{x} + n^2x = 0$$

and (7) is its complete solution.

Now, if a force $A\cos mt$ act on the pendulum bob, the differential equation of motion is

$$\ddot{x} + n^2x = A\cos mt, \tag{8}$$

supposing the motion unresisted by friction. To solve this equation, suppose $x = B\cos mt$ and substitute. We get the condition

$$B = \frac{A}{n^2 - m^2}.$$

Hence (8) is satisfied by the value of x, stated in

$$x = \frac{A}{n^2 - m^2}\cos mt. \tag{9}$$

This, though a particular solution of the equation, is not the complete solution. That is obtained by adding to this value of x that given by (7), so that

$$x = \frac{A}{n^2 - m^2}\cos mt + a\cos(nt - \epsilon). \tag{10}$$

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For it is clear that the second term makes no difference to the value of the left-hand side of (8), and it introduces two arbitrary constants, a, ε, which, by the theory of this kind of differential equation, are required for the complete solution.

The first part represents the forced vibration, the second represents a free vibration superimposed on the former. When the motion has become steady the free vibration is not sensible; in practice it is wiped out by frictional resistances, and there is left only the first term. In general the free vibration only manifests itself when the motion is being started or stopped, or when any variation from the forced motion takes place.

Considering, then, only the forced vibration, we see that, if $n^2 > m^2$, the deflexion x and the force have the same sign, and if $n^2 < m^2$, they have opposite signs. In the former case the vibration is said to be direct, in the latter case it is said to be inverted. The force and deflection have always maximum numerical values at the same instant. Also if n^2 be very nearly equal to m^2, the amplitude of the forced vibration is very great—that is, the application of a periodic force, which has very nearly the *free period* (as it is often called) of the system, results in a great amplitude of vibration. If $n = m$, (9) asserts that the amplitude $A/(n^2 - m^2)$ is infinite; but it is to be remembered that when the amplitude has become great, forces not contemplated in the differential equation will have come into play; and these will impose limits on the motion.

573. **Examples of Forced Oscillations.**—There are many examples in practice of the great amplitudes of vibration which result from impulses applied in the natural period of the system. The rolling of a ship in a sea-way, in which the period of the waves is approximately equal to the free period of rolling, is a case in point. The ship will be in great danger of heeling over beyond the limiting angle within which the righting moment is positive (§ 417), unless she is placed with her length at right angles to the wave-crests, so that their effect is only to produce pitching, for which the vessel has a much longer period. A periodic force, even of small amount, applied over a considerable interval of time, to a massive structure, such as a suspension bridge, may, provided the force acts in or nearly in the period of free oscillation of the structure, produce a very large deflection from the equilibrium position. Thus it is usual to cause a body of soldiers, when they are marching across a suspension bridge, to "break step," lest the accumulated effect of the periodic impulses applied to the structure should produce a dangerous deflection of the bridge from the ordinary position of rest.

574. **Forced Oscillations of a System subject to Friction.**—Forced vibrations are modified in a very important manner by friction. Let the vibrations excited be retarded by friction proportional to the velocity of the moving body. Then the equation of motion is

$$\ddot{x} + k\dot{x} + n^2 x = A \cos mt. \tag{11}$$

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The solution of this equation, with the right-hand side zero, is

$$x = ae^{-\frac{1}{2}kt}\cos(\sqrt{n^2 - \tfrac{1}{4}k^2}.t - \epsilon). \qquad (12)$$

The period is thus increased by the action of friction in the ratio of n to $\sqrt{n^2 - \frac{1}{4}k^2}$, and the amplitude diminishes in consequence of the factor $e^{-kt}/2$, in geometrical progression as the time, t, increases in arithmetical progression. This manner of variation of the amplitudes is illustrated in Fig. 32, p. 46. Let there the radius-vector, OP, revolve about O with uniform angular velocity in the direction against the arrow: the length of OP will diminish according to the law stated, with a value of k depending on the constant value of the inclination of the curve to the radius-vector.

Now assume as a particular solution of (11)

$$x = B\cos(mt - \phi)$$

we obtain by substitution in (11)

$$B(n^2 - m^2)\cos(mt - \phi) - kmB\sin(mt - \phi) = A\cos mt.$$

If this be expanded, and coefficients of $\cos mt$, $\sin mt$, on the two sides, equated, it gives $B = A\sin\phi/mk$, and

$$x = \frac{A\sin\phi}{mk}\cos(mt - \phi) \qquad (13)$$

where

$$\tan\phi = \frac{mk}{n^2 - m^2}.$$

Thus we may write for the forced vibration

$$x = \frac{A\cos(mt - \phi)}{\sqrt{n^4 + m^4 - 2m^2(n^2 - \frac{1}{2}k^2)}}. \qquad (13')$$

The complete solution is that obtained by adding to this the solution contained in (12) of the corresponding differential equation for the free oscillation.

It follows from (13′) that if k be not large the value of x is very great, if m be nearly equal to n. Also ϕ is then $\pi/2$, so that the vibration is a quarter of a period behind the force in phase.

If k be exactly zero, we fall back on the solutions already obtained.

575. **Tides considered as a System of Forced Oscillations. Propagation of Waves.**—To apply these considerations to the tides, we remark in the first place that the relative positions of the earth, sun, and moon are continually changing in a periodic manner. If the sun and moon were to remain fixed relatively to the earth's centre, the tidal force applied at a particular place on the earth's surface would vary periodically with the rotation of the earth, producing the diurnal and semi-diurnal tides. But these bodies change their positions in periods of their own: the right ascension

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of the sun passes through all its values in one solar day, that of the moon in a lunar day (the interval between two successive passages in the same direction of the moon across the meridian), the declination of the sun and the sun's distance have a yearly period, and the declination and distance of the moon a monthly period, and there are also variations, referred to above, of still longer periods.

The earth and the waters upon it are, therefore, subject to periodic tidal forces, and the response thereto is not that of the equilibrium theory, but that of a system which has certain natural modes of free oscillation, and which is also subject to friction. To understand the nature of these modes of oscillation, imagine the level of the water surface to begin to be raised and lowered at one end according to the simple harmonic law, that is, so that the height η above the mean level is given by the equation

$$\eta = a\cos nt.$$

After the starting of this disturbance the level of the surface elsewhere will be disturbed, and the disturbance will travel along the canal with a definite speed. After a time, if the canal be long enough, the disturbance will have travelled a certain distance along the canal, and beyond that distance the water will remain undisturbed. Near the source of disturbance the oscillation follows the simple harmonic law both as regards variation of height with time at any one place, and variation of height with distance from the source at any one instant. Thus, if friction be neglected, the wave profile will finally be given at any time or place by

$$\eta = a\cos(nt - mx), \tag{14}$$

so that when $x=0$, $\eta = a\cos nt$, and when $t=0$, $\eta = a\cos mx$, or when t has any value so that $nt=\epsilon$, $\eta = a\cos(mx-\epsilon)$. Whatever value η has at any time t at distance x from the source will be found after an interval τ at a distance ξ farther from the source given by the equation $nt - mx = n(t+\tau) - m(x+\xi)$, so that $\xi = \tau n/m$. The speed of propagation V of the disturbance is thus n/m. Again the same values of η recur along the disturbance at distances $2\pi/m$ apart. A distance along the wave equal to one of these is called the wave-length of the disturbance, and is generally denoted by λ. Thus $\lambda = 2\pi/m$, and the speed of propagation is $n\lambda/2\pi$. The wave-length is the distance travelled by the disturbance in the period, that is, in time $2\pi/n$, or T. Thus the period, wave-length, and speed are connected by the relation $V = \lambda/T$.

576. **Waves in a Canal. Solution for Waves in a Canal of Finite Length.**—The speed of propagation of the wave in the canal depends on the depth of water if the length of the wave be great in comparison with the depth. Let two cross-sections of the canal at distances x, $x+dx$ from a chosen origin for x be considered. Let the former cross-section be called the left-hand section, and let the wave move from left to right. If p be the pressure at the former of these at any height above the bottom of the canal, the pressure at

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the same level in the other section (if the value of η, and therefore the motion, be everywhere very small) is

$$p+\partial p/\partial x.dx=p+g\rho\partial\eta/\partial x.dx.$$

Also the motion being all in the direction of x (the vertical component being taken as negligible) we obtain for the equation of motion $\dot{u}=-(\partial p/\partial x)/\rho$, or since $\partial p/\partial x=g\rho\partial\eta/\partial x$,

$$\dot{u}=-g\frac{\partial\eta}{\partial x}.$$

But the increase of depth above the mean depth is η between the cross-sections considered, and the volume of liquid above the mean-level is $\eta b dx$, if b be the breadth of the canal. The rate of flow across the left-hand section is hbu, and the rate of flow across the right-hand section is $hb(u+\partial u/\partial x.dx)$. The rate of flow into the space between the sections is thus $-bh\partial u/\partial x.dx$. This must be the time-rate of increase of $b\eta dx$, that is, $b\dot{\eta}dx$. Hence $\dot{\eta}=-h\partial u/\partial x$, or if the time-rate of variation of both sides be taken, $\ddot{\eta}=-h\partial\dot{u}/\partial x$. Eliminating $\partial\dot{u}/\partial x$ from this by the equation of motion written above we get

$$\frac{\partial^2\eta}{\partial t^2}=gh\frac{\partial^2\eta}{\partial x^2} \tag{15}$$

which is the equation of wave-propagation in a straight canal with vertical sides and of uniform depth.

If we write

$$\eta=f_1(x-Vt)+f_2(x+Vt) \tag{16}$$

where f_1, f_2 are arbitrary functions, and $V=\sqrt{gh}$, the equation is satisfied. The functions f_1, f_2 can be chosen so as to satisfy any possible circumstances or form of the wave. They represent two waves travelling with velocity V along the canal, but in opposite directions. For if x be increased by a quantity ξ and t by τ where $V\tau=\xi$ it will be seen that the value of f_1 is not changed On the other hand, if f_2 is to remain unchanged, x must change by $-V\tau$ when t changes by τ.

In a similar way, if ξ denote the horizontal displacement of the water at any place in the canal, we can show that the equation of a free unresisted wave is

$$\frac{\partial^2\xi}{\partial t^2}=c^2\frac{\partial^2\xi}{\partial x^2} \tag{17}$$

where c^2 is put for gh. This has the same form of solution as the equation in η already obtained. If we assume that the wave is simple harmonic in the variation of ξ in time, we may write

$$\xi=f(x)\cos(nt+\epsilon) \tag{18}$$

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where $f(x)$ denotes the function which gives the value of ξ for different values of x. Hence we obtain by substitution in (17)

$$\frac{\partial^2 f}{\partial x^2}+\frac{n^2}{c^2}f=0, \tag{19}$$

so that

$$f=A\cos\frac{n}{c}x+B\sin\frac{n}{c}x \tag{20}$$

where n is to be determined by the terminal conditions of the canal. Hence

$$\xi=\left(A\cos\frac{n}{c}x+B\sin\frac{n}{c}x\right)\cos(nt+\epsilon). \tag{21}$$

The most general solution is obtained by adding together all possible solutions of this form.

For example, consider a canal of length l with vertical ends. Let the origin be at one end: then we must have $\xi=0$, when $x=0$, and when $x=l$, for all values of t. This gives $A=0$, and $B\sin nl/c=0$, or $n=k\pi c/l$ where k is any integer. The general solution is, therefore, for this case

$$\xi=\Sigma B_k\sin\frac{k\pi x}{l}\cos\left(\frac{k\pi ct}{l}+\epsilon_k\right) \tag{22}$$

where Σ denotes summation for all values of k.

577. **Forced Waves in a Canal.**—Consider now a *forced* wave in a canal. For this the differential equation is

$$\frac{\partial^2\xi}{\partial t^2}=c^2\frac{\partial^2\xi}{\partial x^2}+X. \tag{23}$$

where X is a periodic applied force.

We put in (23) $X=C\cos(nt+mx+\phi)$, and assume

$$\xi=D\cos(nt+mx+\phi).$$

Substituting in (23) we get $D=-C/(n^2-c^2m^2)$, and therefore

$$\xi=-\frac{C}{n^2-c^2m^2}\cos(nt+mx+\phi). \tag{24}$$

The wave is, therefore, direct or inverse; that is, has the phase of the force or the opposite phase according as $c^2>$ or $<n^2/m^2$. The velocity c is that of a free wave, and n/m is that of propagation of the periodic force and of the forced wave. The forced wave is therefore direct or inverted according as the velocity of propagation of the force is less or greater than the velocity of a free wave.

578. **Waves in an Equatorial Canal.**—Now let the canal be one with vertical sides and uniform depth round the equator. This may be taken as a straight canal, and the formulæ found above for wave-propagation may be used for it, provided g be corrected for the

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effect of the earth's rotation as in § 281. The periodic force applied may be regarded as that due to the moon and anti-moon, as explained in § 561, both being on the celestial equator. Therefore the period of the force must be half the period of rotation of the earth relatively to the moon. Let the point of the canal under consideration be at a distance x measured eastwards from a meridian of reference, and let the angle between this meridian and the meridian through the moon, measured eastward from the latter meridian (NH and YN of Fig. 291), be $n't$, so that n' is the angular velocity of the earth relatively to the moon. The whole angle through which the radius to the point considered has turned from the direction of the moon is $n't+x/a$, where a is the earth's radius. We therefore write, since the force X goes through two periods in one rotation of the earth relatively to the moon, and taking account of the fact that the horizontal force is zero under the moon, and at points on either side of that position acts towards it,

$$X = -4C\sin 2\left(n't+\frac{x}{a}\right).$$

Hence for the horizontal displacement of the water in the forced tide we obtain

$$\xi = \frac{Ca^2}{n'^2a^2-c^2}\sin 2\left(n't+\frac{x}{a}\right). \tag{25}$$

In consequence of the displacement ξ the thickness of a vertical stratum has been changed from dx to $dx(1+\partial\xi/\partial x)$, and its depth has been increased from h to $h+\eta$. Since the volume must be the same as before, this gives $\eta+h\partial\xi/\partial x=0$, which is equivalent to the relation $\dot{\eta}=-h\partial u/\partial x$ used in § 575 above. Hence from (25) we find

$$\eta = -\frac{2Cha}{n'^2a^2-c^2}\cos 2\left(n't+\frac{x}{a}\right). \tag{26}$$

If $\mathfrak{h}$ denote the utmost range of the rise and fall of the semi-diurnal tide according to the equilibrium theory, it has been shown in § 566 that $\mathfrak{h}=\frac{3}{2}Ma^4/ED^3$, and since by (1) $4C=\frac{3}{2}kMa/D^3$, and $g=kE/a^2$ approximately, we have $2Cha=\frac{3}{4}Ma^4gh/ED^3=\frac{1}{2}\mathfrak{h}c^2$. Thus (26) may be written

$$\eta = -\frac{1}{2}\frac{\mathfrak{h}c^2}{n'^2a^2-c^2}\cos 2\left(nt+\frac{x}{a}\right). \tag{27}$$

The tide is thus direct or inverse according as $c^2>$ or $<n'^2a^2$. In the former case the velocity of the forced wave is less than that of the free wave, and there is high water under the moon; in the latter case the velocity of the forced wave is greater than that of the full wave, and there is low water under the moon.

The time in which a free wave would travel round the canal is given in hours by the formula $\sqrt{5280}.\ L/\sqrt{32}.\ 3600\sqrt{h}$, where L

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and h are the length of the canal and the depth both in miles. Hence that the wave may travel half-way round the canal in twelve solar hours, the depth must be about $13\frac{3}{4}$ miles, which is much in excess of the depth of the ocean anywhere. For twelve lunar hours a somewhat smaller depth, about thirteen miles, would be required.

579. **Canal Theory of the Tides.**—It is clear from what precedes that, as the earth turns under the tidal force due to the sun, a forced wave of elevation exists on each side of the earth and travels half round it in twelve solar hours. Hence at each point a semi-diurnal tide is experienced. If the depth be less than $13\frac{3}{4}$ miles, the period of the forced solar tide is less than the free period, and the tide is inverted, there is low water under the sun.

A similar result holds for the lunar tide due to the lunar tidal force in the period of twelve lunar hours. For a depth lying between the critical depth for the lunar tide and that for the solar tide, there would be inversion of the solar tide, while the lunar tide would be direct. Also in consequence of the near coincidence of the free and forced periods both tides would be very great.

If, then, the ocean consisted of a canal round the equator we should have with any actual depth of the water an inverted semi-diurnal tide—that is, there would be low water under the moon.

For a canal parallel to the equator in latitude θ the case is different. Instead of the term x/a in the applied periodic force, a term $x/a\cos\theta$ appears, and the formula for the forced tide in this case is obtained by substituting $a\cos\theta$ for a. Thus the tide will be direct or inverse, according as $c^2 >$ or $< n'^2a^2\cos^2\theta$. If $c > n'a$, the tide would always be direct, and if $c < n'a$ the tide would only be inverted if $n'a\cos\theta$ did not fall below c.

Thus, according to the canal theory, for a certain depth of water less than the critical depth there would be a latitude beyond which the tide in the canal would be direct. If the canal were in a lower latitude the tide would be inverted, and if the ocean consisted of a series of canals of equal depth, separated from one another by partitions parallel to the equator, and covered the whole earth, the tides would all be direct in higher latitudes and inverted in lower latitudes.

580. **General Problem of the Tides.**—The difficulty of the problem of the tides is immensely increased when these partitions are supposed removed, and it is impossible here to give any adequate account of it. The partitions confined the motion of the water to backward and forward vibrations along the canal, but it is obvious that in the more general case just supposed there must be flow in the south and north directions as well. Further, water flowing from the poles towards the equator comes into a region of greater velocity along the surface, and flows in a westward direction relatively to the earth, while water flowing from the equator towards the poles acquires an easterly velocity relatively to the earth. All this makes the movements exceedingly complicated,

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and it is impossible here to give even an outline of the system of tidal currents which would exist in such an ocean.

The general conclusion that the tides are inverted in low latitudes, and direct in high latitudes, is, however, correct. It might appear that at the latitude of change from inverted to direct tide there would be an infinite tide. This, however, is not the case; on the contrary, a complete discussion of the subjects shows that the latitude of transition is one at which there is neither rise nor fall. The complete discussion cannot be entered on here, however, and the reader is referred to the *Mécanique Céleste* of Laplace, and the article on *Tides* by Professor G. H. Darwin, in the 19th Edition of the *Encyclopædia Britannica*, for further information.*

581. **Harmonic Analysis of Tides. Observations by Tide-gauges.**—The problem is immensely more complicated, even than that just suggested, by the configuration of land and water, and it is only possible to predict the rise and fall at any place by observing by self-registering apparatus the rise and fall at the place in question over a long period of time, and analysing the periodic changes into their simple harmonic constituents. With a very short account of the analysis and process of prediction, we conclude the present discussion.

Tidal curves are drawn at each of the principal ports by tide-gauges. These curves may be regarded as compounded of a number of simple harmonic variations of the height of the water of different periods, the forced vibrations produced by the applied tidal forces. The *ordinates of the curves* are tide-heights or proportional to them, the abscissæ are times from a chosen zero of reckoning. From these curves can be obtained by measurement of a sufficient number of ordinates, the data required to enable the amplitude period and epoch of each of the components to be determined. Then these data are available for the calculation of the tides for future time, and the formation of tide-tables for the ports for which they have been obtained.

A machine called a Tidal Analyser, which may be regarded as complementary to the Tide-predicting Machine about to be described, has been invented by Lord Kelvin. It is only necessary to place the tide-gauge record on a cylinder, follow the curve by a tracing point on the machine, and the constants of the various harmonic constituents can be read off from the machine. For a description see Thomson and Tait's *Nat. Phil.* vol. i. Part I.

582. **Tide-predicting Machine.**—It is clear that the combination of tidal heights could be obtained by drawing the curve of each component on the same scale of time and height, laying them down along the same datum line in the proper positions as determined by the epochs, and then adding the heights for each abscissa together to obtain the resultant height there, and the resultant curve for all the abscissæ. Hence it is theoretically possible to make a machine which will draw the resultant curve. It is only necessary to have a number

* A most excellent popular account of the subject is contained in Professor Darwin's book *The Tides*. London. John Murray.

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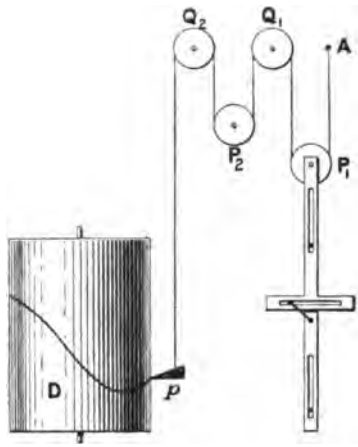
of simple harmonic motions performed by points of the machine, all geared together and driven by one handle or driving-weight.

Such a machine has been constructed by Lord Kelvin, and the diagram illustrates its mode of action. A simple harmonic slide, of the kind shown at p. 34, is worked by a crank, the throw of which represents half the amplitude of one of the tidal constituents as obtained by the tide gauge. At the upper end the slide carries a pulley P, the total vertical range of motion of which is of course twice the throw of the crank. A cord attached at one end to a fixed point A, passes round P, over the fixed pulley Q_1, round the pulley P_2, over the fixed pulley Q_2, round the pulley P_3, and so on, and after passing finally over a fixed pulley is attached to a pencil *p*, which bears on a moving ribbon of paper. This ribbon is in process of being unrolled from one drum and wrapped on another at a rate equal to that at which the drum-surface is moving.

FIG. 294.



If the pulleys P_2, P_3,..., are all fixed P_1 has a motion representing a single tidal constituent. If, however, P_2, P_3, ... are carried by simple harmonic slides geared with the handle driving P_1, so that while P_1 describes its motion P_2, P_3, ... describe in proper period and in proper phase other constituents of the tidal rise and fall, the end of the cord at the pencil will have a displacement at each instant which is twice the algebraical sum of the displacements of the pulleys P_1, P_2, P_3, ... The drums moving the paper are driven at the same time with a speed proportional to that with which the driving-handle of the machine is being driven. The part of the paper opposite the pencil, therefore, represents a particular instant of time, and the length of paper that has then passed the pencil since a particular vertical line on the paper was opposite the marking point, represents the interval of time between a chosen zero of reckoning and the instant in question.

Let then P_1, P_2, P_3, ... be sufficient in number to represent the principal tidal constituents, and have been set by means of information given by tidal records taken at the place so as to have the proper displacements and directions of motion at a given time, and be geared so as to be driven at the proper relative rates. The pencil will move up and down so as to be at the proper distance from the mean level to represent the tidal rise or fall for the instant of time represented by the part of the paper ribbon opposite the marking

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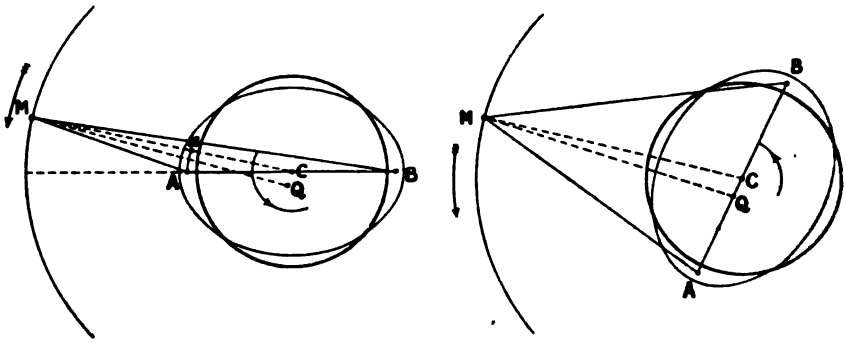
point. The tidal curve for future time will thus be drawn on the paper ribbon, and by measuring its ordinates and recording their lengths tide-tables for the place can be constructed.

One of these machines, which is now at the India Office, was constructed under the care of Mr. Edward Roberts, of the *Nautical Magazine* Office. It has harmonic slides representing twenty-four tidal constituents or partial tides. Tidal information obtained by tide-gauges for thirty-seven Indian ports is sent home by the Indian

FIG. 295.



Government, and is used in the prediction, by means of the machine, of the tides for each of these ports. The machine is set by Mr. Roberts, and driven by a weight, runs off the curves for a year in about four hours, writing automatically at intervals the date corresponding to the position of the pencil so as to prevent all risk of error in taking off the heights from the curve.

583. **Tidal Friction. Retarding Couple on the Earth.**—We now consider very shortly the action of tidal friction on the motions of the earth and the moon. In the first place, by the results obtained in §§ 574–579 the effect of friction when the forced tidal wave is direct is to turn the long axis of what we continue to call the tidal spheroid forward in the direction of the earth's rotation beyond the line CM joining the centres of the earth and moon, as shown on the left in Fig. 295. As the earth revolves the tidal spheroid remains fixed in position relatively to the moon. The force on the moon exerted by the nearer part of the tidal protuberance is greater than that exerted by the farther part, and consequently the resultant force on the moon is in the line MQ instead of being in the direction MC. This causes to be exerted on the moon a force towards the centre C of the earth, and a tangential component in the direction of the moon's orbital motion. On the earth there is exerted, as we shall see, a force along the line CM and a couple retarding the earth's rotation, and producing dissipation of the earth's rotational energy.

On the other hand, when the tidal wave is inverted the effect of

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the friction is to throw the tidal protuberance back, as shown in Fig. 295 on the right, and as before the tidal spheroid remains fixed in position relatively to the moon. If the spheroid were thrown forward the dissipation of energy which obtains in the former case would be replaced here by an increase of the earth's kinetic energy, and the moon would move inward. In this case again there is a couple applied to the earth in the direction to retard the earth's rotation and causing dissipation of energy, and a tangential force to the moon in the direction of its orbital motion. The effect is therefore the same whether the wave is direct or inverted, and it is the tidal distortion in the equatorial regions which is most effective in the way just described. Hence it will be sufficient in studying the result to consider the case represented on the left in Fig. 295.

First, then, as to the calculation of the tangential force on the moon and the couple just referred to. Instead of considering the whole spheroid, it will be sufficient for a proper apprehension of the matter to consider the couple due to the attractions of the moon on unit mass at A and B, and the tangential force due to the attractions of these particles on M.

Denoting the angle ACM by θ, the distance CA by a, and CM by r, we have approximately

$$\overline{AM}^3 = r^3\left(1 - 3\frac{a}{r}\cos\theta\right), \quad \overline{BM}^3 = r^3\left(1 + 3\frac{a}{r}\cos\theta\right),$$

$$\cos AMC = (r - a\cos\theta)/\overline{AM}, \quad \cos BMC = (r + a\cos\theta)/\overline{BM}$$

$$\sin AMC = a\sin\theta/\overline{AM}, \quad \sin BMC = a\sin\theta/\overline{BM}.$$

Now suppose applied to every particle of the earth a force per unit mass of amount kM/r^2, in the direction MC; then on the unit particle at A there acts a force $kM/\overline{AM}^2$ towards M, and a force kM/r^2 parallel to $\overline{MC}$. On the particle at B there act similarly $kM/\overline{BM}^2$ towards M, and kM/r^2 parallel to $\overline{MC}$. Resolving along $\overline{CM}$, and at right angles to $\overline{CM}$ (that is, parallel to Aa), we obtain

Force at A: component parallel to $CM = \dfrac{kM}{r^3}2a\cos\theta$;

component parallel to $Aa = \dfrac{kM\sin\theta}{r^4}a(r + 3a\cos\theta)$.

Force at B: component parallel to $CM = -\dfrac{kM}{r^3}2a\cos\theta$;

component parallel to $Aa = -\dfrac{kM\sin\theta}{r^4}a(r - 3a\cos\theta)$.

The distance between the lines of action of the first pair of parallel forces is $2a\sin\theta$, and that between the lines of action of the second

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pair is $2a\cos\theta$. Neglecting the terms in $\cos\theta$ in the second pair of forces, we obtain for the total couple applied to the two particles

$$3\frac{kMa^2}{r^3}\sin 2\theta.$$

The tangential force on the moon is evidently equal to the algebraic sum of the two forces parallel to Aa, namely,

$$3\frac{kMa}{r^3}\sin 2\theta.$$

In this way the couple and tangential force due to the whole tidal spheroid may be found by summing those due to pairs of particles. It will be clear that the couple and force will vary as the inverse sixth power of the distance r. For, as we have seen, the couple per unit mass varies as the inverse third power, and the whole tidal rise or fall varies also as the inverse third power.

584. **Apparent "Acceleration" of the Moon.**—The action of the moon on the tides is thus to diminish the earth's angular velocity of rotation, and therefore to increase the length of the day. An interval of time measured in solar days will then appear shorter than it really is, and the observed place of the moon will, after any interval of time, be found in advance of the place which has been calculated for the end of the interval on the supposition of no change in the length of the day. Let n be the earth's angular velocity in radians per second, at the beginning of an interval T of time, and $-\dot{n}$ be the rate of change, supposed constant, of n, so that $\dot{n}$ is positive. After any interval t, the angular velocity, in radians *per original second*, is $n-\dot{n}t$. Hence in any short element of time dt the earth will then turn through a smaller angle than it would have turned through if its angular velocity had remained unaltered, by an amount $\dot{n}tdt$. Hence in the interval T the earth turns through an angle smaller by

$$\int_0^T \dot{n}tdt = \tfrac{1}{2}\dot{n}T^2.$$

Thus $\dot{n}$ could be found if the angle by which the moon appears to be in advance of its predicted place in consequence of the retardation of the earth's rotation were known. [It is usual to refer to this as an "acceleration" of the moon's motion. This name is not, however, in accordance with the signification of the term acceleration in dynamics, and would be better called *advance* as Lord Kelvin has suggested. A negative advance might be called a *lag*.]

The advance, however, could not be used with the supposition that there is no alteration of the angular velocity of the moon's orbital motion about the earth. It has been shown in § 537 above that the action of a tangential disturbing force is to cause the moon to recede from the earth, and to move in its orbit about the

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earth with a diminishing linear velocity. Thus both on account of the increase of distance and the diminished velocity the moon's orbital angular velocity is diminished, and the length of the month is increased. This retardation, however, is small in comparison with the apparent acceleration $\dot{n}$ due to the slowing of the earth. If the amount of the tangential force were known, the whole lag of the moon due to this cause might easily be calculated by finding the loss of velocity and increase of distance in the manner indicated in § 537. [See Thomson and Tait's *Natural Philosophy*, App. G.]

585. **Moment of Momentum of Earth-Moon System.**—It is also to be observed that the rates of increase of length of the month and length of the day are connected. The moment of momentum of the earth is about an axis inclined at 23°27′ 10″ to a perpendicular to the plane of the ecliptic. The moon may be taken as moving in the ecliptic. The moment of momentum of the system consists of the moment of momentum of the two bodies due to the motion of their centres, and the moment of momentum due to their rotations. The mass of the earth is 81 times that of the moon, and its distance from the common centroid $\frac{1}{82}$ of the whole distance of the moon from the earth. The angular velocity of the moon's orbital motion is about $n/27{\cdot}3$, if n be that of the earth's rotation. The moment of momentum due to the motion of the centres, taking place it is assumed in the ecliptic, is roughly $En60^2a^2/27{\cdot}3 \times 82$, and that of the rotation of the earth, which is about an axis inclined at 23°27′ 10″ to a perpendicular to the ecliptic, is $\frac{1}{5}Ea^2n$, according to an estimate * which has been formed of the distribution of matter in the earth. Thus the ratio of the former to the latter moment of momentum is about 4·8. The resultant moment is about 5·73 times the earth's rotational momentum, and is about an axis through the centroid of the earth and moon inclined at an angle of about 19°26′ to the earth's axis. Fig. 296 illustrates these statements; AB represents the earth's rotational momentum, AC the orbital moment of momentum. The angle BAC represents the angle between the earth's axis and a perpendicular to the plane of the orbit.

Since, therefore, the rotational momentum of the earth is undergoing diminution in consequence of tidal friction, the moment of momentum of the motions of the centres of the bodies must be increasing at such a rate as to leave the direction and amount of the resultant moment unaltered.

We shall for the present make the assumption that the planes of the moon's orbit and the equator are coincident. This will bring the two sides of the parallelogram in Fig. 296 into line and make $AB + AC = AD$. The plane in which the centres of the two bodies move becomes the invariable plane, and the total moment of momen-

* For it has been found by Serret (*Annales de l'observatoire de Paris*, p. 324, 1859), that if C be the moment of inertia of the earth about its polar axis and A that about an equatorial axis, $(C-A)/A = {\cdot}00327$. This gives, by a table given in § 824 of Thomson and Tait's *Natural Philosophy*, $A = \frac{1}{5}Ma^2$, approximately.

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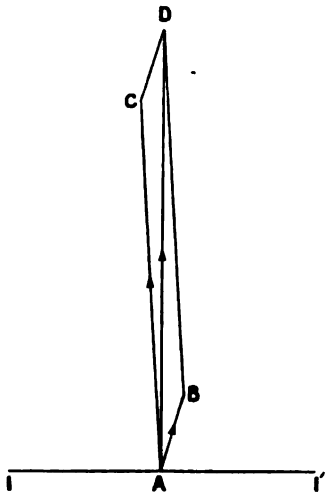
tum 5·8 times the rotational momentum of the earth, only slightly different from the actual value.

586. **Future History of Earth and Moon.**—The moon continues to recede from the earth with transference of moment of momentum from the rotational to the orbital motion, until finally the two bodies turn round their common centroid as if they were parts of a rigid system. The approximate period of this revolution and the distance of the two bodies apart are easily calculated. Let the angular velocity of the orbital motion be n at the beginning of the time considered, and have become n' after the equalisation of the rotational angular velocity of the earth and the orbital angular velocity, while the distance has increased from r to r'. Let h and H denote the orbital moment of momentum at the beginning and end of the time, respectively. Then H is approximately the whole moment of momentum of the system. For, as will be seen at once, the final orbital moment of momentum of the earth is to that of the moon in the ratio M/E, that is, 1/81, about. Also the rotational moment of momentum of the earth after the equalisation is about $\frac{1}{3}$ of that due to the earth's orbital motion, that is, is about $\frac{1}{243}$ of that due to the orbital motion of the moon. We shall therefore neglect the rotational motion of the earth in forming an estimate of the angular velocity of the bodies and their distance apart. We have therefore $nr^2ME/(M+E)=h$, $n'r'^2ME/(M+E)=H$, so that $n^2r^4/n'^2r'^4=h^2/H^2$. Again by the third law of Kepler $n^2/n'^2=r'^3/r^3$, or $n^2r^3/n'^2r'^3=1$. Hence

FIG. 296.



$$\frac{r'}{r}=\frac{H^2}{h^2}=\frac{5\cdot8^2}{4\cdot8^2}=1\cdot46\,;$$

or the distance of the centres of the two bodies apart has increased from about 60 radii of the earth to about 87·6 radii.

Also we obtain $n'^2/n^2=r^3/r'^3=1/1\cdot46^3$, or $n'=n/1\cdot77$. The length of the month, which, when this change has been effected, is also the length of the day, is thus 1·77 times as long as it was formerly, that is, is equal to about 48·36 days of the former length. The tidal spheroid has its long axis directed to the moon, and the moon has ceased to produce tidal retardation of the earth's rotational velocity.

This state of things, however, once attained will not be permanent

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The action of the sun on the solar tides tends to retard the rotation of the earth and bring its period of rotation into coincidence with the length of the year. But owing to the greater distance of the sun this action is insensible compared with that of the moon. When the earth and moon revolve as a rigid body, and the action of the moon has ceased, that of the sun continues to diminish the earth's angular velocity, and the period of rotation of the earth becomes greater than that of the moon. The moon, however, continues to turn the same face towards the earth, and her orbital velocity is not directly affected by the action of the sun. But the terrestrial tidal spheroid is so placed relatively to the moon that the tidal couple, which the attraction of the moon produces, opposes the slowing of the rotation of the earth and a tangential force *opposing* the moon's orbital motion is applied to the moon. This causes the moon to approach the earth in a very gradual spiral and ultimately to fall into the earth.

587. **Past History of Earth and Moon.**—This is only a very slight sketch of a subject which has been discussed fully in some remarkable papers by Professor G. H. Darwin,* who has been able, by dynamical reasoning, to unravel in great measure the probable past history of the system of the earth and moon as well as to predict its future configurations. He has found that there are two sets of circumstances in which motion of the two bodies as a rigid system is possible, one corresponding to a large amount of energy, the other to a comparatively small amount. The former existed when the moon revolved round the earth, almost in contact with its surface, in a period of from three to five hours; the other is the case which has just been considered, in which the moon has been carried out in consequence of the frictional lag of the tides by the reaction of the tidal spheroid.

To the earlier configuration of rigidity the history of the moon can be traced back by a perfectly satisfactory dynamical process, but beyond that point it is impossible to do more than speculate. The earth is losing heat at the present moment from the interior at a very considerable rate, and the interior is at present very hot in comparison with the surface. This loss of heat has gone on for ages, and therefore, the earth must have once been so hot as to be in a plastic condition in which it yielded readily to distorting forces. Consider then the semi-molten, plastic earth rotating in a period of about three hours about an axis inclined at about 12° to a perpendicular to the ecliptic. The planet yielding to centrifugal force is flattened out very much round the equator, and tides are produced by the action of the sun. Pieces fly off from it round the equator like portions of the rim of an over-driven fly-wheel, and one of these larger than the rest, or it may be a mass made up of several pieces consolidated together, forms the moon. The earth and moon then exist revolving about their common centroid, and rotating about their axes in very nearly the same period as that of revolution, so that each turns very nearly the same face towards the other.

* *Loc. cit.* p. 588.

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The exact cause of the breaking off of the moon is uncertain. But Professor Darwin suggests that the free period of vibration of the earth to change of shape was almost equal to half its period of rotation. The free period of a homogeneous globe of the mass of the earth when subjected to change of shape is for the slowest mode of vibration 1 hour 34 minutes. The earth is not homogeneous, and the period is probably somewhat longer. If, then, the earth was rotating in a period somewhere between three and five hours, the solar semi-diurnal tides would have a period of half that amount, and this might be very exactly the free period of oscillation of the body. The tides in the earth would be greatly exaggerated by the agreement of periods, the *resonance* so to speak, and a breaking up of the earth become inevitable.

The two bodies in such close proximity would excite enormous tides in one another, and their revolution with the same face of each toward the other would be unstable. If the period of the satellite in its orbit were a little less than that of rotation of the primary, the satellite, as explained above, would fall back into the primary; if the contrary were the case the satellite would move farther off with diminishing orbital velocity. The existence of the moon shows that immediately after it was thrown off it revolved in its orbit in a somewhat greater period than that of the earth's rotation. As time went on the length of the day and the number of days in the month both increased. The former, however, will continue to increase until the day and month are equal, the latter has already passed its maximum value (about 29), and is at present slowly falling off towards the value one, which it will reach when the day and the month have been equalised.

588. **Action of Couple due to Tidal Friction on Direction of Earth's Axis.**—It is possible to consider the action of tidal friction on the direction of the earth's axis. In consequence of the non-coincidence of the long axis of the tidal spheroid with the plane of the equator, there acts on the earth a couple tending to make it rotate about an axis in the equator, and another couple retarding the rotation about the axis of figure. For let Fig. 297 represent a section through the polar axis ON, and the long axis AB of the tidal figure supposed spheroidal. The axis AB is in advance of the position of the moon, as already explained; the moon revolves in the plane through AB, inclined to the equator EE at the angle ADE, and intersecting the plane of the equator in the line OD. The couple applied by the moon is represented by the two forces F, F applied at A and B in the plane of revolution of the moon. That couple resolves into two components, one about the axis OE, the other about the axis of rotation ON. The action of the latter is to retard the earth's rotation, of the former to produce moment of momentum about OE. As the moon revolves the mean position of the tidal spheroid follows it, and the axis OE changes in correspondence. But, as shown in § 260 above, the result of the action of a couple about an axis moving, as OE does, with the body, is to produce

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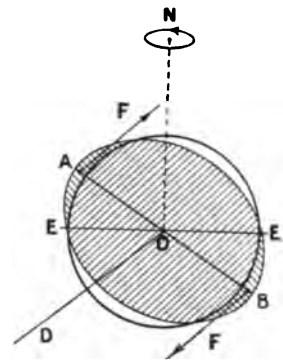
turning of the body about an axis at right angles both to OE and to the axis of rotation. Thus, as will be obvious from the directness of the rotation and the couple, the axis ON turns towards OA, that is, the inclination of the equator to the ecliptic increases. Going back to Fig. 296 (and supposing that AC, the greater of the sides of the parallelogram meeting at A, now represents the *rotational* moment), it will be obvious that as the obliquity of the plane of the equator to the invariable plane increases, the inclination of the plane of the orbit to the invariable plane also increases, unless the rotational moment AC diminishes fast enough to counteract the effect of its change of direction. If the period of rotation, as here supposed, is small, and the viscosity is great enough, the couple produces an increase of the inclination of the axis, and so the inclination of the plane of the orbit to the invariable plane is also increased. The rotational moment AC is great and the angle BAD small. Hence the increase of the angle BDA causes an increase of the angle BAD even if AC be diminishing. The opposite will happen at a much later stage in the history of the earth and moon. Then the rotational moment is very small in comparison with the orbital moment, which is now very nearly equal to the total moment. Thus, diminution of the rotational moment even with increase of angle BDA will mean diminution of the angle BAD. Of course all this is affected by the action of the sun, which has not so far been referred to. We can only state here that the inclination of the lunar orbit to what Professor Darwin calls the "proper plane" (that is, the plane which replaces the invariable plane when the sun's influence is taken into account) at first increased until it reached a maximum, then began to diminish, and is still diminishing. The diminution will continue until the two bodies revolve as a rigid system about their common centroid.

FIG. 297.



589. **Effect of Tidal Friction on Eccentricity of Moon's Orbit.**—One other effect of tidal friction may be referred to here—the change of eccentricity of the orbit which it produces. As the moon revolves about the earth its distance changes, and therefore also the magnitude of the tide produced. At perigee, when the moon is nearest to the earth, the tide is greatest, and at apogee the tide is least. Consequently the effect of tidal friction may be regarded as made up of a constant mean value, and a periodic part which is added to the constant part of the effect at perigee, and subtracted from it at apogee. Hence the moon when coming towards perigee and passing away from it is acted on by a tangential force which increases its distance and diminishes its angular velocity, and does so beyond the average. When the moon comes to apogee it is therefore farther

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out than it would have been if the average effect had been produced; thus the apogeal distance is increased. In the same way the moon is acted on while in apogee by a tangential force less than the average, and in consequence swings round to perigee at a smaller distance than it would have had if the mean force had acted in apogee. Thus so far the perigeal distance is diminished, and the apogeal increased, that is, the eccentricity is increased.

But it is also to be remembered that the moon moves faster when in perigee and slower in apogee than the average. If the period of revolution of the moon be equal to that of rotation of the earth the moon will in perigee outstrip and fall behind the terrestrial tidal spheroid in turning. But the tangential force on the moon is against or with the orbital revolution of the moon according as the moon is behind or ahead of the tidal spheroid. Hence the moon coming round after being delayed in apogee is behind the tidal spheroid, and therefore has its distance—that is, the perigeal distance—increased. In the same way the apogeal distance is diminished. Thus the eccentricity is diminished. It appears, therefore, that there must be some period of revolution for which the eccentricity remains constant.

There are other causes which modify the eccentricity, but for these and for fuller information on the subjects of which a rapid sketch is given here, reference must be made to Professor Darwin's papers* and to his article on "The Tides" in the *Encyclopædia Britannica.*

590. **Amount of Tidal Retardation of Earth's Rotation.**—As to the amount of the tidal retardation of the earth's rotation there is still a good deal of uncertainty. Astronomers are not agreed as to the amount of the acceleration of the moon's motion due to secular changes in the eccentricity of the earth's orbit. According to the late Professor J. C. Adams, the acceleration of the motion in mean longitude as given by calculation from the times at which certain eclipses, which are recorded as having been observed at certain places, must have occurred, is 11·4 seconds of angle per century, so that in a century the mean longitude is 5·7 seconds greater than it would have been without this acceleration. This advance was increased by Delaunay by the addition of small terms to 6·1 seconds of angle.

Now the observed value of the advance is about 12", and is thus about double the theoretical value as calculated by Adams. It was suggested first by Delaunay that the difference could be accounted for by retardation of the earth's rotation; and if 6" be put down to this cause the earth would in a century, according to an estimate communicated by Adams to the authors of Thomson and Tait's "Natural Philosophy," fall about 22 seconds behind a perfect time-keeper, accurately adjusted at the beginning of the century to keep mean solar time. It was, however, Professor Adams' opinion that the exact amount of tidal retardation was quite uncertain. See *Thomson and Tait*, Part II., App. G, a.

* *On the Tides of a Viscous Spheroid* (*Phil. Trans.* Part 1, 1879), and *Memoirs in the Proc. Roy. Soc.* 1879 to 1881.

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CHAPTER XV.

ELASTICITY.

591. **Preliminary Notions and Definitions. Homogeneous Strain.**—Elasticity been defined in chap. ix., and explanations regarding elastic forces, and the distinction between them and forces due to viscosity, also definitions of such terms as stress, strain, isotropy, homogeneity, &c., have been stated, which it is unnecessary to repeat. We shall give here such a sketch of the elasticity of isotropic bodies as may be serviceable in enabling experimental researches on the subject to be understood, leaving the complete theoretical discussion to be studied in the standard works on the mathematical theory of elasticity.*

An elastic body or substance is said to be subjected to homogeneous strain when its particles (not the ultimate molecules, but the smallest elements considered as having the properties of the substance) are so displaced that parallel straight lines in it marked out by particles remain parallel straight lines, and all distances between particles in it, taken parallel to any one chosen direction, are altered in the same ratio. This ratio is, however, in the general case different for different directions, but if given for two non-parallel directions, say two directions at right angles to one another, it is given, of course, for all directions parallel to a plane containing these, and if given for three directions, no two of which are parallel, say three directions at right angles to one another, it is given for all directions in space.

Since parallel lines remain parallel lines, parallel planes remain also parallel planes. A circle becomes an ellipse, and two diameters of a circle at right angles to one another are strained into diameters of the ellipse, which are parallel each to the tangent drawn to the ellipse at the extremity of the other; that is, these diameters of the circle are strained into conjugate diameters of the ellipse.

Also it is clear that homogeneous strain changes a sphere into an ellipsoid, and that any three diameters of the sphere at right angles to one another are strained into three conjugate diameters of the ellipsoid. In general the angles between straight lines in the body are changed by the strain, and a rectangular system of straight lines

* See *Math. Th. of Elasticity*, by A. E. H. Love. *Elasticité*, by Clebsch, edited by Barre de St. Venant,. Thomson and Tait's *Nat. Phil.*; *Hist. of Math. Th. of Elasticity*, by Todhunter, continued and completed by Pearson.

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becomes an oblique system, but there are, as we shall see in § 593, always three straight lines at right angles to one another in the unstrained body which remain at right angles to one another after the strain, and are therefore parallel to the principal axes of the ellipsoid into which the strain alters any spherical portion of the body. That ellipsoid is called the *strain ellipsoid.* An ellipsoid, of course, remains an ellipsoid unless it is changed into a sphere.

Since the state of strain of a body is not altered by displacement of the body as a whole, or rotation of the body about an axis, it will not be affected by supposing a point O of the body to remain fixed. Let then the positions of particles of the body be referred to rectangular axes drawn from the fixed point O as origin. Then, if x, y, z represent the original coordinates of the particle, x', y', z' the coordinates of the particle after the strain, and the strain be homogeneous, x', y', z' must be linear functions x, y, z according to the scheme

$$\left.\begin{aligned} x' &= a_1x + b_1y + c_1z \\ y' &= a_2x + b_2y + c_2z \\ z' &= a_3x + b_3y + c_3z. \end{aligned}\right\} \qquad (1)$$

For these equations, as it is easy to see, by the most elementary theorems of solid geometry, fulfil all the conditions of homogeneous strain; and no other form of relations will do so. For example, the linear relation $Ax + By + Cz + D = 0$ is the equation of a plane drawn in the body, and it is clear from (1) that a linear relation of the same form in x', y', z' is also satisfied by the coordinates of the strained position of the particles. Hence the particles still lie in a plane, and it is clear also that particles which lie in parallel planes before strain lie in parallel planes after the strain, and that, therefore, parallel lines remain parallel lines.

By solving for x, y, z in terms of x', y', z' we obtain three equations of the form

$$\left.\begin{aligned} x &= \alpha_1x' + \beta_1y' + \gamma_1z' \\ y &= \alpha_2x' + \beta_2y' + \gamma_2z' \\ z &= \alpha_3x' + \beta_3y' + \gamma_3z'. \end{aligned}\right\} \qquad (2)$$

The coefficients α_1, β_1, &c., are formed by the rules given by the theory of simple equations, or they can be determined directly for any given case.

592. **Strain Ellipsoid.**—Consider now a sphere described with O as centre in the unstrained body. Its equation is

$$x^2 + y^2 + z^2 = r^2.$$

The condition fulfilled by the coordinates x', y', z' of any point of the surface of the sphere after strain is by (2) expressed by the equation

$$(\alpha_1x' + \beta_1y' + \gamma_1z')^2 + \ldots = r^2. \qquad (3)$$

which is the equation of an ellipsoid, in fact of the strain ellipsoid. It may be written in the form

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$$Ax'^2 + By'^2 + Cz'^2 + 2Dy'z' + 2Ez'x' + 2Fx'y' = r^2, \tag{4}$$

if

$$A = a_1^2 + a_2^2 + a_3^2, \ \ldots, \quad D = \beta_1\gamma_1 + \beta_2\gamma_2 + \beta_3\gamma_3, \ \ldots,$$

where the blanks are to be filled up with corresponding values for B, C, E, F.

593. **Principal Axes of Strain Ellipsoid.**—To find the axes of this ellipsoid, we observe that the direction-cosines of a normal are proportional to $Ax' + Fy' + Ez'$, $Fx' + By' + Dz'$, $Ex' + Dy' + Cz'$, and that the direction-cosines of a radius-vector are proportional to x', y', z'. Hence, if the normal at a point on the surface is coincident with a radius-vector drawn to the point, the direction-cosines of the normal are proportional to x', y', z', that is $(Ax' + Fy' + Ez')/x' = (Fx' + By' + Dz')/y' = (Ex' + Dy' + Cz')/z' = k$, say. Hence

$$\left.\begin{aligned} (A-k)x' + \quad Fy' + \quad Ez' &= 0 \\ Fx' + (B-k)y' + \quad Dz' &= 0 \\ Ex' + \quad Dy' + (C-k)z' &= 0, \end{aligned}\right\} \tag{5}$$

which gives by elimination of x', y', z a cubic for the determination of k (see §168 above). This equation, as can easily be shown, has three real roots, and there are, therefore, three directions at right angles to one another which are normal to the surface. The angles between these are not altered by the strain. By (4), since $r'^2 = x'^2 + y'^2 + z'^2$

$$r'^2 - r^2 = (1-A)x'^2 + (1-B)y'^2 + (1-C)z'^2 - 2Dy'z' - 2Ez'x' - 2Fx'y'.$$

If, then, we assign a constant value, K^2, to the right hand side of this equation, we see that for all radii-vectores of the quadric surface

$$(1-A)x'^2 + (1-B)y'^2 + (1-C)z'^2 - 2Dy'z' - 2Ez'x' - 2Fx'y' = K^2, \tag{6}$$

there is the same excess of r'^2 over r^2. Or, if $x'/r' = l'$, $y'/r' = m'$, $z'/r' = n'$, we have

$$(1-A)l'^2 + (1-B)m'^2 + (1-C)n'^2 - 2Dm'n' - 2En'l' - 2Fl'm' = \frac{K^2}{r'^2}$$

The quantity on the left is $(r'^2 - r^2)/r'^2$, and the equation asserts that along different radii of the quadric surface defined by (6) this ratio varies inversely as the square of the radius-vector.

594. **Theory of Small Strains.**—This discussion is, perhaps, sufficient to indicate the mode of treating finite strain. We shall now suppose the displacements relative to the point O to be small, and consider only the changed positions of points near O. This will suffice for most physical purposes to which the theory is applied. Let u, v, w denote the displacements of a point, near O, the coordinates of which relative to fixed axes through O are ξ, η, ζ. The displacements u, v, w are, we suppose, finite continuous functions of ξ, η, ζ. Then, since u, v, w are zero at O, we have

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$$\left.\begin{aligned} u &= \xi\frac{\partial u}{\partial x}+\eta\frac{\partial u}{\partial y}+\zeta\frac{\partial u}{\partial z} \\ v &= \xi\frac{\partial v}{\partial x}+\eta\frac{\partial v}{\partial y}+\zeta\frac{\partial v}{\partial z} \\ w &= \xi\frac{\partial w}{\partial x}+\eta\frac{\partial w}{\partial y}+\zeta\frac{\partial w}{\partial z}. \end{aligned}\right\} \quad (7)$$

The quantities $\partial u/\partial x$, &c., have the values for the point O, and are regarded as invariable.

The strain, the only limitation imposed on which has been that it is to be small, is therefore homogeneous to the degree of approximation considered.

Now, if $\xi'\ \eta',\ \zeta'$, denote the coordinates of P after the displacement, $\xi'=\xi+u$, $\eta'=\eta+v$, $\zeta'=\zeta+w$, and the last equation may be written

$$\left.\begin{aligned} \xi' &= \xi\left(1+\frac{\partial u}{\partial x}\right)+\eta\frac{\partial u}{\partial y}+\zeta\frac{\partial u}{\partial z} \\ \eta' &= \xi\frac{\partial v}{\partial x}+\eta\left(1+\frac{\partial v}{\partial y}\right)+\zeta\frac{\partial v}{\partial z} \\ \zeta' &= \xi\frac{\partial w}{\partial x}+\eta\frac{\partial w}{\partial y}+\zeta\left(1+\frac{\partial w}{\partial z}\right). \end{aligned}\right\} \quad (8)$$

But, since ξ, η, ζ and $\partial u/\partial x$, &c., are all small, $\xi\partial u/\partial x$, $\eta\partial u/\partial y$, ... are to the second order of small quantities the same as $\xi'\partial u/\partial x$, &c., so that we may write

$$\left.\begin{aligned} \xi &= \xi'\left(1-\frac{\partial u}{\partial x}\right)-\eta'\frac{\partial u}{\partial y}-\zeta'\frac{\partial u}{\partial z} \\ \eta &= -\xi'\frac{\partial v}{\partial x}+\eta'\left(1-\frac{\partial v}{\partial y}\right)-\zeta'\frac{\partial v}{\partial z} \\ \zeta &= -\xi'\frac{\partial w}{\partial x}-\eta'\frac{\partial w}{\partial y}+\zeta'\left(1-\frac{\partial w}{\partial z}\right). \end{aligned}\right\} \quad (9)$$

595. **Unital Elongation of any line. Elongation Quadric.**—If $OP=\rho$ and $OP'=\rho'=\rho+\delta\rho$ we have $\rho\,\delta\rho=\xi\delta\xi+\eta\delta\eta+\zeta\delta\zeta$, and if l, m, n be the direction-cosines of $O\,P$ so that $l=\xi/\rho$, ... we have

$$\frac{\delta\rho}{\rho}=l\frac{\delta\xi}{\rho}+m\frac{\delta\eta}{\rho}+n\frac{\delta\zeta}{\rho}.$$

But $\delta\xi=u$, $\delta\eta=v$, $\delta\zeta=w$, so that by (7) and the values $(l, m, n)=(\xi/\rho, \eta/\rho, \zeta/\rho)$,

$$\frac{\delta\rho}{\rho}=l\left(l\frac{\partial u}{\partial x}+m\frac{\partial u}{\partial y}+n\frac{\partial u}{\partial z}\right)+m\left(l\frac{\partial v}{\partial x}+m\frac{\partial v}{\partial y}+n\frac{\partial v}{\partial z}\right)$$
$$+\,n\left(l\frac{\partial w}{\partial x}+m\frac{\partial w}{\partial y}+n\frac{\partial w}{\partial z}\right).$$

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$$\left.\begin{aligned}&= l^2\frac{\partial u}{\partial x}+m^2\frac{\partial v}{\partial y}+n^2\frac{\partial w}{\partial z}+mn\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)\\&\qquad+nl\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)+lm\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right).\end{aligned}\right\}\tag{10}$$

This is frequently written in the form

$$\frac{\delta\rho}{\rho}=el^2+fm^2+gn^2+2s_1mn+2s_2nl+2s_3lm,\tag{10'}$$

where $e, f, g, 2s_1, 2s_2, 2s_3$ are the values of the co-efficients of l^2, m^2, &c., in the previous expression.

The quantity $\delta\rho/\rho$ is the unital elongation of the line OP. It is usual to call it simply the *elongation* of OP. If a constant value k^2 be assigned to $\rho\,\delta\rho$ we have from (10′) by multiplying by ρ^2

$$e\xi^2+f\eta^2+g\zeta^2+2s_1\eta\zeta+2s_2\zeta\xi+2s_3\xi\eta=k^2\tag{11}$$

which is the equation of a quadric surface. For all radii-vectores of this surface $\rho\,\delta\rho$ is a constant k^2, and therefore $\delta\rho/\rho$ is inversely proportional to ρ^2. This quadric surface is called the *elongation quadric.*

596. **Cone of No Elongation and Cone of Constant Elongation.**—If $\delta\rho$ be negative for a certain range of directions (for example, if the substance is extended along all lines in one direction, and compressed along all lines in a direction at right angles to the former), the conjugate quadric

$$e\xi^2+f\eta^2+g\zeta^2+2s_1\eta\zeta+2s_2\zeta\xi+2s_3\xi\eta=-k^2\tag{12}$$

will represent the (negative) elongation for these directions. The two regions are separated by the cone

$$e\xi^2+f\eta^2+g\zeta^2+2s_1\eta\zeta+2s_2\zeta\xi+2s_3\xi\eta=0,\tag{13}$$

which touches both these quadrics at an infinite distance from O. This is called the *cone of no elongation.*

The direction-cosines of the radii-vectores for which the elongation has a given value ϵ satisfy the equation

$$el^2+fm^2+gn^2+2s_1mn+2s_2nl+2s_3lm=\epsilon,$$

or since $l^2+m^2+n^2=1$

$$(e-\epsilon)l^2+(f-\epsilon)m^2+(g-\epsilon)n^2+2s_1mn+2s_2nl+2s_3lm=0.$$

They are therefore generators of the cone

$$(e-\epsilon)\xi^2+(f-\epsilon)\eta^2+(g-\epsilon)\zeta^2+2s_1\eta\zeta+2s_2\zeta\xi+2s_3\xi\eta=0.\tag{14}$$

597. **Alteration of Direction of Line. Lines of No Rotation.**—If the direction-cosines of a line be altered from l, m, n to $l+\delta l$, $m+\delta m$, $n+\delta n$, the line has been turned through a small angle $\delta\theta$ given by

$$\delta\theta=(\delta l^2+\delta m^2+\delta n^2)^{\frac{1}{2}}.\tag{15}$$

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or

$$\cos\delta\theta = l(l+\delta l) + m(m+\delta m) + n(n+\delta n),$$

and therefore, since $1-\cos\delta\theta = 2\sin^2\frac{1}{2}\delta\theta = \frac{1}{2}\delta\theta^2$ we have

$$\begin{aligned}\delta\theta^2 &= -2(l\delta l + m\delta m + n\delta n)\\ &= l^2+m^2+n^2-\{(l+\delta l)^2+(m+\delta m)^2+(n+\delta n)^2\}\\ &\qquad +\delta l^2+\delta m^2+\delta n^2;\end{aligned}$$

or

$$\delta\theta^2 = \delta l^2 + \delta m^2 + \delta n^2,$$

since the first line of the expression on the right is $1-1$, or zero.

Now we have from $l = \xi/\rho$, &c., $d\xi = u$, &c., and (7),

$$\delta l = \delta\xi/\rho - \xi\delta\rho/\rho^2 = l\,\partial u/\partial x + m\,\partial u/\partial y + n\,\partial u/\partial z - \epsilon l,$$

and similarly for δm, δn. Hence

$$\left.\begin{aligned}\delta\theta^2 = &\left\{l\left(\frac{\partial u}{\partial x}-\epsilon\right)+m\frac{\partial u}{\partial y}+n\frac{\partial u}{\partial z}\right\}^2\\ &+\left\{\frac{\partial v}{\partial x}+m\left(\frac{\partial v}{\partial y}-\epsilon\right)+n\frac{\partial v}{\partial z}\right\}^2\\ &+\left\{l\frac{\partial w}{\partial x}+m\frac{\partial w}{\partial y}+n\left(\frac{\partial w}{\partial z}-\epsilon\right)\right\}^2.\end{aligned}\right\}\quad(16)$$

If $\delta\theta$ vanish, each of the three terms of this equation must vanish. Hence we obtain

$$\left.\begin{aligned}&l\left(\frac{\partial u}{\partial x}-\epsilon\right)+m\frac{\partial u}{\partial y}+n\frac{\partial u}{\partial z}=0\\ &l\frac{\partial v}{\partial x}+m\left(\frac{\partial v}{\partial y}-\epsilon\right)+n\frac{\partial v}{\partial z}=0\\ &l\frac{\partial w}{\partial x}+m\frac{\partial w}{\partial y}+n\left(\frac{\partial w}{\partial z}-\epsilon\right)=0,\end{aligned}\right\}\quad(17)$$

which leads to the determinant

$$\left.\begin{vmatrix}\frac{\partial u}{\partial x}-\epsilon, & \frac{\partial u}{\partial y}, & \frac{\partial u}{\partial z}\\ \frac{\partial v}{\partial x}, & \frac{\partial v}{\partial y}-\epsilon, & \frac{\partial v}{\partial z}\\ \frac{\partial w}{\partial x}, & \frac{\partial w}{\partial y}, & \frac{\partial w}{\partial z}-\epsilon\end{vmatrix}=0\right\}\quad(18)$$

which is a cubic for the determination of the elongation ϵ. This cubic, of course, must have one real root at least. The other two roots may be either both imaginary or both real. Each root gives values of the ratios of l, m, n.

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A line which remains unchanged in direction is called a line of no rotation. There is always one such line at least.

598. **Superposition of Small Variations.**—If two sets of small variations u_1, v_1, w_1, u_2, v_2, w_2 be given successively to ξ, η, ζ, it is easy to see that, when quantities of a higher order of smallness than those to which the approximations here made are limited are neglected, the total resultant variation is u_1+u_2, v_1+v_2, w_1+w_2, and is independent of the order in which the variations are imposed. This the reader may easily verify for himself. We go on to the analysis of an ordinary small strain into a *pure strain* and a rotation.

599. **Ordinary Small Strain resolved into a Pure Strain and a Rotation.**—According to a usual notation we write $2\theta_1$, $2\theta_2$, $2\theta_3$ for the quantities $\partial w/\partial y-\partial v/\partial z$, $\partial u/\partial z-\partial w/\partial x$, $\partial v/\partial x-\partial u/\partial y$. Hence, by the values of s_1, s_2, s_3 stated above

$$\partial w/\partial y=s_1+\theta_1,\quad \partial u/\partial z=s_2+\theta_2,\quad \partial v/\partial x=s_3+\theta_3,$$
$$\partial v/\partial z=s_1-\theta_1,\quad \partial w/\partial x=s_2-\theta_2,\quad \partial u/\partial y=s_3-\theta_3.$$

Thus we obtain by (7)

$$\left.\begin{aligned} u&=e\xi+s_3\eta+s_2\zeta+(\theta_2\zeta-\theta_3\eta)\\ v&=s_3\xi+f\eta+s_1\zeta+(\theta_3\xi-\theta_1\zeta)\\ w&=s_2\xi+s_1\eta+g\zeta+(\theta_1\eta-\theta_2\xi).\end{aligned}\right\}\qquad(19)$$

Now, by the discussion in § 361 above, in which it was shown that, u, v, w denoting component velocities of the fluid, the quantities $\frac{1}{2}(\partial w/\partial y-\partial v/\partial z)$, &c., were the components of spin about the direction l, m, n there specified, it will be clear that θ_1, θ_2, θ_3 denote components of angular displacement of a small portion of the body in the neighbourhood of O, about an axis the direction cosines of which are proportional to θ_1, θ_2, θ_3. The element is to be considered as turned without change of relative position of the particles, and the parts of u, v, w in brackets on the right of (19) are the relative shifts of P due to the rotation. The shifts due to change of relative position of the particles are the remaining parts of the expressions in (19), and are called the components of *pure strain*. We shall denote the latter by u_1, v_1, w_1, and the rotational shifts by u_2, v_2, w_2. Hence

$$u=u_1+u_2,\quad v=v_1+v_2,\quad w=w_1+w_2,$$

and these different parts may be imposed in any order without affecting the result.

600. **Pure Strain made up of Three Elongations and Three Shears. Specification of Shearing Strain.**—It is important to consider a pure strain as made up of six partial strains according to the following scheme,

(1) $u_1=e\xi$, $v_1=0$, $w_1=0$,
(2) $u_2=0$, $v_2=f\eta$, $w_2=0$,
(3) $u_3=0$, $v_3=0$, $w_3=g\zeta$,
(4) $u_4=0$, $v_4=s_1\zeta$, $w_4=s_1\eta$,
(5) $u_5=s_2\zeta$, $v_5=0$, $w_5=s_2\xi$,
(6) $u_6=s_3\eta$, $v_6=s_3\xi$, $w_6=0$.

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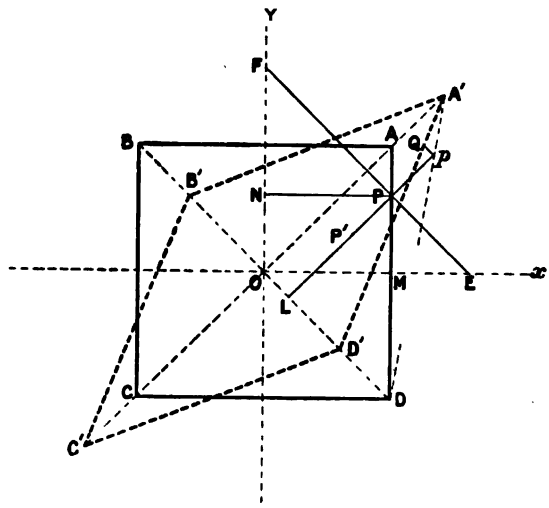
The first three partial strains represent simple elongations of the body in the directions of the axes of x, y, z respectively. The unital elongation or simply the elongation in the first case is e, in the second f, in the third g. We call the strains, therefore, simple elongations of amounts e, f, g. These may be either positive or negative of course, and if negative may be referred to either as "negative elongations" or as contractions.

In consequence of the increments of ξ, η, ζ involved in these strains the volume of the rectangular element, the edges of which are ξ, η, ζ, is changed from $\xi\eta\zeta$ to $\xi(1+e)\eta(1+f)\zeta(1+g)$, that is, is increased per unit of its former amount by $e+f+g$.

Let Fig. 298 represent a section of the body through the point

FIG. 298.



P (at which the strain is considered) by a plane perpendicular to Oz. OM, MP represent the co-ordinates ξ, η of P. Through P draw the square $ABCD$, the sides of which are parallel to the axes of x and y, and the centre of which is at O. This may be regarded as a section of a cubical portion of the body, the centre of which is at the fixed point which is taken as the origin of co-ordinates. Without loss of generality we may take P as situated in the plane of x, y, so that the centre of the section supposed is really the origin.

Now, through P draw a line meeting the axes Ox, Oy in E and F, so that $OE = OF$, and resolve the component displacements $s_3\eta$, $s_3\xi$ (u_6, v_6), of P along and at right angles to this line. The first resolution gives $s_3(\eta - \xi)/\sqrt{2}$ as the displacement of P in the direction from E towards F. The second resolution gives $s_3(\xi + \eta)/\sqrt{2}$ as the displacement of P parallel to OA. Draw PL parallel to AO to meet OD in L; the lengths of OL, LP are obviously $(\xi - \eta)/\sqrt{2}$, $(\xi + \eta)/\sqrt{2}$

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OL, LP represent the displacements of P, which are therefore a displacement $s_3.OL$ from L towards O, and another $s_3.LP$ from L towards P. Thus, if P lie in the diagonal OA, the former displacement is zero, and if P be on the diagonal OD, the latter is zero. The unital displacement, it is important to notice, is s_3 in each case.

The same result would be found for any other position of P, say P', by making a similar construction and proceeding in the same way. Thus in consequence of the displacements considered the body near O sustains a (unital) elongation s_3 along every line parallel to OA, and an equal contraction along every line parallel to OD.

A cubical portion of the body takes the shape indicated in Fig. 298 by the rhombus $A'B'C'D'$, which shows, for distinctness in the diagram, the effect of a large shearing strain. The point P is moved to Q by the displacements Pp, pQ of the amounts and in the directions represented by these lines. This is called a *shearing strain*, or simply a *shear*, and OA, OD are called the axes of the shear.

The numerical amount of the shear is taken as $2s_3$, which for a small strain can easily be seen to be the amount by which after the strain the angle $B'A'D'$ is less than $\pi/2$, or by which the angle $C'D'A'$ exceeds $\pi/2$. It will be observed that to the degree of approximation here employed this strain does not alter the volume of the body.

In the same way the other two pairs of strains v_4, w_4, u_5, w_5 can be shown to be shears in planes perpendicular to the axes of x and y respectively, and to have axes along the lines bisecting the angles between the axes which lie in these planes.

601. **Analytical Conditions of Pure Strain. Principal Axes of Elongation-Quadric. Strain Invariants.**—If the strain be pure we have $\theta_1 = \theta_2 = \theta_3 = 0$, that is $\partial v/\partial z = \partial w/\partial y$, &c. This proves that u, v, w are the differential co-efficients $\partial\phi/\partial x$, $\partial\phi/\partial y$, $\partial\phi/\partial z$ of a function ϕ of x, y, z—a fact we shall make some use of presently. The components of displacement then reduce to $e\xi + s_3\eta + s_2\zeta$, &c., and the equation of the elongation quadric is

$$e\xi^2 + f\eta^2 + g\zeta^2 + 2s_1\eta\zeta + 2s_2\zeta\xi + 2s_3\xi\eta = k^2. \tag{20}$$

Hence to find the direction-cosines of the principal axes of the elongation-quadric we have

$$\left.\begin{aligned} el + s_3m + s_2n &= \kappa l \\ s_3l + fm + s_1n &= \kappa m \\ s_2l + s_1m + gn &= \kappa n. \end{aligned}\right\} \tag{21}$$

which give a cubic for κ, namely,

$$\left.\begin{aligned} \kappa^3 - (e+f+g)\kappa^2 + (fg + ge + ef - s_1^2 - s_2^2 - s_3^2)\kappa \\ - efg - 2s_1s_2s_3 + es_1^2 + fs_2^2 + gs_3^2 = 0. \end{aligned}\right\} \tag{22}$$

The roots of this cubic equation are all real, and by taking them in

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succession and substituting in (20) we obtain the three sets of values of l, m, n required for the axes.

The determinant

$$\begin{vmatrix} e-\kappa, & s_3, & s_2 \\ s_3, & f-\kappa, & s_1 \\ s_2, & s_1, & g-\kappa \end{vmatrix}$$

obtained by eliminating l, m, n from (21) is called the *discriminant* of

$$(e-\kappa)\xi^2+(f-\kappa)\eta^2+(g-\kappa)\zeta^2+2s_1\eta\zeta+2s_2\zeta\xi+2s_3\xi\eta,$$

for the reason that its vanishing is the necessary and sufficient condition that the last-written expression may be resolvable into two factors of the form $\lambda\xi+\mu\eta+\nu\xi$. Now it is clear that if this expression is changed by transformation of co-ordinates without change of origin it must still be resolvable into factors for the same value of κ, inasmuch as $\kappa(\xi^2+\eta^2+\zeta^2)$ remains unchanged. Thus the values of κ do not change because of the transformation and consequently the co-efficients in the cubic (21) remain unchanged. Thus, however the elongation-quadric may be transformed, by being referred to new rectangular axes from the same origin, the quantities

$$e+f+g,$$
$$fg+ge+ef-s_1^2-s_2^2-s_3^2, \quad efg+2s_1s_2s_3-es_1^2-fs_2^2-gs_3^2,$$

remain unchanged. These are called *invariants* of the strain.

602. **Cubical Dilatation.**—If the equation of the elongation-quadric be transformed to its principal axes it becomes

$$\epsilon_1\xi^2+\epsilon_2\eta^2+\epsilon_3\zeta^2=k^2. \tag{23}$$

The quantities ϵ_1, ϵ_2, ϵ_3 are called the principal elongations. If they are all positive the quadric is an ellipsoid. By the first invariant relation indicated above we have

$$e+f+g=\epsilon_1+\epsilon_2+\epsilon_3. \tag{24}$$

We may now find with reference to the same axes the equation of the locus on which after the strain lie the particles which before the strain lay on a sphere with its centre at O. Before the strain the co-ordinates of any point were ξ, η, ζ which satisfied the relation $\xi^2+\eta^2+\zeta^2=k^2$. After the strain the co-ordinates of the same point have become $\xi'=(1+\epsilon_1)\xi$, $\eta'=(1+\epsilon_2)\eta$, $\zeta'=(1+\epsilon_3)\zeta$, so that, the accents being dropped, the equation of the locus is

$$\frac{\xi^2}{(1+\epsilon_1)^2}+\frac{\eta^2}{(1+\epsilon_2)^2}+\frac{\zeta^2}{(1+\epsilon_3)^2}=k^2. \tag{25}$$

This denotes an ellipsoid whatever the signs of ϵ_1, ϵ_2, ϵ_3 may be. The ratio of the volume of the ellipsoid to the volume of the sphere is $(1+\epsilon_1)(1+\epsilon_2)(1+\epsilon_3)$, or $1+\epsilon_1+\epsilon_2+\epsilon_3$ to quantities of the first order

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of smallness. The ratio of the increase of volume to the original volume is therefore $\epsilon_1+\epsilon_2+\epsilon_3$. This is called the *dilatation*. Equation (24) shows that the dilatation has the same form of expression whatever system of rectangular axes is chosen.

603. **Equation of Strain Ellipsoid.**—The amount of the dilatation and the form of the strain ellipsoid would not be affected by imposing the rotations θ_1, θ_2, θ_3 which would merely turn the ellipsoid round as a whole. The equation of the ellipsoid referred to the original axes is got by putting for ξ, η, ζ in the equation of the sphere

$$(1-e)\xi-s_3\eta-s_2\zeta,\quad -s_3\xi+(1-f)\eta-s_1\zeta,\quad -s_2\xi-s_1\eta+(1-g)\zeta,$$

and is

$$(1-2e)\xi^2+(1-2f)\eta^2+(1-2g)\zeta^2-4s_1\eta\zeta-4s_2\zeta\xi-4s_3\xi\eta=k^2. \quad (26)$$

604. **Simple Elongation represented by Dilatation and Two Shears.**—By (19) it is clear that the displacement in a pure strain is capable of being represented by three simple elongation displacements in the directions of the principal axes of the elongation-quadric. We can now show that a simple elongation is capable of being represented by a uniform dilatation and two shears. Thus consider the scheme

$$\begin{array}{lll} \frac{1}{3}(\epsilon_1\xi, & \epsilon_1\eta, & \epsilon_1\zeta) \\ \frac{1}{3}(\epsilon_1\xi, & -\epsilon_1\eta, & 0) \\ \frac{1}{3}(\epsilon_1\xi, & 0, & -\epsilon_1\zeta). \end{array}$$

The first line contains the component displacements due to a uniform dilatation ϵ_1, the other two lines contain the displacements due to two shears each of amount $\frac{2}{3}\epsilon_1$, one in the plane of ξ, η, the other in the plane of ξ, ζ.

605. **Pure Strain resolved into Dilatation and Three Shears.**—In the same way the displacement $\epsilon_2\eta$ may be shown to be equivalent to the displacements due to a uniform dilatation of amount ϵ_2, and two shears of amount $\frac{2}{3}\epsilon_2$, in the planes of η, ζ and ξ, η. Similarly the displacement $\epsilon_3\zeta$ may be dealt with.

The three displacements are therefore equivalent to the scheme of displacements

$$\begin{array}{lll} \frac{1}{3}\{(\epsilon_1+\epsilon_2+\epsilon_3)\xi, & (\epsilon_1+\epsilon_2+\epsilon_3\eta), & (\epsilon_1+\epsilon_2+\epsilon_3)\zeta\} \\ \frac{1}{3}\{0, & (\epsilon_2-\epsilon_3)\eta, & -(\epsilon_2-\epsilon_3)\zeta\} \\ \frac{1}{3}\{(\epsilon_1-\epsilon_3)\xi, & 0, & -(\epsilon_1-\epsilon_3)\zeta\} \\ \frac{1}{3}\{(\epsilon_1-\epsilon_2)\xi, & -(\epsilon_1-\epsilon_2)\eta, & 0. \quad \} \end{array}$$

In this way a pure strain is resolved into a uniform dilatation and three shears.

It is important to notice that any two pure strains applied in succession to a body do not as a general rule produce a pure strain. This is at once seen by first changing, by § 599, ξ, η, ζ to ξ', η', ζ' by pure strain, then applying a pure strain to ξ', η', ζ'. The relation of co-efficients necessary for pure strain will not

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then be fulfilled by the final component displacements obtained, except in special cases. It is, however, always possible to apply a third pure strain which, with the other two, will give a pure strain. We have only to find a strain of general type which produces an equal and opposite rotation to that produced by the two preceding pure strains.

Similarly three pure strains can be specified which together will produce rotation merely without distortion of the substance.

606. **Specification of Stress. Forces on Elementary Tetrahedron of the Body.**—When an elastic body is subjected to strain it is held in equilibrium by stresses set up within the body, which depend at each instant upon the amount of the strain which there exists. To fix the ideas in the specification of this stress suppose drawn through the fixed point O, relative to which the displacements already considered have been regarded as produced, three co-ordinate planes perpendicular to the axes Ox, Oy, Oz as in Fig. 299, and a fourth plane intersecting the axes in points A, B, C so that the four planes cut off a tetrahedron of the body. This tetrahedron has, in consequence of the strain, forces applied to it over the four faces by the surrounding matter, and applies equal and opposite forces to that matter. Thus across the face there acts a stress one aspect of which is a force on the tetrahedron, the other aspect a force on the surrounding matter. These stresses vary, of course, from point to point of the areas, but will each have a certain average value per unit area of the face considered. Thus if S be the area of a face and PS be the total force in the direction Ox, exerted on the face by the surrounding matter of the body, P is the average value of the stress per unit area which acts parallel to Ox across the face.

Across each face, then, there acts a stress which gives average component forces on the element and on the surrounding matter in the directions of Ox, Oy, Oz. Taking the face perpendicular to Ox, we shall denote these, each reckoned per unit area, by X_x, Y_x, Z_x. The suffix indicates that the face considered is perpendicular to the axis of x, and the X, Y, Z the directions of the components. We shall consider the components as acting on the tetrahedron.

Now, in general there will be forces distributed over the face which if the element were perfectly rigid would be equilibrated by three component forces applied at the centroid of the face and three couples about the co-ordinate axes. When the element is taken small the moments of these couples involve the dimensions of the face in a power one degree higher than do the forces, and therefore in the limit are negligible in comparison with the latter. Also as the element is made smaller and smaller the average force per unit area, multiplied by the area dS of any small element of the surface, becomes more and more nearly equal to the force actually applied to the element dS in consequence of the existing stress. Thus taking a sufficiently small element we neglect the couples and take only the forces per unit area, which we regard as having the values they approach as the element is diminished indefinitely while its apex is

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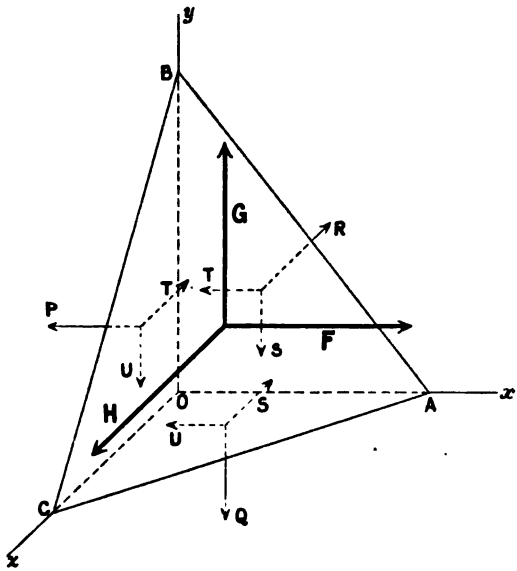
kept at O. Although X_x, Y_x, Z_x, &c., have not the dimensions of "force in the true dynamical sense" we shall call them "forces" for brevity. The dimensional formula of each is however $[ML^{-1}T^{-2}]$.

We shall take X_x, Y_y, Z_z the forces perpendicular to the faces as tractions applied to the faces, that is when they act as shown (by P, Q, R) in Fig. 299. The other forces X_y, X_z, &c., are tangential forces applied to the three faces of the tetrahedron, which are now considered, and are taken as acting in the directions opposite to those

FIG. 299.



in which the axes are drawn, as shown in Fig. 299 by S, T, U. (It is shown below that $Z_y = Y_z$, $X_z = Z_x$, $Y_x = X_y$, and therefore these forces are denoted by S, T, U respectively.) But it will be observed that normals drawn outward from the faces of the tetrahedron to which these forces are applied would have directions opposite to the axes. But in consequence of the same system of stress the forces applied to faces, the outward-drawn normals to which are in the same directions as the axes, would have in the typical or normal case the same directions as the axes. The tangential forces are often called *shearing forces*.

Across the face ABC we suppose a stress to act which gives for the component forces per unit area, F, G, H acting as shown in Fig. 299. Also there will act on the matter of the tetrahedron what are usually called *body forces*, that is, forces applied to the particles within the element due to outside agencies, for example, forces due to gravity, electric or magnetic attraction and repulsion, &c. These

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will have to be considered when we come to discuss the equilibrium of a body under stress; but at present it will be seen that for the vanishing tetrahedron such forces, which depend on the mass of the tetrahedron, and therefore on its volume, vanish ultimately in comparison with the forces applied to its faces in consequence of the stresses.

607. **Equilibrium of Tetrahedron.**—We may now consider the equilibrium of the tetrahedron. If l, m, n be the direction-cosines of the outward-drawn normal to the face ABC and Δ be the area of that face, the areas of the faces perpendicular to Ox, Oy, Oz are $l\Delta$, $m\Delta$, $n\Delta$ respectively. Thus, for the total force in the direction opposed to Ox applied to these faces, we have

$$X_x l\Delta + X_y m\Delta + X_z n\Delta.$$

This is balanced by $F\Delta$, and similarly the other forces may be dealt with. Thus we obtain the three equations of equilibrium

$$\left.\begin{aligned} F &= lX_x + mX_y + nX_z \\ G &= lY_x + mY_y + nY_z \\ H &= lZ_x + mZ_y + nZ_z. \end{aligned}\right\} \quad (27)$$

These equations, it is to be observed, give the components of the stress at any point across any plane, the direction-cosines of the normal to which are l, m, n, in terms of the stresses across planes drawn through the same point at right angles to the co-ordinate axes.

Moreover, and this is of very great importance, if the triangle ABC of Fig. 299 be part of the bounding surface of the body, F, G, H are the components of forces which must be balanced by the forces applied to that portion of the boundary to produce equilibrium. Thus the forces applied to the boundary can be expressed in terms of the stresses over each element of the surface.

608. **Equilibrium of Rectangular Parallelepiped. Equations of Stresses.**—We now consider the equilibrium of a small rectangular parallelepiped (Fig. 300) of the substance with pairs of faces perpendicular to the co-ordinate axes, with its centre at O, and having edges of lengths dx, dy, dz. In consequence of the stresses in the material, the forces applied to the faces will be as indicated by the arrows. The directions of these on the pairs of opposite faces agree, as will be seen, with the convention as to positive and negative indicated above. The force along Ox, applied to the right hand force of the element, is a traction in the positive direction, that parallel to Ox on the left hand face is a traction outwards also, but regarded as a force on the element is in the negative direction.

If X_x, X_y, X_z be the numerical values of the component stresses across a plane drawn through O perpendicular to the axis of x, the component stresses across the right-hand and left-hand faces can be shown, by a process similar to that used in § 355, to be

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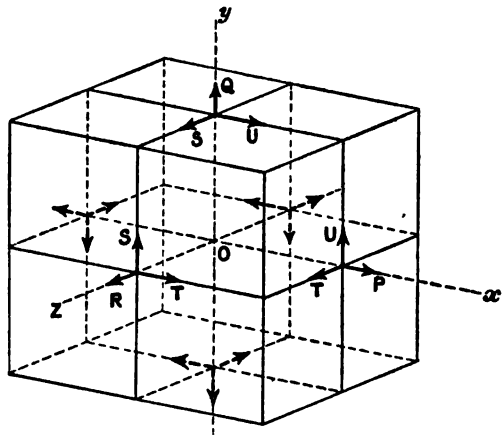
$X_x \pm \frac{1}{2}\partial X_x/\partial x.dx$, $X_y \pm \frac{1}{2}\partial X_y/\partial x.dx$, $X_z \pm \frac{1}{2}\partial X_z/\partial x.dx$, where the positive sign is taken for the right-hand side and the negative sign for the left-hand one. Similarly components of Y_x, Z_x across these two faces are obtained. In the same way the stress across the other two pairs of faces can be calculated. If we denote components of forces per

FIG. 300.



unit volume by X, Y, Z without suffixes and the density of the element by ρ we obtain for the total forces in the directions of Ox, Oy, Oz,

$$\rho X\,dx\,dy\,dz, \quad \rho Y\,dx\,dy\,dz, \quad \rho Z\,dx\,dy\,dz.$$

Now the x-component of force due to the stress across the right-hand face perpendicular to Ox is positive and is

$$(X_x + \tfrac{1}{2}\partial X_x/\partial x.dx)\,dy\,dz$$

in amount, that across the left-hand face acts in the negative direction and is $(X_x - \frac{1}{2}\partial X/\partial x.dx)dy\,dz$ in amount. Hence the former is numerically greater than the latter by $\partial X_x/\partial x.dx\,dy\,dz$. Similarly from the two faces perpendicular to the axis Oy comes a force $\partial X_y/\partial y.dy\,dz\,dx$, and from the two faces perpendicular to Oz a force $\partial X_z/\partial z.dz\,dx\,dy$. Thus the total force in the direction Ox applied to the element is

$$\{\rho X + \partial X_x/\partial x + \partial X_y/\partial y + \partial X_z/\partial z\}\,dx\,dy\,dz,$$

and since the element has no displacement along Ox this must vanish. Similarly the other two directions are dealt with. We thus get the three equations of equilibrium as regards translatory motion.

$$\left.\begin{aligned} \rho X + \frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z} &= 0 \\ \rho Y + \frac{\partial Y_x}{\partial x} + \frac{\partial Y_y}{\partial y} + \frac{\partial Y_z}{\partial z} &= 0 \\ \rho Z + \frac{\partial Z_x}{\partial x} + \frac{\partial Z_y}{\partial y} + \frac{\partial Z_z}{\partial z} &= 0. \end{aligned}\right\} \quad (28)$$

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Besides these there are three equations which must hold if there is no rotation about the co-ordinate axes. It is to be observed that any moments about the axes which the body forces may exert are, if the element is infinitesimal, small in comparison with those due to the stresses across the surfaces, and can be made as small as we please in comparison with the latter by making the element sufficiently small. Hence in the limit it is only necessary to consider the stresses across the faces. Further, the moments of the variations of these from the values across parallel sections of the element through O may be neglected in comparison with the moments of the latter. Thus we obtain, taking moments about Ox, $Y_z \times \frac{1}{2}\, dzdxdy - Z_y \times \frac{1}{2}\, dydxdz = 0$, and similarly for the moments about the other two axes. Hence we have

$$Y_z = Z_y, \quad Z_x = X_z, \quad X_y = Y_x. \tag{29}$$

Hence we may denote Y_x or Z_y by S, Z_x or X_z by T, and X_y or Y_z by U. Also to avoid any confusion with the body forces we denote X_x, Y_y, Z_z by P, Q, R respectively.

609. **Equations of Equilibrium and Motion of an Elastic Solid.**—The equations of equilibrium (28) now become

$$\left.\begin{aligned} \rho X + \frac{\partial P}{\partial x} + \frac{\partial U}{\partial y} + \frac{\partial T}{\partial z} &= 0 \\ \rho Y + \frac{\partial U}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial S}{\partial z} &= 0 \\ \rho Z + \frac{\partial T}{\partial x} + \frac{\partial S}{\partial y} + \frac{\partial R}{\partial z} &= 0, \end{aligned}\right\} \tag{30}$$

while the three equations (29) are involved in the identification of Y_z with Z_y, &c., in the formation of (30) from (28). It may be noticed at once that if the element is under acceleration the quantities on the left in (30) each multiplied by $dxdydz$ are now, if $\ddot{u}$, $\ddot{v}$, $\ddot{w}$ are the accelerations of the centroid of the element, respectively equal to $\ddot{u}\rho\, dx\, dy\, dz$, $\ddot{v}\rho\, dx\, dy\, dz$, $\ddot{w}\rho\, dx\, dy\, dz$. The equations of motion are therefore

$$\left.\begin{aligned} \rho(\ddot{u} - X) &= \frac{\partial P}{\partial x} + \frac{\partial U}{\partial y} + \frac{\partial T}{\partial z} \\ \rho(\ddot{v} - Y) &= \frac{\partial U}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial S}{\partial z} \\ \rho(\ddot{w} - Z) &= \frac{\partial T}{\partial x} + \frac{\partial S}{\partial y} + \frac{\partial R}{\partial z}. \end{aligned}\right\} \tag{31}$$

610. **The Stress-Quadric. Cones of Shearing and Normal Stress.**—If the notation P, Q, R, S, T, U be substituted in (27) for the forces X_x, Y_y, ... as diminished in number by (29) we obtain

$$\left.\begin{aligned} F &= lP + mU + nT \\ G &= lU + mQ + nS \\ H &= lT + mS + nR. \end{aligned}\right\} \tag{32}$$

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These multiplied in order by l, m, n and added give

$$lF + mG + nH = l^2P + m^2Q + n^2R + 2mnS + 2nlT + 2lmU. \tag{33}$$

Imagine now the quadric surface

$$P\xi^2 + Q\eta^2 + R\zeta^2 + 2S\eta\zeta + 2T\zeta\xi + 2U\xi\eta = K^2, \tag{34}$$

having its centre at O, to be constructed. If l, m, n be the direction-cosines of a radius-vector to the surface, the direction-cosines of the normal drawn to the surface at the extremity of this radius-vector are proportional to $lP + mU + nT$, $lU + mQ + nS$, $lT + mS + nR$, that is to F, G, H, and $lF + mG + nH$ is the normal component of the stress across a diametral plane perpendicular to the radius-vector in the direction l, m, n. This normal component is for different radii-vectores of the quadric, of which the equation is (34), inversely proportional to the squares of their lengths. The quadric is therefore called the *stress-quadric*. The resultant of F, G, H across the same plane is perpendicular to the tangent plane at the extremity of the radius-vector l, m, n.

Where this normal stress is a traction exerted by the matter on one side of the plane on the matter on the other side, K^2 is positive, but where this stress is a thrust, we must take for the value of the left-hand side of (34) the value $-K^2$, which gives the conjugate quadric.

If this normal stress vanishes in certain directions, points on lines drawn through O in these directions are given by the equation

$$P\xi^2 + Q\eta^2 + R\zeta^2 + 2S\eta\zeta + 2T\zeta\xi + 2U\xi\eta = 0, \tag{35}$$

which is called the cone of zero normal stress, or, more frequently, since the stress is everywhere tangential to the surface, the *cone of shearing stress*. It separates the directions along which the normal stress across diametral planes of the quadric is a traction from those along which this stress is a thrust.

The lines through O along which the normal stress has a constant value k are given by the equation

$$l^2(P-k) + m^2(Q-k) + n^2(R-k) + 2mnS + 2nlT + 2lmU = 0, \tag{36}$$

so that these directions also are the generating lines of a cone.

611. **Resultant Stress across any Plane.**—If V denote the resultant stress across the plane l, m, n and λ, μ, ν, its direction cosines we have

$$\left.\begin{aligned} F &= lP + mU + nT = \lambda V \\ G &= lU + mQ + nS = \mu V \\ H &= lT + mS + nR = \nu V. \end{aligned}\right\} \tag{37}$$

If ϖ denote the length of the perpendicular let fall from the centre of the quadric on the tangent plane at the extremity of the radius l, m, n, and ρ denote the length, $\sqrt{\xi^2 + \eta^2 + \zeta^2}$, of that radius, these equations give by multiplication by l, m, n in order and addition

$$V = \frac{K^2}{\varpi\rho}. \tag{38}$$

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612. **Reciprocal Relation of Stresses across Two Planes. Hydrostatic Stress.**—The component stress parallel to any line l', m', n' is $ll'P+mm'Q+nn'R+(m'n+mn')S+(n'l+nl')T+(l'm+lm')U$. The symmetry in l, m, n, l', m', n', of this expression asserts that the stress across any plane l, m, n, resolved along the normal to the plane l', m', n', is equal to the stress across the latter plane resolved along the normal to the former.

If the stress is normal to the plane l, m, n, we have λ, μ, ν equal to l, m, n, and the stress resolved at right angles to the plane l', m', n' is $V(ll'+mm'+nn')$, that is $V\cos\theta$, where θ is the angle between the normals.

Hence, by the theorem just stated, the stress V' across the plane l', m', n', resolved perpendicular to the plane l, m, n, gives the component $V\cos\theta$. It follows that if V' be also normal to its plane this component is also $V'\cos\theta$, and therefore $V'=V$. If, therefore, the stress across every plane through C is normal to the plane the stress across all planes has the same value. Thus, when the stress is of the nature of thrust, we have the hydrostatic theorem that the pressure throughout a fluid in equilibrium has the same value in all directions. The stress-quadric is then a sphere, as indeed is every surface of the second degree by which the distribution of stress can be graphically represented.

613. **Principal Axes of Stress. Stress Invariants.**—The principal axes of the stress-quadric can be found by the same process as that used for the elongation quadric in § 601 above. For any plane l, m, n to which the resultant stress is normal we obtain the equations

$$lP+mU+nT=lV,\quad lU+mQ+nS=mV,\quad lT+mS+nR=nV, \tag{39}$$

so that we obtain for V the determinantal cubic

$$\left.\begin{vmatrix} P-V, & U, & T \\ U, & Q-V, & S \\ T, & S, & R-V. \end{vmatrix}=0\right\} \tag{40}$$

This gives three real values of V for the determination by (39) of three sets of values of l, m, n. The three directions thus obtained are perpendicular to one another (§ 168), and give the three principal axes of the stress quadric. The planes at right angles to them are the principal planes, and the stresses across these planes are the principal stresses.

If P', Q', R' now denote the principal stresses, the equation of the stress-quadric is

$$P'\xi^2+Q'\eta^2+R'\zeta^2=K^2, \tag{41}$$

and if a, b, c be the three semi-axes of this quadric, the principal stresses are *numerically* equal to K^2/a^2, K^2/b^2, K^2/c^2. The stress invariants are

$$\left.\begin{aligned} P+Q+R&=P'+Q'+R' \\ QR+RP+PQ-S^2-T^2-U^2&=Q'R'+R'P'+P'Q' \\ PQR+2STU-RU^2-QT^2-PS^2&=P'Q'R'. \end{aligned}\right\} \tag{42}$$

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614. **Stress-Ellipsoid.**—If now, accents dropped, P, Q, R be the principal stresses, the equation

$$\frac{\xi^2}{P^2}+\frac{\eta^2}{Q^2}+\frac{\zeta^2}{R^2}=1, \tag{43}$$

where x, y, z are distances measured along the principal axes, denotes an ellipsoid. The components of the stress across the plane the direction cosines of which are ξ/P, η/Q, ζ/R are $F=\xi$, $G=\eta$, $H=\zeta$, and the component of stress in the direction of the normal to this plane is $\xi^2/P+\eta^2/Q+\zeta^2/R$. The resultant stress across the same plane is $\sqrt{\xi^2+\eta^2+\zeta^2}$, that is, it is numerically equal to the radius-vector ρ, and acts along it. The ellipsoid (43) is called the *stress-ellipsoid.*

The radii-vectores representing the resultant stresses across three co-orthogonal planes, that is planes which are at right angles to one another are not themselves co-orthogonal, but it is easy to see that they are conjugate axes of the ellipsoid (43). For, let ξ_1, η_1, ζ_1, ξ_2, η_2, ζ_2, ξ_3, η_3, ζ_3 be the co-ordinates of their extremities, the three stresses re-represented by $\sqrt{\xi_1^2+\eta_1^2+\zeta_1^2}$, &c., are the resultant stresses across the planes the direction cosines of which are ξ_1/P, η_1/Q, ζ_1/R, &c., and if these planes be co-orthogonal the relations

$$\frac{\xi_1\xi_2}{P^2}+\frac{\eta_1\eta_2}{Q^2}+\frac{\zeta_1\zeta_2}{R^2}=0, \tag{44}$$

and two others got by cyclic changes of suffixes, hold. But these are the relations that hold between conjugate semi-diameters: the first affirms that the point ξ_1, η_1, ζ_1 lies on a plane parallel to the tangent plane at the extremity of ξ_2, η_2, ζ_2, and so for the others.

615. **The Director Quadric. Cones of Shearing Stress and Zero Normal Stress.**—The quadric

$$\frac{\xi^2}{P}+\frac{\eta^2}{Q}+\frac{\zeta^2}{R}=K'^2, \tag{45}$$

in which P, Q, R denote the quantities represented by P', Q', R' in § 613, is called the *reciprocal* of the quadric (41). The stress across a diametral plane of this quadric has components

$$F,\ G,\ H=(\xi,\ \eta,\ \zeta)/(\xi^2/P^2+\eta^2/Q^2+\zeta^2/R^2)^{\frac{1}{2}},$$

that is $(\xi, \eta, \zeta)\ p/K'^2$. The resultant stress across this plane is therefore $p\rho/K'^2$ where $\rho=\sqrt{\xi^2+\eta^2+\zeta^2}$, and is in the direction of ρ. Hence the quadric is sometimes called the *director-quadric.*

The component along the normal [the direction cosines of which are $(p\xi/P,\ p\eta/Q,\ p\zeta/R)K'^2$] is

$$(Fp\xi/P+Gp\eta/Q+Hp\zeta/R)/K'^2=p^2(\xi^2/P+\eta^2/Q+\zeta^2/R)/K'^4=p^2{}_1K.$$

If the normal component is everywhere positive this quadric is an ellipsoid, and if the normal component is everywhere negative the quadric is again an ellipsoid, but with the equation

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$$\frac{\xi^2}{P}+\frac{\eta^2}{Q}+\frac{\zeta^2}{R}=-K^2. \tag{46}$$

If the stress is a traction in some directions and a pressure in others, (45) holds for the former and (46) for the latter. The equations then represent two conjugate hyperboloids separated by the cone

$$\frac{\xi^2}{P}+\frac{\eta^2}{Q}+\frac{\zeta^2}{R}=0, \tag{47}$$

which is the cone of *zero normal stress.* On the other hand the stress is, as we have seen, at all points tangential to the reciprocal cone

$$P\xi^2+Q\eta^2+R\zeta^2=0^2, \tag{48}$$

which is the cone of shearing stress. Neither of these cones exists when the stress is throughout of one character.

616. **Transformation of Stresses. Expression of General Stresses in Terms of Principal Stresses.**—The transformation of the specification of stress from one set of axes to another is important, especially the expression of the general system of stresses P, Q, R, S, T, U in terms of the principal stresses P', Q', R'. Let l_1, m_1, n_1, l_2, m_2, n_2, l_3, m_3, n_3 be the direction cosines of the axes of any general system of stresses P, Q, R, S, T, U with reference to the axes of a given general system P', Q', R', S', T', U'. Substituting in (32) P', Q', &c., for P, Q, &c., and l_1, m_1, n_1, l_2, m_2, n_2, l_3, m_3, n_3 in succession for l, m, n we obtain three sets of values of F, G, H, namely

$$\begin{aligned}F&=l_1P'+m_1U'+n_1T'\\G&=l_1U'+m_1Q'+n_1S'\\H&=l_1T'+m_1S'+n_1R'\end{aligned}$$

and two others obtained from these by the specified changes of suffixes. This first set are the components across the plane the direction cosines of the normal to which are l_1, m_1, n_1, that is the new plane of y, z. The normal component across this plane is therefore

$$\begin{aligned}P=l_1^2P'&+m_1^2Q'+n_1^2R'\\&+2m_1n_1S'+2n_1l_1T'+2l_1m_1U'\end{aligned} \tag{49}$$

with similar formulæ for Q, R to be obtained by changing the suffixes of the cosines.

The values of F, G, H for the new plane of x, y are $l_3P'+m_3U'+n_3T'$, &c. The component across this plane parallel to y is $l_2F+m_2G+n_2H$, and is the value of S. Hence

$$\begin{aligned}S=l_2l_3P'&+m_2m_3Q'+n_2n_3R'\\&+(n_2m_3+n_2m_3)S'+(l_2n_3+n_2l_3)T'+(m_2l_3+l_2m_3)U'\end{aligned} \tag{50}$$

with similar formulæ for T, U.

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If P, Q, R are principal stresses S, T, U are identically zero, and (49) suffices for the transformation. If P', Q', R' are principal stresses P, Q, R, S, T, U are obtained by deleting the second lines of the expressions on the right of (49) and (50). In this case we have

$$\left.\begin{aligned} P &= l_1^2P' + m_1^2Q' + n_1^2R' \\ Q &= l_2^2P' + m_2^2Q' + n_2^2R' \\ R &= l_3^2P' + m_3^2Q' + n_3^2R' \end{aligned}\right\} \quad (49')$$

$$\left.\begin{aligned} S &= l_2l_3P' + m_2m_3Q' + n_2n_3R' \\ T &= l_3l_1P' + m_3m_1Q' + n_3n_1R' \\ U &= l_1l_2P' + m_1m_2Q' + n_1n_2R' \end{aligned}\right\} \quad (50')$$

These formulæ may be used also for the transformation of strain. It is only necessary to put in (49) and (50), e, f, g, a, b, c for P, Q, R, $2S$, $2T$, $2U$ with the corresponding substitutions for P', Q', &c. Equations (49′) and (50′) give $e, f, g, \frac{1}{2}a, \frac{1}{2}b, \frac{1}{2}c$, if $\epsilon_1, \epsilon_2, \epsilon_3$ the principal elongations be put for P', Q', R'.

617. **Equivalence of Traction and Pressure in Perpendicular Directions to System of Shearing Stress.**—We can now easily discover equivalent systems of stress which are of great importance for practical applications. For example, consider a system composed of two principal stresses, a traction P' across all planes perpendicular to the principal axis Ox', and a pressure of equal amount across all planes perpendicular to the principal axis Oy'. These act across planes drawn through a point Q in the directions shown in Fig. 301. The equivalent stress across two planes both parallel to the axis of z, and bisecting the angles between the axes Ox, Oy, can be found at once from the formulas just written. The direction-cosines of the axes Ox', Oy', Oz' with reference to Ox, Oy, Oz (Oz and Oz' being coincident) are $l_1 = 1/\sqrt{2}$, $m_1 = -1/\sqrt{2}$, $n_1 = 0$, $l_2 = 1/\sqrt{2}$, $m_2 = 1/\sqrt{2}$, $l_3 = 0$, $m_3 = 0$, $n_3 = 1$. Hence by (49)

$$\left.\begin{aligned} &P = \tfrac{1}{2}P' - \tfrac{1}{2}P' = 0, \quad Q = 0, \quad R = 0 \\ &S = 0, \quad T = 0, \quad U = -\tfrac{1}{2}P' - \tfrac{1}{2}P' = -P'. \end{aligned}\right\} \quad (51)$$

Thus, by Fig. 301, the forces acting across the new co-ordinate planes of xz and yz at the point Q are the two tangential forces U as shown in Fig. 300; that is, a tangential force (P per unit area) acting over the face perpendicular to the new axis of x in the direction from B towards A, and an equal tangential force over the face perpendicular to the new axis of y and from C towards Q. The original stresses at Q act on the matter within the angle MQN, the equivalent tangential stresses at Q act on the matter within the angle CQB. Thus the given system of stress is equivalent to a system of shearing stress on planes through the point considered inclined at 45° to the axes.

The system of stresses, shown applied to the faces of a rect-

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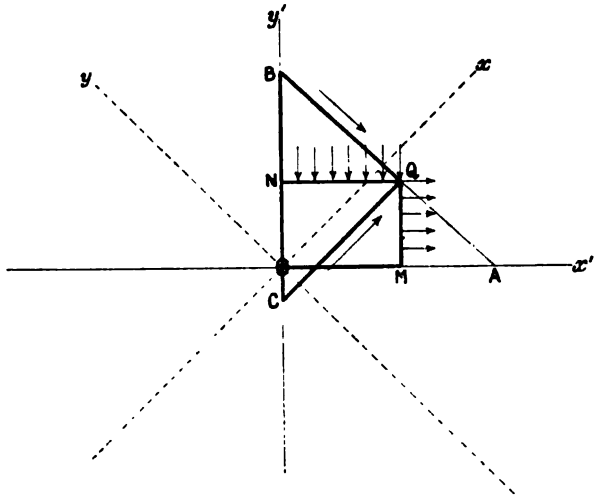
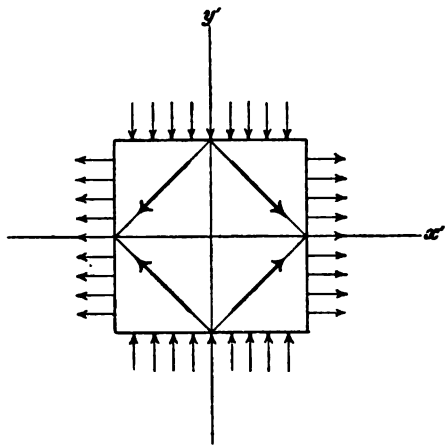
angular element in Fig. 301′, is indicative of the stresses supposed applied by the surrounding matter to any such element taken with

FIG. 301.



its centre at O, the system being that just dealt with. The internal rectangular element is shown affected with the equivalent tangential or shearing stresses. The opposite tangential stresses applied by

FIG. 301′.



the inner element to the triangular prisms outside it, just equilibrate the forces applied to these by the normal stresses across their outer faces.

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618. **Simple Normal Traction reduced to Uniform Dilating Stress and Two Systems of Shearing Stress.**—From this result it is clear that a simple normal traction across one set of parallel planes can be reduced to a uniform normal traction across this set of planes and the other two sets which are co-orthogonal with it, and two systems of stress similar to the system just discussed. For, let the stress be a traction P across all planes perpendicular to the axis of x. Consider a cubical element of the body with its centre at O. Across the pair of opposite faces parallel to yOz there acts the stress P. Suppose applied perpendicular to each of the other two pairs of faces a stress consisting of a traction $\frac{1}{3}P$ and a pressure $\frac{1}{3}P$. The distribution is indicated in Fig. 302. This gives, of course, zero stress

Fig. 302.

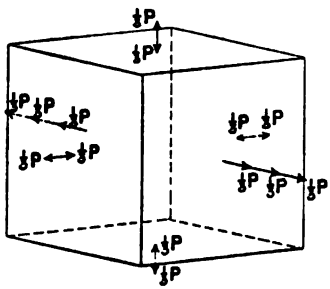
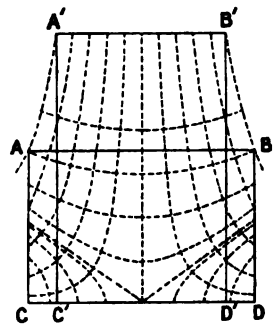


Fig. 303.



for each pair of planes except those perpendicular to Ox. Thus we have parallel to every pair of parallel faces a traction $\frac{1}{3}P$; that is, a uniform dilating stress of this amount acts on the body. Perpendicular to the pair of faces parallel to yOz there acts a traction $\frac{1}{3}P$, and perpendicular to the pair of faces parallel to zOx a pressure of the same amount; a similar system of stresses acts across the two sets of planes perpendicular to yOz and xOy. Each of the two latter systems of stress is reducible, as in § 617, to a system of tangential stresses parallel to the planes bisecting the angles between Ox, Oy, and the two planes bisecting the angles between the axes Ox, Oz. Therefore the simple traction, or elongating stress, acting on an element is reducible to a uniform dilating stress, and two systems of shearing stresses each acting on the element as already explained in § 617.

It follows that any system of stress being reducible to three simple tractions (positive or negative) parallel to the principal axes, is reducible to a uniform dilating stress, and three systems of shearing stress, acting parallel to the three pairs of planes bisecting the angle between the axes.

Fig. 303 shows the lines along which an axial section $ABCD$ of a portion of a wire changes to the configuration $A'B'C'D'$. The longitudinal extension from AC to $A'C'$, and the lateral contraction from CD to $C'D'$ are much exaggerated, and are far from being in the

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proper ratio. But the dotted lines are so drawn that a particle on one of these lines in the original configuration is on the same line in another position in the other configuration. These are therefore lines of displacement of the particles.

619. **Relations between Stress and Strain.**—We have now to consider the relations between stress and strain. In doing so we shall suppose that there are no energy changes, resulting from the application of stress to the body, which are not exactly taken account of when the work done by the applied forces, and the change of potential energy due to change of strain configuration of the elastic substance, are evaluated. In the limited discussion possible here we neglect energy changes due to changes of temperature and other variations of the internal state of the body, and for these reference must be made to treatises on *Thermodynamics*, or to Part II. of this work. But, provided the body is strained either at constant temperature, or subject to the condition that no heat enters or leaves any portion of the strained substance the changes of potential energy are independent of the order of succession of the strain configurations imposed on the body, that is, the work spent on the body in any change depends only on the initial and final configurations of the change. There is thus a work function W which expresses in terms of the strain configurations the work done by the applied forces in carrying the substance from one to the other. In other words, the work done in any small change of strain of any type taken per unit of the change is a perfect differential of a function of the co-ordinate or co-ordinates expressing the strain configuration.

620. **Hooke's Law. Strain due to Simple Traction.**—We shall endeavour to determine on certain assumptions the nature of the (small) strain produced by any type of stress and shall assume that different stresses or elements of a single stress are independent in the sense that each produces its own effect independently of the existence of the others. In other words the strain will be taken as proportional to the existing stress. That this was approximately the case for small strains of a certain type was first noticed by Hooke, and the proportionality which holds between small strains and their corresponding stresses of whatever type is now generally referred to as Hooke's Law.

It will be found that a simple traction in one direction, say a stretching force applied to a rod or wire along its length, does not merely produce elongation, but also causes alteration of the lateral dimensions of the specimen; but that an isotropic body when subject to uniform dilating stress suffers no change of shape, and when under the influence of shearing stress changes in shape without alteration of volume.

621. **Limits of Elasticity. Perfect Elasticity.** When a succession of small strains is applied to a body the accumulated strain of any type may cease to be small, and be no longer proportional to the total stress of that type applied to it. The body is then said to have been strained beyond its limits of elasticity for the kind of strain

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considered. The limits for all bodies are exceedingly narrow, if indeed in strictness the law is fulfilled for any strain however small. We shall later consider results as to elastic limits in particular cases.

A body is said to be perfectly elastic at constant temperature for strain of any type whatever, whether change of volume or change of shape or any definite combination of these, when, if a succession of increments of strain be slowly applied by a succession of increments of stress until a final configuration is reached, and these increments of stress are removed in the reverse order, the successive increments of strain are also removed in the reverse order. In other words, the substance has perfect elasticity of the kind considered if the configuration corresponding to any value of the stress is the same in this cycle of operations when the stress is being applied and when it is being removed. When this is the case the work done on the body by applied forces in producing the strain is exactly equal to the work done by the body against external forces in returning to the original configuration. We shall find that a balance of work is spent on an imperfectly elastic body when it is put through a cycle of changes, and shall show how it is to be graphically estimated.

622. **Work done in Strain.**—We might, following a process due to Green, write down an expression for the work done in any small displacement of the body by the forces applied to its surface and the body-forces, and transform this expression into another in terms of the internal stresses to which these applied forces give rise, and we should arrive at the result that if $e, f, g, 2s_1, 2s_2, 2s_3$, or as we shall write them e, f, g, a, b, c, be the elongations and simple shears specified above, we should obtain for the work done in increasing these by amounts de, df, dg, da, db, dc,

$$dW = \int (Pde + Qdf + Rdg + Sda + Tdb + Vdc)d\varpi, \qquad (52)$$

where $d\varpi$ is an element of volume and the integral is taken throughout the whole volume of the body. The work done per unit volume is thus

$$Pde + Qdf + Rdg + Sda + Tdb + Vdc.$$

P, Q, R thus correspond to the elongation strains de, df, dg, and S, T, U to the shearing strains da, db, dc in the sense that the work due to each stress is the product of the stress and the strain of its own type produced. We shall take these results for granted at present.

Assuming, then, the result that W is a function of the strain co-ordinates e, f, g, a, b, c only, we have under the condition stated, namely, either that the temperature is constant, or that no heat enters or leaves any part of the body,

$$P = \frac{\partial W}{\partial e}, \quad Q = \frac{\partial W}{\partial f}, \quad R = \frac{\partial W}{\partial g}, \quad S = \frac{\partial W}{\partial a}, \quad T = \frac{\partial W}{\partial b}, \quad U = \frac{\partial W}{\partial c}. \qquad (53)$$

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623. **Moduli of Elasticity of an Isotropic Body. Young's Modulus.**—We define two principal moduli or constants of elasticity, the *bulk modulus*, and the *modulus of rigidity*. In each case the modulus is measured by the ratio of the applied stress of the proper type to the (small) strain produced by it. We shall denote the bulk modulus by k, the rigidity modulus by n.

Besides these we have also Young's modulus, or the modulus for simple longitudinal stress, and the modulus for simple longitudinal strain. Each of these is measured by the ratio of the longitudinal stress to the (small) unital elongation produced by it, but the former is taken irrespectively of change of lateral dimensions, the latter irrespectively of any lateral stress imposed to ensure the prevention of change of lateral dimensions. We shall denote them by E and E' respectively and shall presently find expressions for them in terms of k and n.

624. **Stress-Strain Equations. Determination of Co-efficients.**—We assume that the principal axes of stress and the principal axes of strain in the body coincide and are drawn from the origin O at which the state of the body is considered. The principal strains are denoted by e, f, g and the principal stresses by P, Q, R. In accordance with what has been stated above each stress must be a linear function of the strains. Now it is clear that in one equation for the stress P, say, the strains f, g which are in the two directions at right angles to P, must appear with the same co-efficient of proportionality, since the body is isotropic. The strain e will, however, have a different co-efficient. Similarly the other principal stresses are related to the strains. Hence we write

$$\left.\begin{aligned} P &= \lambda(e+f+g)+\mu e \\ Q &= \lambda(e+f+g)+\mu f \\ R &= \lambda(e+f+g)+\mu g, \end{aligned}\right\} \qquad (54)$$

where λ and μ are constants.

We can determine λ and μ in terms of k and n as follows. First let $P=Q=R$, then clearly $e=f=g$. But (54) give by addition

$$P=(3\lambda+\mu)e, \text{ or } 3\lambda+\mu=P/e.$$

Now the strain in this case is evidently a dilatation of amount $\Delta=e+f+g=3e$. But if k be the bulk modulus, $k=P/\Delta$, so that

$$\lambda+\tfrac{1}{3}\mu=k. \qquad (55)$$

Again, let $P=-Q$, and $R=0$, then by (54) we have

$$P=(\lambda+\mu)e+\lambda(f+g)=-(\lambda+\mu)f-\lambda(e+g),$$
$$(\lambda+\mu)g+\lambda(e+f)=0.$$

The stress applied in this case is a shearing stress of amount P, and we assume that the strain is a shear having the same axis and of amount P/n where n is the rigidity modulus. But as has been

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shown above (§ 600) the shear is numerically equal to twice the unital elongation in the direction of P or twice the contraction in the direction of Q. Hence $P/n = 2e = -2f$, and $e = -f$. This by the relation $(\lambda+\mu)g + \lambda(e+f) = 0$ gives $g = 0$, which agrees with the supposition as to strain. Hence the equation $P = (\lambda+\mu)e + \lambda(f+g)$ becomes $P = (\lambda+\mu)e - \lambda e = \mu e$. Hence, since $P/2e = n$, we have $\mu = 2n$. Substituting in (55) we find that this gives

$$\lambda = k - \tfrac{2}{3}n. \tag{56}$$

Equations (54) can now be written

$$\left.\begin{aligned} P &= \lambda(e+f+g) + 2ne \\ Q &= \lambda(e+f+g) + 2nf \\ R &= \lambda(e+f+g) + 2ng, \end{aligned}\right\} \tag{57}$$

where λ has the value $k - \frac{2}{3}n$.

625. **Expression of Young's Modulus in Terms of Bulk Modulus and Rigidity Modulus. Poisson's Ratio.**—Now consider the case of simple longitudinal stress. By (54) we have since $Q = R = 0$,

$$\begin{aligned} P &= \lambda(e+f+g) + 2ne \\ 0 &= \lambda(e+f+g) + 2nf \\ 0 &= \lambda(e+f+g) + 2ng, \end{aligned}$$

so that $f = g$, and $f = -\frac{1}{2}e\lambda/(\lambda+n)$. Hence $P = en(3\lambda+2n)/(\lambda+n)$.

But if E be Young's modulus $E = P/e$, and, therefore,

$$E = \frac{n(3\lambda+2n)}{\lambda+n} = \frac{9nk}{3k+n} \tag{58}$$

by the value $k - \frac{2}{3}n$ of λ. Thus Young's modulus is a composite modulus involving both the principal moduli of an isotropic body.

The ratio $\frac{1}{2}\lambda/(\lambda+n)$ of the numerical amount of the lateral contraction to the longitudinal extension is known as *Poisson's ratio.* We shall denote it by η, and write for reference

$$\eta = \tfrac{1}{2}\frac{\lambda}{\lambda+n} = \frac{3k-2n}{2(3k+n)}. \tag{59}$$

The modulus of simple longitudinal strain e is obtained from (57) by putting $f = g = 0$, so that

$$\begin{aligned} P &= (\lambda+2n)e = (k+\tfrac{4}{3}n)e \\ Q = R &= \lambda e \quad = (k-\tfrac{2}{3}n)e. \end{aligned} \tag{60}$$

Thus

$$E' = k + \tfrac{4}{3}n. \tag{61}$$

The stress required to prevent lateral contraction is thus a traction in each of the directions perpendicular to the extension of amount $(k - \frac{2}{3}n)e$ or $P(k - \frac{2}{3}n)/(k + \frac{4}{3}n)$.

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*626. **Expression of General Stresses in Terms of Principal Stresses.**—The values of the stresses of the system P, Q, R, S, T, U can be found by the formulæ of transformation (49′), (50′) from the principal stresses, which we now denote for distinction by P', Q', R'. The principal strains (e', f', g', say) are also transformed by similar formulæ to the system e, f, g, a, b, c. Hence, if the reader will go through the process, writing down the values of P, &c., in terms of P', &c., by (49′) above (for example, $P = P'l_1^2 + Q'l_2^2 + R'l_3^2$) substituting the values of P', Q', R' from (57), and taking account of the application of (49) and (50) to the transformation of strain as stated in § 616, and the invariance of $e' + f' + g'$, he will find

$$\left.\begin{array}{c} P = \lambda\Delta + 2ne,\ \ Q = \lambda\Delta + 2nf,\ \ R' = \lambda\Delta + 2ng, \\ S = na,\ \ T = nb,\ \ U = nc, \end{array}\right\} \qquad (62)$$

where Δ denotes the dilatation $\partial u/\partial x + \partial v/\partial y + \partial w/\partial z\,(= e + f + g)$.

*627. **Equations of Motion of an Elastic Body.**—Substituting in these $\partial u/\partial x$ for e, &c., $\partial w/\partial y + \partial v/\partial z$ for a, &c., we obtain values of P, Q, R, S, T, U for use in the transformation of (31) to equations in terms of the displacements u, v, w. If then ∇^2 denote the operator $\partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$ the equations of small motions become

$$\left.\begin{array}{l} n\nabla^2 u + (\lambda + n)\dfrac{\partial\Delta}{\partial x} + \rho X = \rho\ddot{u} \\ n\nabla^2 v + (\lambda + n)\dfrac{\partial\Delta}{\partial y} + \rho Y = \rho\ddot{v} \\ n\nabla^2 w + (\lambda + n)\dfrac{\partial\Delta}{\partial z} + \rho Z = \rho\ddot{w} \end{array}\right\} \qquad (63)$$

which become the equations of equilibrium when $\ddot{u}$, $\ddot{v}$, $\ddot{w}$ are put equal to zero. By introducing the rotations θ_1, θ_2, θ_3 the reader may convert them into

$$(\lambda + 2n)\frac{\partial\Delta}{\partial x} - 2n\left(\frac{\partial\theta_3}{\partial y} - \frac{\partial\theta_2}{\partial z}\right) + \rho X = \rho\ddot{u}, \qquad (63')$$

with two similar equations for $\ddot{v}$, $\ddot{w}$.

From the latter form of the equations it is clear that if the strain be without rotation the equations are of the comparatively simple form

$$(\lambda + 2n)\frac{\partial\Delta}{\partial x} + \rho X = \rho\ddot{u}, \qquad (64)$$

with two similar equations in y and z.

628. **Longitudinal Strain without Change of Lateral Dimensions.**—It will be evident that if a narrow straight or conical bar of the material is subjected to extension along the axis of the rod or along lines through the vertex of the cone, while the lateral boundary remains unchanged, that is to say, when there is longitudinal exten-

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sion without lateral contraction, there is no rotation. The equations of small motions just written are then obvious, for the case at least of a straight bar of uniform cross-section. For consider the narrow straight bar, and let its length be in the direction of the axis of x. There is no motion of any particle in the direction of y or z, and the extension along x gives a pure dilatation. The strain at the centre of a slice perpendicular to x and of length dx being Δ, the strain at the right- or the left-hand end of the slice is $\Delta \pm \frac{1}{2}\partial\Delta/\partial x\, dx$, according as the positive or negative sign is taken. Since the modulus of simple longitudinal strain is $\lambda + 2n = (k + \frac{4}{3}n)$, the force (traction) per unit area applied to the left-hand end of the slice by the matter of the bar farther to the left is

$$(\lambda + 2n)(\Delta - \tfrac{1}{2}\partial\Delta/\partial x.dx),$$

and that applied to the other end by the matter of the bar to the right is $(\lambda + 2n)(\Delta + \frac{1}{2}\partial\Delta/\partial x.dx)$ so that there is a balance of traction to the right of $(\lambda + 2n)\partial\Delta/\partial x.dx$, and if σ be the cross-section the total force to the right on the element is $(\lambda + 2n)\partial\Delta/\partial x.\sigma dx$. But the body-force per unit mass towards the right along x being X, there is a total force towards the right of amount $X\rho\sigma dx$, since $\rho\sigma dx$ is the mass of the element. The sum of these two forces must be equal to $\rho\sigma dx\ddot{u}$ if there is acceleration of the element, or to zero if there is equilibrium, so that we have

$$(\lambda + 2n)\frac{\partial\Delta}{\partial x} + \rho X = \rho\ddot{u}. \tag{65}$$

In this case it is to be observed $\Delta = \partial u/\partial x$.

629. **Radial Strain.**—In the other case of radial strain of a narrow cone we take a slice contained between two spherical surfaces of radii r and $r + dr$ described from the vertex of the cone as centre. This slice is we suppose displaced outwards a distance w, and the radial strain at the centre of the element is therefore $\partial w/\partial r$. But this is not the only strain at that point. There are two tangential strains at right angles to each other and to the radius, each of amount w/r, so that the dilatation is $\partial w/\partial r + 2w/r$. To prove this it is only necessary to consider the slice of thickness dr, subjected at its centre to a radial displacement w. The radial displacement at the inner face is $w - \frac{1}{2}\partial w/\partial r.dr$, and at the outer face it is $w + \frac{1}{2}\partial w/\partial r.dr$. The outer face is therefore displaced relatively to the inner through a distance $\partial w/\partial r.dr$, so that the thickness of the slice has become $dr + \partial w/\partial r.dr$. But the cross-section of the slice has been increased by its transference along the axis of the cone through the distance w in the ratio of $(r + w)^2$ to r^2, that is in the ratio of $(1 + 2w/r)$ to unity.

The ratio of the new volume of the element to the old volume is $(1 + \partial w/\partial r)(1 + 2w/r)/1$, so that the unital increase of volume is $(1 + \partial w/\partial r)\ (1 + 2w/r) - 1 = \partial w/\partial r + 2w/r$, since w is small. This is the dilatation. Hence by equation (64), taking r in the direction of z, we have since the rotations are manifestly zero

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$$(\lambda+2n)\frac{\partial}{\partial r}\left(\frac{\partial w}{\partial r}+2\frac{w}{r}\right)+\rho Z=\rho\ddot{w}, \tag{66}$$

where Z is the body force outwards per unit of mass at the centre of the element. If the strained body is in equilibrium $\ddot{w}=0$, and we have

$$(\lambda+2n)\frac{\partial}{\partial r}\left(\frac{\partial w}{\partial r}+2\frac{w}{r}\right)+\rho Z=0. \tag{67}$$

630. **Strain in Spherical Shell under given Internal and External Pressures.**—A very important case of this equation is that in which $Z=0$, that is in which a substance not acted upon by body-forces is subjected to radial strain and the accompanying tangential strains. This is the case, for example, of an isotropic spherical shell subjected to pressure applied to its inner and outer surface. In this case the differential equation fulfilled by the radial displacement is

$$\frac{\partial}{\partial r}\left(\frac{\partial w}{\partial r}+2\frac{w}{r}\right)=0. \tag{68}$$

The complete solution of this equation is

$$w=Ar+\frac{B}{r^2}, \tag{69}$$

where A and B are constants. This gives

$$\frac{\partial w}{\partial r}=A-\frac{2B}{r^3},\quad \frac{w}{r}=A+\frac{B}{r^3};$$

and therefore by (57) the radial stress

$$R=\lambda\left(\frac{\partial w}{\partial r}+2\frac{w}{r}\right)+2n\frac{\partial w}{\partial r}=(3\lambda+2n)A-4n\frac{B}{r^3}, \tag{70}$$

and the tangential stress

$$P=\lambda\left(\frac{\partial w}{\partial r}+2\frac{w}{r}\right)+2n\frac{w}{r}=(3\lambda+2n)A+2n\frac{B}{r^3}. \tag{71}$$

Let the external and internal radii be r_1, r_0, the external and internal pressures p_1, p_0 respectively; then for the determination of A and B we have

$$p_1=-(3\lambda+2n)A+4n\frac{B}{r_1^3}$$

$$p_0=-(3\lambda+2n)A+4n\frac{B}{r_0^3};$$

and, therefore,

$$A=\frac{1}{3\lambda+2n}\,\frac{p_0r_0^3-p_1r_1^3}{r_1^3-r_0^3},\quad B=\frac{p_0-p_1}{4n}\,\frac{r_0^3r_1^3}{r_1^3-r_0^3}.$$

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Hence the radial displacement at distance r from the centre is

$$w = \frac{r}{r_1^3 - r_0^3}\left\{\frac{p_0 r_0^3 - p_1 r_1^3}{3\lambda + 2n} + \frac{p_0 - p_1}{4n}\,\frac{r_0^2 r_1^3}{r_1^3 - r_0^3}\,\frac{1}{r^3}\right\}. \tag{72}$$

The radial strain is

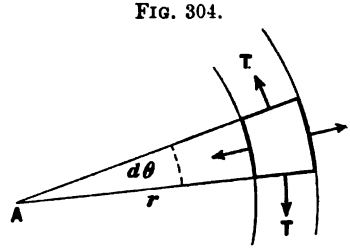
$$\frac{\partial w}{\partial r} = \frac{1}{r_1^3 - r_0^3}\left\{\frac{p_0 r_0^3 - p_1 r_1^3}{3\lambda + 2n} - \frac{p_0 - p_1}{2n}\,\frac{r_0^3 r_1^3}{r_1^3 - r_0^3}\,\frac{1}{r^3}\right\}. \tag{73}$$

The tangential strain w/r is at once given by (72).

These results will be of service in the discussion of some results of Regnault and others on the compressibility of liquids with which we shall deal in vol. ii.*

631. **Cylindrical Boiler. Stiffening of Flexible Tube by Hydrostatic Pressure.**—Now consider a tube bounded by co-axial right cylindrical surfaces and flat or round ends. Let the internal radius of the cylinder be r_0, the external radius r_1, the internal pressure p_0, the external pressure p_1. Further, suppose, what is approximately the case in practice, that the pull $\pi(p_0 r_0^2 - p_1 r_1^2)$ applied by the pressures on each end to the walls of the tube is uniformly distributed over the cross-section so that its amount per unit area is $(p_0 r_0^2 - p_1 r_1^2)/(r_1^2 - r_0^2)$. Let then R be the radial stress (as before supposed to be numerically positive when a traction), T the circumferential stress or "hoop-stress" as it is frequently called, and P the longitudinal traction, all taken at a point in the wall of the tube at distance r from the axis. Consider the equilibrium of an element (Fig. 304) bounded by two co-axial surfaces of radii $r - \frac{1}{2}dr$, $r + \frac{1}{2}dr$, two radial planes from the axis A at an angle $d\theta$, and two planes at right angles to the axis at unit distance apart. The inward radial force on the inner curved surface of the element is

FIG. 304.



$$Rrd\theta - \tfrac{1}{2}d(Rr)dr.dr.rd\theta,$$

and the outward radial force on the other curved face is

$$Rrd\theta + \tfrac{1}{2}d(Rr)/dr.dr.rd\theta,$$

so that there is a balance of outward force of amount $d(Rr)/dr.drd\theta$. The hoop-stress gives (§ 205) an inward force of amount $Tdr/r.rd\theta$, or $Tdrd\theta$. Hence for equilibrium we have the condition

$$\frac{d(Rr)}{dr} = T. \tag{74}$$

* The problem of the strain of a solid sphere under its own gravitation is important in connection with terrestrial physics, and is easily solved. The reader may consult Professor Love's Treatise, vol. i. r. 219.

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If w denote as before the radial displacement, the radial and tangential strains are $\partial w/\partial r$ and w/r respectively. Let the longitudinal strain be g. Then the equations (57) connecting the strains and stresses may be written

$$\left.\begin{aligned} R &= \lambda\left(\frac{\partial w}{\partial r} + \frac{w}{r} + g\right) + 2n\frac{\partial w}{\partial r} \\ T &= \lambda\left(\frac{\partial w}{\partial r} + \frac{w}{r} + g\right) + 2n\frac{w}{r} \\ P &= \lambda\left(\frac{\partial w}{\partial r} + \frac{w}{r} + g\right) + 2ng. \end{aligned}\right\} \tag{75}$$

The value of P is assigned and therefore we have from the third equation

$$g = -\frac{\lambda}{\lambda + 2n}\left(\frac{\partial w}{\partial r} + \frac{w}{r}\right) + C, \tag{76}$$

where C is the constant, $P/(\lambda + 2n)$. Using this value of g in the first and second of (75) and substituting from these in (74) we obtain easily the condition $\partial(\partial w/\partial r + w/r)/\partial r = 0$, or

$$\frac{\partial w}{\partial r} + \frac{w}{r} = 2c, \tag{77}$$

where c is a constant. This equation gives at once

$$w = cr + \frac{c'}{r}, \tag{78}$$

where c' is another constant.

At the inner surface $r = r_0$ and $-R = p_0$, at the outer surface $r = r_1$ and $-R = p_1$. Substituting these values of r, R [with the corresponding value of w given by (78)] in succession in the first of (75) we obtain two equations to determine c, c'. The reader, remembering that $3\lambda + 2n = 3k$ and $P = (p_0 r_0^2 - p_1 r_1^2)/(r_1^2 - r_0^2)$, may verify that

$$c = \frac{1}{3k}\frac{p_0 r_0^2 - p_1 r_1^2}{r_1^2 - r_0^2}, \quad c' = \frac{1}{2n}\frac{p_0 - p_1}{r_1^2 - r_0^2} r_0^2 r_1^2.$$

Hence (78) gives

$$w = \frac{1}{3k}\frac{p_0 r_0^2 - p_1 r_1^2}{r_1^2 - r_0^2} r + \frac{1}{2n}\frac{p_0 - p_1}{r_1^2 - r_0^2} r_0^2 r_1^2 \frac{1}{r}; \tag{79}$$

and therefore

$$\frac{\partial w}{\partial r} = \frac{1}{3k}\frac{p_0 r_0^2 - p_1 r_1^2}{r_1^2 - r_0^2} - \frac{1}{2n}\frac{p_0 - p_1}{r_1^2 - r_0^2} r_0^2 r_1^2 \frac{1}{r^2}. \tag{80}$$

$$\frac{w}{r} = \frac{1}{3k}\frac{p_0 r_0^2 - p_1 r_1^2}{r_1^2 - r_0^2} + \frac{1}{2n}\frac{p_0 - p_1}{r_1^2 - r_0^2} r_0^2 r_1^2 \frac{1}{r^2}. \tag{81}$$

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$$y = \frac{1}{3k}\frac{p_0 r_0^2 - p_1 r_1^2}{r_1^2 - r_0^2}. \tag{82}$$

The strain in the wall therefore consists of a uniform dilatation of amount $(p_0 r_0^2 - p_1 r_1^2)/k(r_1^2 - r_0^2)$, and a shear in each cross-section of the tube of amount $(p_0 - p_1) r_0^2 r_1^2 / n(r_1^2 - r_0^2) r^2$. The dilatation will be negative, that is, there will be a contraction of the substance of the wall if $p_1 r_1^2 p > {}_0 r_0^2$. Thus if $p_1 > p_0$ there will be voluminal contraction in every part of the walls. If there be only internal pressure the volume strain is everywhere a dilatation.

When the thickness of the walls is great in comparison with the internal radius, and the pressure is wholly internal, the shear at the inner surface is p_0/n. Thus, so far as the yielding to this shear is concerned, there is little to be gained by making the tube of very small bore or very thick.

Equation (75) gives for $p_1 = 0$, $T - p_0 r_0^2 (1 + r_1^2/r^2)/(r_1^2 - r_0^2)$, a tension. Hence the tube may be strengthened to resist internal pressure by hoops shrunk round it so as to place it under circumferential thrust. Internal pressure produces relief of the inner tube from this strain and tends to stretch the outer parts. The strength of guns to resist bursting is increased in this way.

For the radial strain we have, if there be no external pressure,

$$\frac{\partial w}{\partial r} = \frac{p_0 r_0^2}{r_1^2 - r_0^2}\left(\frac{1}{3k} - \frac{1}{2n}\frac{r_1^2}{r^2}\right), \tag{83}$$

and if there be no internal pressure,

$$\frac{\partial w}{\partial r} = -\frac{p_1 r_1^2}{r_1^2 - r_0^2}\left(\frac{1}{3k} - \frac{1}{2n}\frac{r_0^2}{r^2}\right). \tag{84}$$

For most substances k is greater than n, and therefore $1/3k < 1/2n$; and therefore, as $r < r_1$, the value of $\partial w/\partial r$ for zero external pressure is always negative, that is, there is always in that case radial contraction. On the other hand, if there be no internal pressure and the wall be thick enough, we may have $r_0^2/r^2 > 2n/3k$, and the radial strain, outside the cylinder of radius $r_0\sqrt{3k/2n}$, is a contraction, and at all points within this cylinder is an extension. If, however, the wall be not thick enough the radial strain is a contraction at all points.

For steel k is about 1500×10^9 dynes per square centimetre, or 22×10^6 pounds per square inch, n is 730×10^9 dynes per square centimetre, or $10{\cdot}6 \times 10^6$ pounds per square inch. Thus, $3k/2n$ for steel is roughly equal to 3. For flint glass the ratio is more nearly 2·6.

It is worthy of notice in this connection, that stiffness could be given to an otherwise flexible tube by means of hydrostatic pressure within it. Thus if a tube of circular section were closed at the ends and internal pressure p were applied an extension of amount $pr_0^2/3k(r_1^2 - r_0^2)$ would be produced in the direction of the length of the tube. If then care were taken to keep the tube always under tension it might be used as a strut to support applied thrust. Also the tube would resist bending if the resulting compression were not

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so great as to overcome the extension due to the internal pressure. The stiffness of the inflated tyre of a bicycle is a case in point. It has been suggested (see Perry's *Applied Mechanics*, p. 472), by Professor FitzGerald, that easily portable supports for military bridges and other structures could be made by utilising these principles.

632. **Torsion of a Cylindrical Rod.**—It will be seen without difficulty that a pure shearing strain may be given to a cube, as indicated in § 346 above, by displacing a set of planes of a cubic portion of the material parallel to one set of four edges of the cube through spaces proportional to their perpendicular distances from one of the two terminal planes. One diagonal is increased in length, the other diminished by an amount equal to half the measure of the shear, and they remain at right angles to one another. The only distinction between this and the shear described in § 600 lies in the fact that the diagonals are changed in position, being turned round through the small angles ACA' or BDB'.

To every part of a right cylindrical rod of material which has elastic properties the same in all directions at right angles to the length strain of this kind may be imagined given in the following manner. Let the ends of the rod be made planes at right angles to the length, and cover each of them by a rigid plate cemented firmly over the whole end surface by unyielding cement. Then, keeping the rod straight, say by suspending it vertically, and keeping one end plate fixed, turn the other plate about the axis of the rod through any angle θ. Every cross-section will be turned in its own plane through an angle proportional to its distance from the fixed end, and there will be the same distribution of shear throughout the rod.

This mode of applying the shear would not be convenient in practice. In general a cross-bar is soldered or clamped firmly to one end of the rod, while the other is held fixed by being soldered or clamped to a fixed support. The cross-bar is then rotated and the rod is subjected to shear, which, though certainly not uniform very near to either fastening, is uniformly distributed through the rest of the rod, and may in practice, if the rod is long, be taken without serious error as uniform throughout the portion between the fastenings.

633. **Shear in Cross-Section of a Rod.**—We shall now investigate the distribution of the shear over each cross-section. It is to be clearly understood that the rod is either solid with a circular cylindrical boundary, or is a tube bounded by two coaxial cylindrical surfaces. It is clear that if the rod be of uniform elastic material the strain must be uniformly distributed, otherwise there could not be equilibrium. For if at any cross-section the strain were different from what it is at another, the stress applied at one of these to the portion of the rod between them by the matter beyond the cross-section, would not be balanced by the stress applied at the other end in the same way. The angle through which any cross-section has been turned about the axes relatively to a cross-section at unit distance from it on either side is therefore θ/l if l be the length of the rod. This is called the *twist* of the rod.

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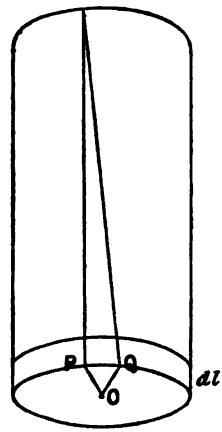
Consider two concentric circles, of radii r and $r+dr$, described in a chosen cross-section from the intersection of the axis with it as centre. The radius to a point P of a cross-section (Fig. 305) has been turned to Q, so that angle $POQ=\theta$. Hence the angle through which a radius to P has been turned relatively to the cross-section at distance dl is $\theta dl/l$. The distance through which P has been moved parallel to the cross-section at distance dl is thus $r\theta dl/l$, and the angle which a line through P, formerly parallel to the axis, now makes with the axis is $r\theta/l$. This is the measure of the shear which has been given to the small approximate parallelepiped bounded by the two cross-sections at distance apart dl, the two radial planes at distance $rd\phi=r\theta dl/l$ apart on the circle of radius r, and the two coaxial cylindrical surfaces of radii r and $r+dr$. The shear at any point in the cross-section is therefore proportional to the distance of the point from the axis. This fact is important, as a rod may be subjected to so great a twist that the surface is strained beyond its elastic limits, while the strain throughout the greater part of the internal substance is well within these limits.

FIG. 305.



634. **Torsional Strains and Stresses.**—To obtain the components of displacement let axes of x and y be drawn from the fixed end of the axis of the cylinder and parallel to the cross-section. The displacements u, v, w along these axes due to the twist θ/l (or τ) are $u=-yz\tau$, $v=xz\tau$, $w=0$. Hence the strains e, f, g, that is, $\partial u/\partial x$, $\partial v/\partial y$, $\partial w/\partial z$ are all zero. Also

$$a=\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}=x\tau,\ b=\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}=-y\tau,\ c=\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}=0. \qquad (85)$$

By (62) the stresses are

$$P=Q=R=0,\ S=nx\tau,\ T=-ny\tau,\ U=0. \qquad (86)$$

To find the stresses on any element supposed subjected to uniform twist, instead of the element referred to in last section, imagine a small parallelepiped of the substance one edge of which is dx, another dy, and the third dz, the three edges being drawn parallel to the axes from the point x, y, z. By Fig. 300, S is (1) a tangential stress acting on the strained matter parallel to the axis Oy, and distributed over those faces of the element which are parallel to the plane yOx and (2) is a tangential stress parallel to the axis Oz, and distributed over the faces parallel to the plane xOz. Similarly, T is (1) a tangential stress parallel to Oz and distributed over the pair of faces parallel to the plane yOz, and (2) a tangential stress parallel to Ox

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and distributed over the faces parallel to xOy. Of course the forces on opposite faces are oppositely directed as shown in Fig. 300. They are required to hold the matter in the state of strain, and are applied by the matter external to the element, except at the bounding surface, where they are applied from without by external bodies.

Consider the stress at the element across a plane parallel to the axis of z, and so situated that its normal makes an angle a with the axis of x. The direction cosines of such a plane are $\cos a$, $\sin a$, 0. Hence by equation (32)

$$F=0,\ G=0,\ H=T\cos a+S\sin a=-n\tau(y\cos a-x\sin a). \quad (87)$$

The only stress is therefore a shearing stress in the plane and parallel to the axis of z.

If the lateral surface of the cylinder is to be free from applied stress we must have $y\cos a-x\sin a=0$, that is the normal to the surface must pass through the axis of z. The cylinder, therefore, to be in equilibrium under pure torsion without tangential stress applied to the lateral surface must be of circular section.

635. **Moment of Tangential Stresses in Cross-Section about Axis. Torsional Vibrations.**—The tangential stress then required for this shear in the case of the right circular cylinder is $nr\theta/l$, and this is the tangential stress exerted across any cross-section. Its direction at any point is perpendicular to the radius. This tangential stress gives a total tangential force exerted on one face of the element referred to in § 633 of $nr\theta\,dr\,rd\phi/l$, or $nr^2drd\phi.\theta/l$, and the moment of the force about the axis is $nr^3drd\phi.\theta/l$. The moment about the axis of the wire of the tangential stresses exerted on the elements of the ring between the two circles of radii r and $r+dr$ is therefore $2\pi nr^3dr\theta/l$, and the total moment for the cross-section is $\pi nr^4\theta/2l$.

This is the moment, over each cross-section, of the internal forces tending to bring the rod back toward its unstrained state. If the rod be suspended vertically, and a cross-bar, or better, a massive cylinder coaxial with the wire, be attached rigidly at its lower end, and the lower end with the attached bar or cylinder be turned relatively to the upper end through an angle θ, the total moment of the forces acting so as to send the cylinder back towards the original position is $\pi nr^4\theta/2l$. If the system is left to itself the wire will untwist, the cylinder will receive angular acceleration until the couple upon it due to twist has vanished, and will continue then to move in the same direction until stopped by the reverse acceleration due to the growing up of twist in the opposite direction. Then it will return, and so on, and vibrations in a definite period will be performed.

To simplify the problem we might suppose that the motion is so slow that the wire has throughout the same twist at each instant. In strictness we have to consider a wave of change of twist continually running up the wire and returning after reflection at the fixed end, so that the wire is kept in a state of stationary torsional vibration. We can discuss this point only very briefly.

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Consider two cross-sections at distance dz' apart and distance z from the lower end of the rod. Let the twist at the lower cross-section be $\partial\theta/\partial z'$; then that at the upper section is $\partial\theta/\partial z' + \partial^2\theta/\partial z'^2.dz'$. Hence the resultant couple on the portion of the rod between the cross-sections is by what we have seen above $\frac{1}{2}\pi nr^4\,\partial^2\theta/\partial z'^2.dz'$. And, since the moment of inertia of the slice about the axis of the wire is $\pi r^2\rho r^2 dz'/2(=\pi r^4\rho dz'/2)$, the rate of change of moment of momentum of the slice is $\frac{1}{2}\pi\rho r^4 dz'\,\partial^2\theta/\partial t^2$. By equating this to the couple we get the equation of propagation of a wave of torsion along the wire,

$$\frac{\partial^2\theta}{\partial t^2} = \frac{n}{\rho}\frac{\partial^2\theta}{\partial z'^2}. \tag{88}$$

The velocity of propagation of a wave of torsional displacement along the wire is thus $\sqrt{n/\rho}$. This result is important; and the reader may note here that the rate of propagation of a wave of shearing strain is always equal to the square root of the ratio of the rigidity modulus to the density of the material.

Now, let M be the mass of the vibrator attached at the lower end of the wire and k its radius of gyration. The rate of change of moment of momentum of the vibrator at any instant, that is $Mk^2\partial^2\theta/\partial t^2$, must be equal to the couple at the lower end due to the twist existing there. Thus

$$Mk^2\frac{\partial^2\theta}{\partial t^2} = \tfrac{1}{2}\pi nr^4\frac{\partial\theta}{\partial z'}$$

for $z'=0$. Also at the upper end of the wire where $z'=l$,

$$\theta = 0.$$

These two conditions with (88) give the motion of the wire and vibrator. Assume as at § 576 above

$$\theta = \left(A\cos\frac{m}{c}z' + B\sin\frac{m}{c}z'\right)\cos(mt+\epsilon)$$

to be a solution of (88), so that each cross-section of the wire (and the vibrator at the lower end) oscillates in simple harmonic motion in period $2\pi/m$. The value of c is easily seen to be $\sqrt{n/\rho}$ the velocity of propagation of the wave of torsional displacement along the wire. The two terminal conditions give

$$A = -\frac{\pi nr^4}{2cMk^2m}B, \text{ and } A = -B\tan\frac{m}{c}l,$$

and therefore for $m\,(=2\pi/T)$ we have the equation

$$\tan\frac{m}{c}l = \frac{\pi r^4 n}{2cMk^2m}. \tag{89}$$

If Mk^2 be very great ml/c will be very small, and we may write $\tan ml/c = ml/c$. Hence, since $A/B = -(\sin ml/c)/(\cos ml/c)$, the solution becomes, B' being another constant,

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$$\theta = B'\sin\frac{m}{c}(l - z')\cos(mt + \epsilon).$$

$$= B'\frac{m}{c}(l - z)\cos(mt + \epsilon) \tag{90}$$

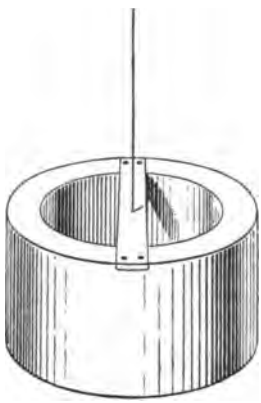
since ml/c is small. Hence it follows that for any given value of t the angle through which a cross-section is turned relatively to the upper end is proportional to the distance of the cross-section from that end, that is, the twist is uniform at each instant during the oscillation. The length of the wire in fact represents only a small part of the stationary wave, of which the fixed end is a node. Putting ml/c for $\tan ml/c$ we get from (89) $4\pi^2 l/T^2 = \pi r^4 n/2Mk^2$, since $m = 2\pi/T$ where T is the period of oscillation of the vibrator. Hence

$$n = \frac{8\pi M k^2 l}{r^4 T^2}. \tag{91}$$

636. **Experimental Determination of Rigidity Modulus.**—Thus by observation of the period of torsional oscillation of a vibrator of known moment of inertia attached to a rod or wire the rigidity modulus of the material can be determined.

The objection to this method is the fact that the fourth power of the radius of the wire appears in the formula for n. Any inaccuracy of measurement of r will be much magnified in r^4, for example, an inaccuracy of 1 per cent. will become about 4 per cent. in r^4.

FIG. 306.



The form of vibrator usually employed is a hollow circular cylinder of brass or copper (Fig. 306) attached so that the axis of the cylinder and the wire are in line. The cylinder is hung on a horizontal cross-bar soldered or clamped to the lower end of the wire. A ready and yet rigid means of attachment (not shown in the diagram) is afforded by two holes nearly fitting the cross-bar, one on each side of the cylinder. The ends of the bar project outward through these holes, and two grooves cut in the upper face of the bar receive the upper edges of the holes, so that there is no motion of the cylinder relatively to the lower end of the wire in the oscillations.

The moment of inertia of this form of vibrator can be determined with great ease. The moment of inertia of the cylinder of mass M_c and internal and external radii r, r' is easily found to be $M_c(r^2 + r'^2)/2$, and a correction can easily be made on this for the pieces cut out to form apertures for the supporting bar. The moment of inertia of the bar if of mass m, length $2l$, and breadth $2b$, is $m(l^2 + b^2)/3$. For the notches near its ends made to receive the

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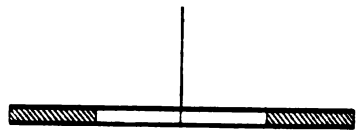
upper edges of the holes, correction is best made when making the correction for the holes in the cylinder, by adding the depth of the notch to the vertical breadth of the holes. The moment of inertia of the bar may then be taken as if the notches did not exist. The value of Mk^2 to be used in (88) is then $M_c(r^2+r'^2)/2+m(l^2+b^2)/3$.

637. **Maxwell's "Vibration-Needle."**—Another method of proceeding is by the use of Maxwell's *vibration-needle* (Fig. 307). This consists of a hollow tube of brass in which four equal tubes of brass each $\frac{1}{4}$ of the length of the large tube can be fitted. Two of these are empty, the other two are filled with lead. The two empty tubes are placed in the middle and the two filled tubes at the ends. The outer tube is rigidly attached to the wire by soldering or clamping the latter to a rigid stem projecting from the end of the tube, and the period of oscillation is determined. The filled tubes are next placed in the middle and the empty tubes at the ends, and the period redetermined. If m be the mass of lead in each of the smaller filled cylinders, that is the difference in mass of the filled and empty cylinders, and $2s$ the length of each of the four cylinders, the difference of the moments of inertia μ_1, μ_2 of the two arrangements is

Fig. 307.



$$\mu_1-\mu_2=16ms^2.$$

If the periods in the two experiments were T_1, T_2 the formula for n gives

$$n=\frac{8\pi\mu_1 l}{r^4T_1^2}=\frac{8\pi\mu_2 l}{r^4T_2^2}=\frac{8\pi(\mu_1-\mu_2)l}{r^4(T_1^2-T_2^2)}$$

$$=\frac{128\pi mls^2}{r^4(T_1^2-T_2^2)}. \qquad (92)$$

638. **Torsional Rigidity of a Rod of Circular Section.**—So far the cylinder twisted about its axis has been supposed to be a right circular cylinder, and the torsional couple required to give a twist θ/l has been found to be $\frac{1}{2}\pi nr^4\theta/l$. The factor $\frac{1}{2}\pi r^4$ may be put in the form $\pi r^2.\,r^2/2$ where πr^2 is the area of cross-section, and $r^2/2$ is the square of the radius of gyration of a uniform circular disk about an axis at right angles to its plane and passing through its centre. The factor $\frac{1}{2}\pi nr^4$ is therefore equal to the moment of inertia of a disk of radius r, and of mass per unit of area n, about its axis. This factor is called the torsional rigidity of the rod. It was proved first by Coulomb, and this result has been amply confirmed by later experiments, that the elastic couple is proportional to the twist θ/l, when elastic limits have not been exceeded. Thus Hooke's law has been proved to be valid for shear.

It will be observed that a given torsional rigidity can be obtained

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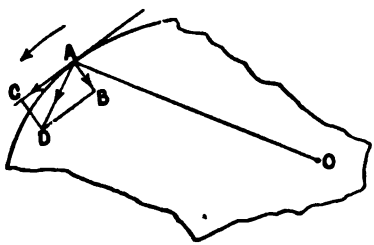
with less material by making the rod a hollow cylindrical tube of circular section. The tube must not, however, be made too thin, or if subjected to bending stress it may collapse.

The fact that the couple is equal to the product of n, the moment of inertia of the cross-section about the axis, and the twist, has been called Coulomb's law. This law is only valid, however, for rods or tubes of circular section.

639. **Torsion of a Non-Circular Cylinder. Warping of Cross-Sections.**—The moment of inertia of the cross-section cannot thus be used as a factor to give the torsional rigidity of a cylinder of any other form than a right circular cylinder. For only in that case is a line drawn in a cross-section from the axis to the surface at right angles to the cross-sectional boundary. Consider the case shown in Fig. 308; there the line OA makes an obtuse angle with the tangent AC to the cross-sectional boundary at A. A displacement of the point A of the cross-section at right angles to OA, which is the kind of displacement which takes place when each cross-section is turned round in its own plane about the axis of the cylinder, can be resolved into two component displacements, one AB at right angles to the tangent at A, and one along the tangent AC to the cross-section. The former of these displacements is a shear of the substance in the inward direction from the outer lateral surface of the cylinder. To effect this it will be seen from § 634 that tangential stress must be applied to the surface in order that a strain of pure torsion, that is twist without warping of the cross-sections, may exist. If we take the axis of x outward from the axis of the cylinder at right angles to the tangent at A, and the axis of y forward along the tangent in the direction of twist, the angle α is zero and we have by (87) $H = -nry$. This is a tangential force parallel to z along the lateral surface of amount nry per unit of area and directed towards the fixed end. It is the force required to preserve the planarity of the cross-section. If no such force acts any point A of the cross-section will be displaced in the direction towards the free end, the rotation being in the direction of the arrow. Thus if Fig. 308 represent the cross-section as seen from the free end, the point A would rise above the paper, and, in the absence of the tangential force at the surface, points between A and O would be raised, but to different distances depending on their position.

Fig. 308.



On the other hand, in a part of the cross-section at which the angle OAC is acute the different points would be depressed below the paper. For example, a cross-section of an elliptic cylinder would be lowered and raised in alternate quadrants as shown in Fig. 309,

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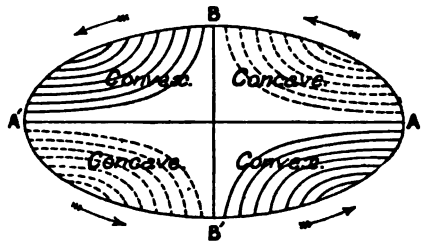
by the dotted and full curves. These are lines of intersection of the warped cross-section by planes perpendicular to the axis of the cylinder. The axes AA', BB' remain in one plane; the distances to which other points are deflected are in the order shown by the curves, beginning with the outside dotted curves where there is greatest concavity, and ending with the outside full curves where there is greatest convexity.

*640. **De St. Venant's Theory of Torsion.**—St. Venant has given a theory of torsion in which the warping of the sections is taken into

FIG. 309.



account. The displacements u, v are taken as in § 634, while w is taken as a function of x, y to be determined by the nature of the boundary. Hence

$$u = -yz\tau,\ v = xz\tau,\ w = \phi(x, y),$$

and $\partial u/\partial x = \partial v/\partial y = \partial w/\partial z = 0$ as before. But now

$$a = x\tau + \frac{\partial \phi}{\partial y},\ b = -y\tau + \frac{\partial \phi}{\partial x},\ c = 0. \tag{93}$$

Hence

$$S = n\left(x\tau + \frac{\partial \phi}{\partial y}\right),\ T = -n\left(y\tau - \frac{\partial \phi}{\partial x}\right). \tag{94}$$

We have $\cos\alpha = dy/ds$, $\sin\alpha = -dx/ds$, and therefore

$$T\cos\alpha + S\sin\alpha = -n\left(x\tau + \frac{\partial \phi}{\partial y}\right)\frac{dx}{ds} - n\left(y\tau - \frac{\partial \phi}{\partial x}\right)\frac{dy}{ds}.$$

But since by hypothesis there is no force along the surface

$$T\cos\alpha + S\sin\alpha = 0.$$

Hence

$$\tau(xdx + ydy) + \frac{\partial \phi}{\partial y}dx - \frac{\partial \phi}{\partial x}dy = 0. \tag{95}$$

Let ψ be a function such that

$$\frac{\partial \psi}{\partial x} = \frac{\partial \phi}{\partial y},\ \frac{\partial \psi}{\partial y} = -\frac{\partial \phi}{\partial x};$$

then

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$$\tfrac{1}{2}\tau(x^2+y^2)+\psi=const, \tag{96}$$

is the equation of the boundary of the wire. Thus ϕ and ψ are what have been defined above (§ 365) as conjugate functions of the co-ordinates, and ψ must fulfil the condition at every point:

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=0. \tag{97}$$

As an example, the simplest possible, we take $\phi(=w)=-kxy$ where k is a constant. This gives

$$\partial\phi/\partial y=-kx=\partial\psi/\partial x,\ \partial\phi/\partial x=-ky=-\partial\psi/\partial y.$$

Hence

$$d\psi=-k(x\,dx-y\,dy),\text{ and }\psi=\tfrac{1}{2}k(y^2-x^2),$$

the equation of the boundary (96) is thus

$$(\tau-k)x^2+(\tau+k)y^2=C, \tag{98}$$

which if k is properly chosen is an ellipse. If the equation be written in the usual form

$$\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}=1,$$

we have

$$\alpha^2(\tau-k)=\beta^2(\tau+k),\text{ or }k=\frac{\alpha^2-\beta^2}{\alpha^2+\beta^2}\tau. \tag{99}$$

The strains are therefore

$$a=x\tau-kx,\quad b=-y\tau-ky,\quad c=0.$$

The strain at the extremity of the major axis is thus $(\tau-k)\alpha$, and at the extremity of the minor axis is $-(\tau+k)\beta$. The ratio of the numerical values of the strains is therefore $\beta(\tau+k)/\alpha(\tau-k)=\alpha/\beta$. Thus the strains at the extremities of the axes are inversely as the lengths of the axes. The elliptic rod is thus strained most at the ends of the minor axis and least at the ends of the major axis.

We have no space for further examples of this most important theory. The reader should, however, consult *The Elastical Researches of Barré de St. Venant*, by Prof. Karl Pearson, Thomson and Tait's *Natural Philosophy*, or Love's *Treatise on Elasticity*.

*641. **Bending and Torsion of Thin Wires.**—We must now consider briefly the finite bending and twist of initially straight wires of small uniform cross-section, which is assumed to be either rectangular or circular. We shall then deal very shortly with the theory of the bending of rods of prismatic shape, in different planes.

The wire is supposed to be held fast at one end while to the other is applied a force and couple both of given amount and direction.

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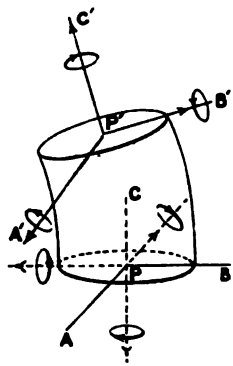
Consider a slice of the wire between two cross-sections which were parallel in the unstrained state of the wire. The material on either side applies to each cross-section a force and resultant couple. By the methods explained at § 176 above each force and couple can be converted into a force at the centroid of the cross-section and a couple about a definite axis. In the wire in the unstrained state let axes be chosen in each cross-section, one of them a tangent to the line of centroids, and the other two at right angles to each other, and also to the sides of the cross-section, if the rod is of rectangular shape.

Thus two rectangular planes intersecting the axis will be obtained. These will be called principal planes of flexure of the wire.

*642. **Equations of Stresses.**—The axes thus drawn at right angles to the axis of the wire are called *transverses*. We may choose a positive direction along the axis of the wire, and take that as axis of z. Then with the other two axes we obtain for any cross-section the usual system (Fig. 3) of three axes of x, y, z. Now let the wire be bent in the plane of xz, or yz. The only effect is that the axis of z ceases to be a straight line, and the axes of x, y at the different sections cease to be parallel. But if the radius of curvature of the wire at each cross-section be very great in comparison with the dimensions of cross-section of the wire, the sections will remain without distortion and the axes of x and y at each will still be at right angles to one another. Again, if the wire be given a small degree of twist τ about the axis of z, such that its reciprocal $1/\tau$ is also great in comparison with the greatest dimension of cross-section, the sections will still remain plane, and they will remain plane under a combination of such strains.

FIG. 310.



Consider then two cross-sections (Fig. 310) at two near points P, P' at distance ds apart in the strained wire: we have a system of axes which we shall call PA, PB, PC at the point P, and a system $P'A'$, $P'B'$, $P'C'$ at P' slightly inclined to the former. The system at P' may be regarded as obtained from that at P by carrying the origin without rotation along the axis through the distance ds, then turning the axes round the axes PA, PB, PC through the angles κds, λds, τds respectively. Let the components of force parallel to the axes of x, y, z (that is parallel to the lines PA, PB, PC) at P be S_1, S_2, T, and at P' be S_1+dS_1, S_2+dS_2, $T+dT$. These act as shown in Fig. 310; those at P' in the positive direction and those at P in the negative direction of the axes, since these forces are applied to the slice by the matter of the wire beyond the cross-section in each case. The forces S_1, S_2 which act in the sections are usually called the *shearing*

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forces, the forces T are usually called the *tensions*. Now resolve the forces at P' parallel to the axes at P. The axis $P'A'$ makes with the axis PC at P the angle $\pi/2+\lambda\,ds$, and the axis $P'B'$ makes with the same axis the angle $\pi/2-\kappa\,ds$. The components S_1+dS_1, S_2+dS_2 therefore have components along the axis PC of amount $-(S_1+dS_1)\lambda\,ds$, $+(S_2+dS_2)\kappa\,ds$, while $T+dT$ along $P'C'$ gives, to quantities of the second order, simply $T+dT$ along PC. The total elastic force in the direction PC on the element is, therefore, $dT-S_1\lambda\,ds+S_2\kappa\,ds$. This with the applied force Zds acting on the slice in the direction PC must be a system in equilibrium. Similarly systems in equilibrium are obtained for the forces parallel to the other two axes.

Hence, if the forces applied from without to the slice be Xds, Yds, Zds, the equations of equilibrium of forces are

$$\left.\begin{aligned}
&\frac{dS_1}{ds}-S_2\tau+T\lambda+X=0\\
&\frac{dS_2}{ds}-T\kappa+S_1\tau+Y=0\\
&\frac{dT}{ds}-S_1\lambda+S_2\kappa+Z=0.
\end{aligned}\right\}\quad(100)$$

In the same way, if G_1, G_2, H denote the couples referred to above, the axes of which are along PA, PB, PC but in the opposite directions, and G_1+dG_1, G_2+dG_2, $H+dH$ denote similarly the couples, the axes of which are along $P'A'$, $P'B'$, $P'C'$, we obtain, remembering that S_1+dS_1, S_2+dS_2 have moments $-(S_2+dS_2)ds$, $+(S_1+dS_1)ds$ about the axes PA, PB respectively,

$$\left.\begin{aligned}
&\frac{dG_1}{ds}-G_2\tau+H\lambda-S_2=0\\
&\frac{dG_2}{ds}-H\kappa+G_1\tau+S_1=0\\
&\frac{dH}{ds}-G_1\lambda+G_2\kappa\qquad=0,
\end{aligned}\right\}\quad(101)$$

G_1, G_2 are called the principal flexural couples, and H the torsional couple. The two couples G_1, G_2 are of course equivalent to a single couple of amount $\sqrt{G_1^2+G_2^2}$ acting about an axis inclined at an angle $\tan^{-1}G_2/G_1$ to the direction of the couple G_1.

Since $\kappa\,ds$, $\lambda\,ds$, $\tau\,ds$ are the angles through which the element is bent about the axes PA, PB, and twisted about the axis PC respectively, it is evident that κ is the curvature which has been given to the element in the plane BPC, and λ the curvature in the plane APC, while τ is the angle through which the transverses in a plane of the slice have been turned round relatively to those of a cross-section at unit distance. We have seen above that $H=C\tau$, where C

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depends on the material and on the extent and shape of the cross-section. For a wire of circular section, or a tube bounded by co-axial circular cylinders, C is the product of n, and the moment of inertia of the cross-section about the axis of the wire or tube.

***643. Flexural Rigidities. Kinetic Analogue of Bending and Torsion of a Thin Wire.**—In the case of bending, as we shall show later, we have

$$G_1 = A\kappa, \quad G_2 = B\lambda,$$

where

A = Young's modulus × moment of inertia of section about PA,
B = " " × " " " " PB.

A, B are called the *principal flexural rigidities* of the wire or rod. Putting these values of the couples in equations (101) we obtain

$$\left.\begin{aligned} A\frac{d\kappa}{ds} - (B-C)\lambda\tau &= S_2 \\ B\frac{d\lambda}{ds} - (C-A)\tau\kappa &= -S_1 \\ C\frac{d\tau}{ds} - (A-B)\kappa\lambda &= 0, \end{aligned}\right\} \quad (102)$$

which are precisely analogous to the Eulerian equations of motion of a rigid body about its principal axes. It will be noticed that since PA, PB, PC are drawn in the directions of the principal axes of cross-section and A, B, C are the moments of inertia of the cross-section about these axes multiplied by certain factors, we may take A, B, C as the moments of inertia of a rigid body. If, then, we suppose the centroid of that body to move along the axis of the body with unit speed, while its principal axes take always the positions which the axes PA, PB, PC, in Fig. 310, have at the successive cross-sections, the values of $d\kappa/ds$, $d\lambda/ds$, $d\tau/ds$ will become the rates at which κ, λ, τ vary with the time as the body moves. We may suppose a second rigid body to remain with one point of its axis fixed and to revolve about its principal axes OA, OB, OC (§ 282) so that these are always parallel to the principal axes of the body which is carried along the wire But κ, λ, τ are the rates ω_1, ω_2, ω_3, say, at each cross-section at which the body is turning round its principal axes, and so $d\kappa/ds$, $d\lambda/ds$, $d\tau/ds$ become $\dot{\omega}_1$, $\dot{\omega}_2$, $\dot{\omega}_3$, and we have precisely the Eulerian equations

$$\left.\begin{aligned} A\dot{\omega}_1 - (B-C)\omega_2\omega_3 &= S_2 \\ B\dot{\omega}_2 - (C-A)\omega_3\omega_1 &= -S_1 \\ C\dot{\omega}_3 - (A-B)\omega_1\omega_2 &= 0. \end{aligned}\right\} \quad (103)$$

Thus the variations of the angular velocities of the second body in its hypothetical motion are precisely similar to those of the angular velocities of the rigid body with one point fixed, under couples round

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the axes of x and y, to which reference is made in § 282, and which in one form as the problem of the top had been previously discussed in the same chapter.

The resultant of κ, λ, τ, which has magnitude $\omega = \sqrt{\kappa^2+\lambda^2+\tau^2}$, is the analogue therefore of the angular velocity of the body the equations of motion of which are given by (103). The motion is about the axis which has direction cosines $(\kappa, \lambda, \tau)/\omega$, which is therefore the analogue of the instantaneous axis.

If S_1, S_2 be zero as in § 287, we obtain as before (from (102)) the equations

$$\left.\begin{aligned} A\kappa^2 + B\lambda^2 + C\tau^2 &= K^2 \\ A^2\kappa^2 + B^2\lambda^2 + C^2\tau^2 &= G^2, \end{aligned}\right\} \quad (104)$$

where K and G are constants.

*644. **Kinetic Analogue. Potential Energy of the Strained Wire.**—It is clear from the considerations already stated that the flexures κ, λ, in the principal planes, and the twist τ may be imposed independently. Hence the work done in producing these strains from zero in the element is, if G_1, G_2, H be the final values of the couples,

$$\tfrac{1}{2}(G_1\kappa + G_2\lambda + H\tau)ds = \tfrac{1}{2}K^2 ds. \quad (105)$$

Thus K^2 is twice the potential energy of the strained wire per unit of length.

The component couples G_1, G_2, H are the moments of momentum of the rigid body about the axis, and represent the impulsive couples required to produce the motion from rest, and G is the resultant impulsive couple. Equations (104) therefore assert that the potential energy per unit of length and the impulsive couple are the same at every cross-section of the wire.

The rigid body may be supposed to have one point fixed and to begin to turn from an initial position in which its principal axes are paralled to the principal axes of a terminal cross-section. If then the body begin to turn with component angular velocities equal to the values of κ, λ, τ for the terminal point and the body is then left to itself, the angular velocities about its principal axes will become in succession the values of κ, λ, τ for successive sections of the wire, so that after a time s it will be moving with the proper angular velocities κ, λ, τ for the section at distance s along the wire from the end.

If there be only couples applied at the extremities there will be zero values of S_1, S_2 all along the wire, and the moment of the resultant couple, and the position of its axis in space will remain unchanged for each cross-section of the body, however the principal axes (which are fixed in the body) and the angular velocities about them may vary.

*645. **Case of Equal Flexural Rigidities. Wire Twisted and Bent into a Helix.** When the two flexural rigidities A, B are equal, and the components S_1, S_2 of force applied to the free terminal are zero, we have a very simple and important case. There remain

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applied to the free terminal of the wire only the component T of force, and the couple. The third equation of (100) shows that T is, in the absence of applied force Z, constant along the wire. Equations (103) are in this case

$$A\dot{\omega} - (A - C)\omega_2\omega_3 = 0,\ A\dot{\omega}_2 - (C - A)\omega_3\omega_1 = 0,\ \dot{\omega}_3 = 0. \qquad (106)$$

The third equation gives $d\tau/ds = 0$, or τ is a constant along the wire Again, since τ is constant the first two equations give

$$A(\kappa^2 + \lambda^2) = const,$$

and the resultant curvature is the same at all points. The curvature in the plane of yz is κ, that in the plane of xz is λ; hence the resultant curvature $1/\rho = \sqrt{\kappa^2 + \lambda^2}$, and since the bending about Px gives convexity towards the positive direction of Py, and the bending about Py gives concavity towards the positive direction of Px, the x and y-direction cosines of the normal are $\lambda\rho$, $-\kappa\rho$. The z-direction cosine is of course zero. The direction cosines of the binormal are therefore $\kappa\rho$, $\lambda\rho$, 0. These are the projections of a unit vector along the binormal at P. At P' the projections of the binormal on Px', Py', Pz' are $\kappa\rho + d(+\kappa\rho)$, &c., and these are component vectors inclined to the axes of Px, Py, Pz at small angles due to the rotations. Since the axes at P' have been turned through the angle $\tau\,ds$ about Pz, through the angle $\kappa\,ds$ about Px, and $\lambda\,ds$ about Py, the projections of the vectors on Px, Py, Pz are to the first order of small quantities

$$\kappa\rho + d(\kappa\rho) - \lambda\rho\tau\,ds,\quad \lambda\rho + d(\lambda\rho) + \kappa\rho\tau\,ds,\quad \lambda\rho\kappa\,ds - \kappa\rho\lambda\,ds,$$

so that the direction cosines with reference to Px, Py, Pz have changed by

$$d(\kappa\rho) - \lambda\rho\tau\,ds,\quad d(\lambda\rho) + \kappa\rho\tau\,ds,\quad 0.$$

The angle, $d\beta$ say, through which the binormal has turned is the square root of the sum of the squares of these quantities. Let ϕ be the angle between the binormal and the axis PA (that is the angle ϕ which OD makes with OA, as shown in Fig. 311), then $\kappa\rho = \cos\phi$, $\lambda\rho = -\sin\phi$. Hence

$$d\beta^2 = (\sin^2\phi + \cos^2\phi)\left(-\frac{d\phi}{ds} + \tau\right)^2 ds^2,$$

or

$$\frac{d\beta}{ds} = -\frac{d\phi}{ds} + \tau. \qquad (107)$$

Taking now the case of a helix we have by the first two of equations (102)

$$A\frac{d\kappa}{ds} - (A - C)\lambda\tau = 0,\quad A\frac{d\lambda}{ds} - (C - A)\tau\kappa = 0.$$

Multiplying the first of these by λ, the second by κ, and subtracting we obtain

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$$\lambda\frac{d\kappa}{ds}-\kappa\frac{d\lambda}{ds}=\frac{A-C}{A}(\kappa^2+\lambda^2)\tau.$$

But since $\phi=\tan^{-1}(-\lambda/\kappa)$

$$-\frac{d\phi}{ds}=\frac{\kappa\frac{d\lambda}{ds}-\lambda\frac{d\kappa}{ds}}{\lambda^2+\kappa^2}=\frac{C-A}{A}\tau.$$

Thus, since τ is constant $d\phi/ds$ is also constant. Further, since $d\beta/ds=-d\phi/ds+\tau$, and is equal to the tortuosity, $1/\sigma$, say, we have

$$\frac{1}{\sigma}=\frac{C}{A}\tau, \tag{108}$$

and this is also constant.

At each cross-section there must act the couple $A\sqrt{\kappa^2+\lambda^2}$ in the osculating plane, and the couple $C\tau$ round the axis of the wire. If as we suppose now the form of the wire be a helix, then $\sqrt{\kappa^2+\lambda^2}=1/\rho$, where ρ is the constant radius of curvature. Hence, if α be the angle at which the wire is inclined at each point to the cross-section of the cylinder, and a the radius of the cylinder

$$\frac{1}{\rho}=\sqrt{\kappa^2+\lambda^2}=\frac{\cos^2\alpha}{a}.$$

Also

$$\frac{1}{\sigma}=\frac{\sin\alpha\cos\alpha}{a},*$$

and therefore

$$\tau=\frac{A}{C}\frac{\sin\alpha\cos\alpha}{a}. \tag{109}$$

*646. **Wire held in form of Helix by Terminal Couple.**—At each cross-section of the wire there act two couples, $C\tau$, which has its axis along the wire, and $A\cos^2\alpha/a$, which has its axis at right angles to the osculating plane. The resultant couple is thus $(C^2\tau^2+A^2\cos^4\alpha/a^2)^{\frac{1}{2}}$. But by the value obtained above for τ this becomes $A\cos\alpha/a$. The axis of the couple $A\cos^2\alpha/a$ is in the plane tangential to the cylinder; hence the axis of the resultant couple is in the same plane inclined at the angle $\tan^{-1}(A\cos^2\alpha/aC\tau)$ to the wire. But $A\cos^2\alpha/aC\tau=\cot\alpha$, and therefore the axis of the resultant couple is parallel to the axis

* These values for the curvature and tortuosity are easily obtained as follows. Imagine a narrow ribbon of paper wound round a cylinder so that the central line of the paper forms a helix. The paper is bent at each point about a generating line of the cylinder. The turning about a generating line can be resolved into two components, one about a line across the ribbon perpendicular to the central line, the other about the central line of the ribbon. The angle turned through about the generating line for a length ds along the central line, is $ds\cos\alpha/a$. Hence the components just specified are $ds\cos^2\alpha/a$, and $ds\cos\alpha\sin\alpha/a$. Hence the curvature and tortuosity are $\cos^2\alpha/a$ and $\sin\alpha\cos\alpha/a$ as stated.

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of the cylinder, that is, the couple acts in a plane at right angles to the axis of the cylinder.

It follows that the couple at the free terminal must be $A\cos\alpha/a$. We see therefore that if one end of a wire of round section be fixed and a couple G be applied at the other end the wire is strained into a helix of curvature $G\cos\alpha/A$ and twist $G\sin\alpha/C$. The angle of the helix and the radius of its cylinder are determined by the magnitude of the couple and the distance of the fixed end of the wire from the plane of the applied couple. If d be this latter distance and l be the length of the wire, we must have $\sin\alpha = d/l$, $\cos\alpha = \sqrt{(l^2 - d^2)}/l$. The radius of the cylinder is thus $G/A\cos\alpha$ or $Gl/A\sqrt{l^2 - d^2}$.

*647. **Wire held in Helix by Axial Force.**—If one extremity of the wire be held fixed while a force R is applied to the other, the wire, provided it be given a proper amount of twist, will be strained into a helix the axis of which is coincident with the line of action of R. This will also be the form of the strained wire, if besides the force R there act at the free extremity a couple the axis of which is parallel to R.

To see this let the helical form be assumed by the wire. The force R gives a couple of moment Ra acting on the portion of wire between the free extremity and the cross-section, and a force equal and parallel to R at the cross-section. The couple just specified must be balanced by the elastic reaction due to the twist and flexure of the wire. Let in addition a couple, with axis parallel to R, be applied at the extremity. Having regard to the direction of twist and flexure of the wire we have

$$G\sin\alpha + Ra\cos\alpha = C\tau$$
$$-G\cos\alpha + Ra\sin\alpha = -A\frac{\cos^2\alpha}{a}.$$

and therefore

$$R = C\tau\frac{\cos\alpha}{a} - A\frac{\cos^2\alpha\sin\alpha}{a^2}, \tag{110}$$

$$G = C\tau\sin\alpha + A\frac{\cos^3\alpha}{a}. \tag{111}$$

If the couple be zero

$$\tau = -\frac{A}{C}\frac{\cos^3\alpha}{\sin\alpha}\frac{1}{a}, \text{ and } R = -A\frac{\cos^2\alpha}{a^2\sin\alpha}.$$

If the force be zero

$$\tau = \frac{A}{C}\frac{\sin\alpha\cos\alpha}{a}, \text{ and } G = A\frac{\cos^2\alpha}{a}.$$

The latter results are those already obtained at § 645 above.

*648. **Kinetic Analogue of Wire bent into Form of Helix.**—The couples acting at the cross-section do not completely exhaust the

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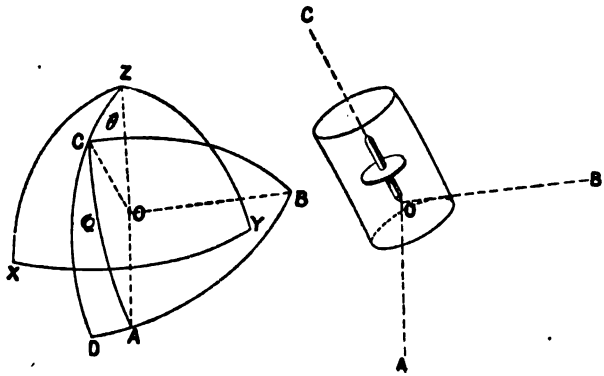
action there. Across each cross-section there acts also the force R, transferred from the terminal by Poinsot's method. The strain in the wire can be studied with great advantage in detail, by supposing a force, and a couple in a plane perpendicular to the line of action of the force, to be applied to one terminal of the wire while the other is held fast. It will be found that all the conditions are fulfilled by the strain which is the elastic analogue of the steady motion of the top under gravity, discussed in chap. v. The reader may verify

FIG. 311.



the following results. Let as in § 282, and Fig. 134 or Fig. 311, θ be taken as the inclination of the axis OC of the top to the vertical OZ, OA, OB, OC be the directions of the principal axes, and θ, ϕ, ψ denote the angles which as shown in Fig. 311 determine the positions of the principal axes OA, OB, OC with reference to fixed axes, OZ vertical and OX, OY horizontal. The force in the kinetic analogue has components S_1, S_2, T parallel to the directions of the principal axes, and if R be the vertical applied force at the free end of the wire, the values of S_2, $-S_1$, which appear on the right of the first two equations of (102), are respectively $R\sin\theta\sin\phi$, and $R\sin\theta\cos\phi$.

If the motion be steady the spin $\omega_3(=\tau)$ about the axis is constant and this is $\dot{\phi}+\dot{\psi}\cos\theta$, where $\dot{\phi}$ is the spin of the body about OC regarded for the moment as a fixed axis, and $\dot{\psi}\cos\theta$ is the component of spin due to the motion of the body as a whole about the vertical OZ. Thus in the elastic analogue $d\phi/ds+d\psi/ds.\cos\theta$ is constant. Further, $\dot{\psi}(=d\psi/ds)$ and θ are both constant in the steady motion, and so $d\phi/ds$ is constant along the wire.

Further results are

$$\sin\theta\frac{d\psi}{ds}=\sqrt{\kappa^2+\lambda^2}=\frac{1}{\rho}, \tag{112}$$

so that $1/\rho$ is constant. Again, $\phi=\tan^{-1}(-\lambda/\kappa)$, and so ϕ is identified with the angle between the binormal OD and the principal axis OA,

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as shown in Fig. 311. Thus since $\cos\theta\, d\psi/ds - \cot\theta/\rho$ we get for the tortuosity $1/\sigma$ the value $\cot\theta/\rho$. The first two equations of (102) with the values of S_2, $-S_1$ given above lead to the result

$$C\tau\rho - A\cot\theta = R\rho^2\sin\theta, \tag{113}$$

which agrees with (109).

*649. **Wire Strained from one Helical Form to Another.**—As a final example of the results for the strain of a spiral spring we may state the values of the force and couple of the wrench required to strain a wire, given without strain in the form of a helix of radius b and inclination β, to a helix of radius a and inclination α, subject to the condition that the wrench is such as to keep the line on the surface of the wire which was in contact with the cylinder of radius b in contact with the cylinder of radius a. In this case κ may be taken as zero, and τ as identical with $1/\sigma$.

The change in τ is therefore $\sin\alpha\cos\alpha/a - \sin\beta\cos\beta/b$, and the change of curvature $\cos^2\alpha/a - \cos^2\beta/b$. The values of R and G for the wrench are obtained by substituting in (110), (111) these differences for the twist and curvature which there appear. Thus

$$\left.\begin{aligned} Ra &= C\cos\alpha\left(\frac{\sin\alpha\cos\alpha}{a} - \frac{\sin\beta\cos\beta}{b}\right) - A\sin\alpha\left(\frac{\cos^2\alpha}{a} - \frac{\cos^2\beta}{b}\right) \\ G &= C\sin\alpha\left(\frac{\sin\alpha\cos\alpha}{a} - \frac{\sin\beta\cos\beta}{b}\right) + A\cos\alpha\left(\frac{\cos^2\alpha}{a} - \frac{\cos^2\beta}{b}\right) \end{aligned}\right\} \tag{114}$$

If the inclinations α, β of the spiral be small the second term on the right in the first of these equations is small in comparison with the first. Neglecting it and noticing that $\cos\alpha/\cos\beta = a/b$, $\sin\alpha = \alpha = h/l$, $\sin\beta = \beta = h_0/l$, where h, h_0 are the axial lengths of the spring in the two cases, and l is the length of the wire, we obtain

$$R = C\cos\alpha\cos\beta\frac{h - h_0}{abl}, \text{ or } h - h_0 = \frac{abl}{C\cos\alpha\cos\beta}R. \tag{115}$$

Of course a and b are in practice very nearly equal, and $\cos\alpha\cos\beta$ is nearly unity.

The last equation expresses a rule given by the late Professor J. Thomson for finding the elongation of a spiral spring of small inclination produced by a given axial force R. Let the wire be straightened out along the axis of the cylinder, and held fixed at one end, while to a circular cap of radius equal to that of the cylinder, attached at its centre at right angles to the wire, is applied tangentially a force R. This force will twist the wire until the torsional reaction just balances the couple Ra. The distance through which a point on the edge of the cap has moved is $h - h_0$.

650. **Flexure of a Uniform Rod. Principal Axes. Elastic Central Line.**—We shall consider here very briefly the bending in one plane of a thin uniform rod which is straight when free from

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strain. It will be supposed as before that the amount of bending is such at every point that the radius of curvature is great in comparison with every dimension of cross-section of the rod, and, moreover, that the breadth of the rod at right angles to the plane of bending is small in comparison with a mean proportional to the radius of curvature and the thickness in the plane of bending. The exceptional case here excluded is that of a broad thin bar, like a flat clock-spring, bent into a circle so small as to make the breadth comparable with the mean proportional specified. When these conditions are satisfied we may, as has been shown by de St. Venant in his theory of the Flexure of Prisms,* as an approximation to actual fact consider the cross-sections as undergoing no distortion.

Now imagine a very thin slice of the rod to be taken at any cross-section with its faces parallel to the cross-section. This slice will have principal axes of moment of inertia at right angles to one another, which we shall refer to as the principal axes of moment of inertia of the cross-section. The centre of inertia of the slice we shall refer to as the centre of inertia of the cross-section. When the rod is bent these axes are no longer straight lines, but curves parallel to the plane of bending.

We shall call that straight line which contained the centres of inertia of the cross-sections the *elastic central-line* or simply the *central-line* of the rod.

651. **Principal Planes of Bending.**—At each cross-section there can be drawn a tangent to the elastic-central line, and through that tangent can be drawn two planes to touch the two curves which in the unstrained rod were the principal axes of moment of inertia of the cross-section. These planes are called the principal planes of flexure of the rod. We shall suppose first the bending to be in a principal plane, that is so that the radius of curvature of each of the lines which were parallel to the unstrained elastic central line are all parallel to a principal plane of flexure.

The effect of the bending is to produce a force and couple with which the matter on one side of an element of a cross-section acts on the matter on the other side of the element. Consider then any cross-section intersecting the elastic central line in a point P, and let us distinguish the matter on the two sides of the cross-section as the matter on the left and on the right of the cross-section respectively. The force and couple applied at each element of the cross-section to the matter on the right, say, can be converted by Poinsot's method into a single force at P, and a couple tending to turn the cross-section round an axis perpendicular to the plane of bending. The force will give two components T along the tangent to the central line at P, and the other N, per unit of length of the central line, towards the centre of curvature of the element of the elastic central line at P. We shall call these components T and N. The latter is along the cross-section and it is usually called the shearing force, the former is a stretching force along the elastic central line.

* See *The Elastical Researches of Barré de St. Venant*, by Karl Pearson.

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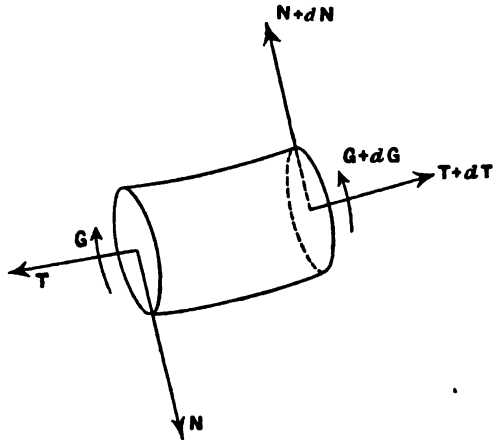
652. **Equations of Equilibrium.**—The equations of equilibrium of a slice of the rod will be three in number, two obtained by resolving the forces tangentially to the elastic central line, and normally to the central line, in the plane of bending, and a third expressing that the moments of forces upon it are in equilibrium. Let, Fig. 312, T be the stretching-force towards the left along the tangent to the central line at the left-hand end of the slice, then $T+dT$ will denote the stretching-force towards the right along the tangent at the right-hand end of the slice. Also let X, Y be the components of applied force along the central line, and inwards towards the centre of curva-

FIG. 312.



ture, each taken per unit length of the element. The shearing-force on the left-hand end of the element is *from* the centre of curvature, that on the right-hand end is *toward* the centre of curvature and of amount $N+dN$. These give a resultant along the tangent to the central line at the right-hand end of the element of amount

$$dT - Nd\theta + Xds.$$

For equilibrium this must vanish. Similarly the resultant force toward the centre of curvature is found to be $dN + Td\theta + Yds$, and this also must vanish. If R be the radius of curvature, $d\theta = ds/R$. Hence we have the two equations

$$\frac{dT}{ds} - \frac{N}{R} + X = 0, \quad \frac{dN}{ds} + \frac{T}{R} + Y = 0. \tag{116}$$

A third equation is obtained from the elastic couples G, $G+dG$ acting on the slice, the couple, Hds say, due to the applied forces X, Y, and the moments of the internal forces T, N. Clearly this equation is

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$$\frac{dG}{ds} + N + H = 0. \tag{117}$$

There are, of course, in every actual case conditions which must be fulfilled at the ends of the rod; these will appear in the examples which follow. The equations just obtained are sometimes of service but in most cases the solution of problems is best obtained by the direct application of first principles.

653. **Estimation of Couple of Bending at Cross-Section.**—It is necessary to explain next how the couple G depends on the amount of bending at the cross-section. No complete treatment of this point is possible without entering into considerations of general theory which lie outside the limits of this work, and it is to be understood that what is stated here requires amplification, and to some extent correction. Strictly speaking, there is no line of particles originally parallel to the axis of the rod which remains unaltered in length, but the elastic central line suffers less change of length than any other line of the rod. Lines of the rod which lie on the convex side of a cylindric surface through the elastic central line at right angles to the plane of bending experience elongation, and those which lie on the inner side of this cylinder experience contraction, relatively to the elastic central line, and the amount of this relative elongation or contraction is proportional to the distance of the point from the cylindric surface. Thus across every element of the cross-section where there is this relative stretch there is applied to the matter on one side A of the cross-section by the matter on the other side B a pull, and to every element where there is relative contraction there is applied to the matter on the side A a thrust by the matter on the side B, in each case relatively to the action across an element at the central line. This is illustrated by Fig. 313. We make the supposition that if the unital elongation or contraction at any point be e, the force on an element of area dS of the cross-section is $EedS$, where E is Young's modulus for the material. This amounts to supposing that each longitudinal filament of the substance expands and contracts laterally with perfect freedom, just as if its lateral surface were free. Thus across each cross-section there acts a couple tending to turn the cross-section round in the plane of bending so as to straighten the rod, and this is balanced and the rod kept bent by the moment at the cross-section of the forces applied to the wire from outside, and by the moment of any applied couples.

That the filaments of the material swell or shrink laterally according as they are shortened or lengthened is shown by the fact that the cross-section is distorted from the rectangular shape shown by the dotted lines in Fig. 314 to the curved shape shown by the full lines. The convex and concave sides of the cross-section are on the concave and convex side respectively of the rod in the plane of bending. In the case of the flat spring referred to in § 650, unless the condition there stated is fulfilled, the half-thickness AX of the bar

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would be much greater than the distance of A' from the line XX, and much less than the distance of B' from the same line. The elongations and contractions at the edges would then be very different from those supposed to exist in the approximate treatment of flexure here given.

For a full discussion reference should be made to St. Venant's researches.

On the supposition that the cross-section remains plain and rectangular, we can easily calculate the couple across the cross-section.

FIG. 313.

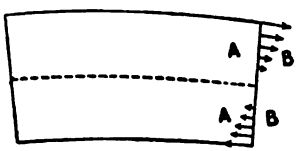
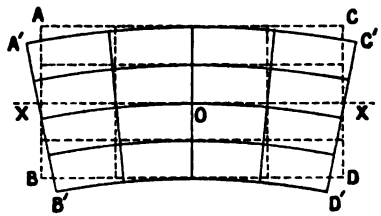


FIG. 314.



Let x be the distance of the filament considered from the principal axis about which the bending takes place, and ds be the distance along the elastic central line from the cross-section under consideration to the other cross-section of a slice of the material. The unitel elongation of a filament at distance x in or parallel to the plane of bending is proportional to x; we shall call it ax. The moment, about the principal axis just specified, of force applied to an element of area dS is therefore Eax^2dS. The amount of the couple is obtained by integrating this over the whole cross-section, since the moment due to contraction tends to turn the cross-section in the same way round. Hence if G be the bending couple

$$G = Ea\int x^2 dS = EaI, \tag{118}$$

where I is the so-called moment of inertia of the cross-section. It remains to determine a. The inclination of one terminal cross-section of the slice to the other is $axds/x$, or ads. But this is the angle $d\theta$ between the radii of curvature of the elastic central line where it meets the cross-sections. Hence $a = d\theta/ds = 1/R$. Hence we have

$$G = EI\frac{1}{R}, \tag{119}$$

or the couple is equal to the product of the Young's modulus, the moment of inertia of the cross-section and the curvature. The quantity EI is frequently called the flexural rigidity of the rod.

If the rod is of rectangular cross-section of dimensions $2a$, $2b$, the flexural rigidity is $4Ea^3b/3$ or $4Eab^3/3$ according as the plane of

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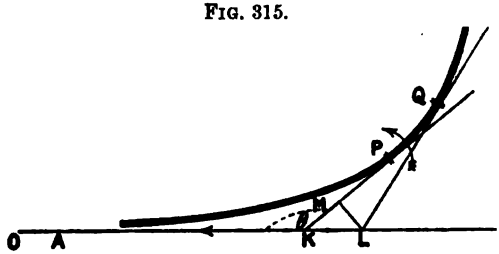
flexure is the plane parallel to the two sides of length $2a$, or to the two sides of length $2b$. Thus the flexural rigidity of a beam, or the couple required to produce unit curvature at any cross-section can be made as great as may be desired by making the dimensions in the plane of bending sufficiently great. The reader may contrast the difficulty of bending a lath in its own plane with the ease with which it can be bent in a plane at right angles to its breadth.

654. **The Linea Elastica. Dynamical Analogue.**—The case of a wire bent by forces applied to its extremities admits of illustration by a dynamical analogue. At a cross-section P at which the curva-

FIG. 315.



ture is $1/\rho$ let a tangent to the wire be drawn and let it make an angle θ with the line of action of a force F applied to an extremity A of the wire, while the wire is held bent by a couple applied to it by the part beyond P, as in Fig. 315. This couple is B/ρ, and has the sense indicated by the curved arrow. But if θ be the angle indicated on the diagram and ds (or PQ) be taken in the direction from P to Q, $1/\rho$ is $-d\theta/ds$. The couple at P is therefore numerically $-Bd\theta/ds$. Also the moment round P of the force F is $Fl\sin\theta$, if l be the distance of P from the point K of intersection of the tangent with the line of F. Hence

$$B\frac{d\theta}{ds}+Fl\sin\theta=0.$$

Differentiating we have

$$B\frac{d^2\theta}{ds^2}+F\frac{dl}{ds}\sin\theta+Fl\cos\theta\frac{d\theta}{ds}=0, \qquad (120)$$

Let the tangent at Q intersect the line of action of F in L, and let LM be perpendicular to KP. Then clearly $dl=ds-KM=ds+LM/\tan\theta=ds-ld\theta\cos\theta/\sin\theta$. Hence $\sin\theta\, dl/ds=\sin\theta-l\cos\theta\, d\theta/ds$ and (120) becomes

$$B\frac{d^2\theta}{ds^2}+F\sin\theta=0. \qquad (121)$$

which is (117) thus obtained by another method.

If ds represent an increment of time (121) becomes the equation

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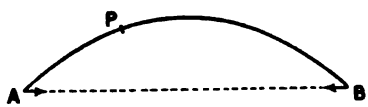
of motion of a simple pendulum of length Bg/F vibrating through a finite angle. If then as the pendulum vibrates the different values of θ be taken at successive intervals of time each numerically equal to ds, the inclination of the thread to the horizontal will give the inclination of the tangent to the line of action of F at the corresponding cross-sections of the wire.

The curve into which the wire is bent is called the *linea elastica*, or elastic curve. Various forms are shown in Figs. 316–321. The elastica with points of inflexion as in Fig. 317 has the directions of its tangents given by an oscillating pendulum with, if necessary for the arc of vibration, a rigid rod for support. When there are no points of inflexion as in Fig. 320 the directions of its tangents are given by a revolving pendulum, that is a rigid pendulum revolving in a vertical plane.

655. **Forms of Elastica.**—We can now solve some simple problems. Take first the case of a uniform bow bent by a stretching-force in the string as shown in Fig. 316.

FIG. 316.



If F be the stretching-force in the cord and x, y be the co-ordinates (from A as origin along and perpendicular to the cord) of the cross-section P considered, we have $G = Fy$. Hence $EI/R = Fy$, or if $EI/F = a^2$ and R be the radius of curvature

$$Ry = a^2. \tag{122}$$

But by geometry $1/R = d^2y/dx^2/\{1 + (dy/dx)^2\}^{\frac{3}{2}}$, and therefore

$$y = a^2 \frac{\dfrac{d^2y}{d^2x}}{\left\{1 + \left(\dfrac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}} = a^2 \frac{\dfrac{dp}{dx}}{(1 + p^2)^{\frac{3}{2}}}$$

if p denote dy/dx. This may be written

$$ydy = a^2 \frac{pdp}{(1 + p^2)^{\frac{3}{2}}},$$

which gives by integration

$$y^2 = C - 2a^2 \frac{1}{(1 + p^2)^{\frac{1}{2}}} \tag{123}$$

or

$$\frac{1}{p} = \frac{dx}{dy} = \frac{C - y^2}{\sqrt{4a^4 - (C - y^2)^2}}. \tag{123'}$$

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Hence

$$x = \int \frac{(C - y^2)dy}{\sqrt{4a^4 - (C - y^2)^2}}, \tag{124}$$

in which either the positive or the negative value of the square root may be taken.

Equation (123) may be written $\pm \sqrt{1 + p^2} = 2a^2/(C - y^2)$. Hence if $p = 0$, $C - y^2 = \pm 2a^2$, and

$$y = \pm \sqrt{C \pm 2a^2}. \tag{125}$$

This gives the points of maximum or minimum value of y according as the upper or lower sign is taken under the radical on the right.

FIG. 317.

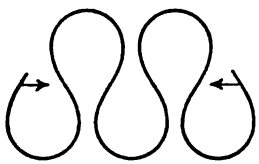
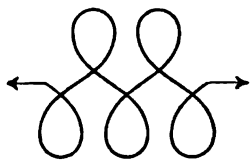


FIG. 318.

FIG. 319.



If C be less than $2a^2$ the negative sign under the radical on the right cannot be taken. Taking then the upper sign we see that the upper and lower signs *before* the radical give equal and opposite numerical maxima of y as in Figs. 317–319. If $C > 2a^2$ there are alternate maxima and minima values as in Fig. 320. Fig. 320 corresponds to the positive sign before the radical; the negative sign would give a similar curve on the opposite side of the axis of x.

If y be small the equation of the curve can be obtained by suc-

FIG. 320.

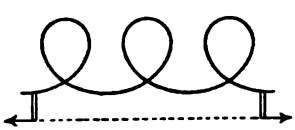
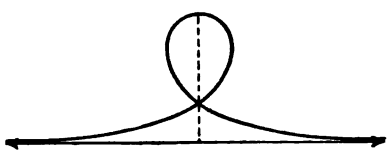


FIG. 321.



cessive approximatiou. For this the reader may consult Thomson and Tait's *Natural Philosophy*, vol. i. part ii., § 611.

Figs. 320 and 321 show different forms of the non-inflexional elastica. Fig. 321 is the curve for the case in which C is put equal to $2a^2$ in (123). The equation of the curve as the reader may verify is, if the axis of y be the dotted line of symmetry,

$$x = \pm \sqrt{4a^2 - y^2} \pm a \log \frac{2a + \sqrt{4a^2 - y^2}}{y}. \tag{126}$$

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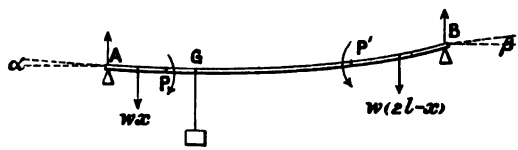
The locus has two branches starting from the vertex, and terminating one at $x = +\infty$, and the other at $x = -\infty$. The equation gives the latter or the former branch according, as the upper or the lower signs are taken on the right. This curve is of physical importance as it is the curve of the capillary surface along a plane plate dipping into a liquid.

656. **Flexure of Beam Supported on Props at its Ends: Ends Free.**—A uniform rod is supported at its ends on props on the same level and bent by its own weight and a weight W attached between the props.

Let A, B, Fig. 322, be the supports and G the point at which the weight is applied. [The diagram requires correction: A, B, should be on the same level.] Let also $2l$ be the length of the rod and a the

FIG. 322.



distance of G from A, w the weight of the rod per unit of length. It is required to determine first the curvature at a cross-section P at a horizontal distance $x(<a)$ from A. The trestle at A bears half the weight of the rod, and the fraction $(2l-a)/2l$ of W. Hence at A there is applied an upward force of amount $wl + W(2l-a)/2l$. At the cross-section halfway between A and P there acts a downward force wx. The horizontal distance of this point from P, on the supposition of only a small amount of bending, is $\frac{1}{2}x$. The total couple applied to the left-hand extremity of the portion of the rod to the right of P, by the portion to the left, is thus

$$\left(wl + \frac{2l-a}{2l}W\right)x - \tfrac{1}{2}wx^2,$$

if the direction of turning represented by the arrow be taken as positive. Measuring y downwards and putting B for the flexural rigidity EI, we get

$$-B\frac{d^2y}{dx^2} = \left\{wl + \frac{2l-a}{2l}W\right\}x - \tfrac{1}{2}wx^2.$$

Integrating this with respect to x we obtain

$$-B\frac{dy}{dx} = \left\{wl + \frac{2l-a}{2l}W\right\}\frac{x^2}{2} - \tfrac{1}{6}wx^3 - B\tan\alpha, \qquad (127)$$

if α be the inclination of the rod at A to the horizontal. Integrating again we get

$$-By = \left(wl + \frac{2l-a}{2l}W\right)\frac{x^3}{6} - \tfrac{1}{24}wx^4 - Bx\tan\alpha, \qquad (128)$$

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and no constant falls to be added since y is zero when x is zero. Put F_1 for $wl + W(2l-a)/2l$, the force applied at A, and F_2 for the corresponding force, $wl + Wa/2l$, applied at B. The same process gives for a point P'

$$-By = \tfrac{1}{6}F_2(2l-x)^3 - \tfrac{1}{24}w(2l-x)^4 - B(2l-x)\tan\beta, \qquad (129)$$

where β is the inclination of the rod to the horizontal as indicated in Fig. 322, and x now denotes the distance AP'.

The values of y and dy/dx given by (128) and (129) must coincide for $x = a$. Thus, putting b for $2l - a$, we obtain for $x = a$

$$\left.\begin{aligned} -By &= \tfrac{1}{6}F_1a^3 - \tfrac{1}{24}wa^4 - Ba\tan\alpha \\ -By &= \tfrac{1}{6}F_2b^3 - \tfrac{1}{24}wb^4 - Bb\tan\beta \end{aligned}\right\} \qquad (130)$$

$$\left.\begin{aligned} -B\frac{dy}{dx} &= \tfrac{1}{2}F_1a^2 - \tfrac{1}{6}wa^3 - B\tan\alpha \\ -B\frac{dy}{dx} &= -\tfrac{1}{2}F_2b^2 + \tfrac{1}{6}wb^3 + B\tan\beta. \end{aligned}\right\} \qquad (131)$$

The first two, (130), give

$$B(a\tan\alpha - b\tan\beta) = \tfrac{1}{6}(F_1a^3 - F_2b^3) - \tfrac{1}{24}w(a^4 - b^4),$$

and the second two, (131),

$$B(\tan\alpha + \tan\beta) = \tfrac{1}{2}(F_1a^2 + F_2b^2) - \tfrac{1}{6}w(a^3 + b^3).$$

Hence

$$B(a+b)\tan\alpha = \tfrac{1}{6}F_1a^2(3b+a) + \tfrac{1}{3}F_2b^3 - \tfrac{1}{24}w(4a^3b + a^4 + 3b^4),$$

and similarly

$$B(a+b)\tan\beta = \tfrac{1}{6}F_2b^2(3a+b) + \tfrac{1}{3}F_1a^3 - \tfrac{1}{24}w(4b^3a + b^4 + 3a^4).$$

But $F_1 = wl + Wb/(a+b)$, $F_2 = wl + Wa/(a+b)$. Substituting in the last two equations and reducing we obtain

$$\left.\begin{aligned} B(a+b)\tan\alpha &= \tfrac{1}{6}Wab(a+2b) + \tfrac{1}{24}w(a+b)^4 \\ B(a+b)\tan\beta &= \tfrac{1}{6}Wab(b+2a) + \tfrac{1}{24}w(a+b)^4, \end{aligned}\right\} \qquad (132)$$

which give the inclinations of the rod at its extremities. The reader may study the results obtained by supposing either W or w negligible.

The value of y for any value of x is given by (128) and (129) when those of $\tan\alpha$ or $\tan\beta$ and F_1 or F_2 are inserted. The reader may verify that the deflection of the rod at any point P due only to the weight W hung at another point Q is equal to the deflection which would be produced at Q if W were hung at P. If W be applied at the middle point so that $a = b = l$ and $F_1 = \tfrac{1}{2}W$ we obtain after reduction

$$By = \tfrac{1}{6}Wl^3 + \tfrac{5}{24}wl^4, \qquad (133)$$

which gives the droop of the rod at the middle point.

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If the weight $2wl$ of the rod itself be negligible in comparison with W, we have

$$y = \tfrac{1}{6}\frac{Wl^3}{B}. \tag{134}$$

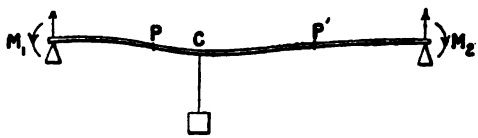
By observing this value of y the Young's modulus $E(=B/I)$ can be determined for the material of which the rod is composed. We have then

$$E = \tfrac{1}{6}\frac{Wl^3}{Iy}. \tag{135}$$

The reader may himself solve the problem in which the rod is hung over a trestle at its middle point and bent by its own weight. It will be found that the ends hang below the middle point a distance $\frac{3}{24}wl^4/B$.

657. **Beams Supported at Ends: Ends Horizontal.**—The reader may also solve the problem of a rod resting at its extremities on two

FIG. 323.



props and bent by a single load applied at any point C, subject to the condition that the ends are constrained to be horizontal. To produce this horizontality couples M_1, M_2 must be supposed applied at the ends as shown in Fig. 323. For a point P, between A and C we obtain

$$B\frac{d^2y}{dx^2} = M_1 - F_1x, \tag{136}$$

and for a point P' between C and B

$$B\frac{d^2y}{dx^2} = M_2 - F_2(2l - x). \tag{137}$$

These integrated give for P and P' respectively

$$By = \tfrac{1}{2}M_1x^2 - \tfrac{1}{6}F_1x^3, \quad By = \tfrac{1}{2}M_2(2l - x)^2 - \tfrac{1}{6}F_2(2l - x)^3. \tag{138}$$

But these values of y must coincide at C, where we suppose $x = a$, $2l - x = b$. Hence

$$a^2(3M_1 - F_1a) = b^2(3M_2 - F_2b).$$

Also dy/dx as obtained from the two equations of (138) must be the same for C, and this condition gives

$$a(2M_1 - F_1a) = -b(2M_2 - F_2b).$$

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Two other equations are obtained by taking moments about A and B, namely

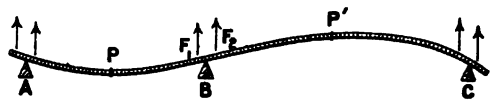
$$F_1 = \frac{Wb + M_1 - M_2}{a+b}, \quad F_2 = \frac{Wa + M_2 - M_1}{a+b},$$

and from these with the preceding equations the values of M_1, M_2, F_1, F_2 can be obtained.

The results substituted in (138) give the deflection at any point in AC, or CB. The points of inflection can be obtained by putting the values of M_1, F_1 in (136) and M_2, F_2 in (137) and equating d^2y/dx^2 to zero.

658. **Clapeyron's Theorem of the Three Moments.**—We shall consider one other problem, which is of importance in engineering Let a rod, bent by its own weight only, rest on any number of supports at points A, B, &c., on the same level. We may start from

FIG. 324.



any prop, say B, if we consider the shearing forces F_1, F_2 applied to the parts of the rod on the two sides of the support, and the couple, M_B applied at B to the matter on either side of the prop by the matter on the other. This is called the bending moment at B. Measuring distance x from B towards the right we obtain for a point P'

$$B\frac{d^2y}{dx^2} = M_B - F_2x + \tfrac{1}{2}wx^2, \tag{139}$$

and again measuring distance towards the left for a point P,

$$B\frac{d^2y}{dx^2} = M_B - F_1x + \tfrac{1}{2}wx^2. \tag{140}$$

Since y is measured downwards, we have from (139) if β be the inclination of the rod to the horizontal at the prop B

$$By = \tfrac{1}{2}M_Bx^2 - \tfrac{1}{6}F_2x^3 + \tfrac{1}{24}wx^4 - Bx\tan\beta, \tag{141}$$

and from (140)

$$By = \tfrac{1}{2}M_Bx^2 - \tfrac{1}{6}F_1x^3 + \tfrac{1}{24}wx^4 + Bx\tan\beta. \tag{142}$$

The x in the former of these means the distance of P' from B, in the latter it is the distance of P from B.

If a be the distance BA, and b the distance BC, M_A, M_C the bending moments at A and C, then since Bd^2y/dx^2 in (140) becomes the moment at A when $x=a$, and Bd^2y/dx^2 in (139) becomes the moment at C when $x=b$,

$$M_A = M_B - F_1a + \tfrac{1}{2}wa^2, \quad M_C = M_B - F_2b + \tfrac{1}{2}wb^2, \tag{143}$$

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from which F_1 F_2 can be determined from the bending moments M_A, M_B, M_C, and the distances a, b.

Again (141) and (142) give since $y=0$ when $x=b$, and when $x=a$

$$\tfrac{1}{2}M_B b - \tfrac{1}{6}F_2 b^2 + \tfrac{1}{24}wb^3 - B\tan\beta = 0$$
$$\tfrac{1}{2}M_B a - \tfrac{1}{6}F_1 a^2 + \tfrac{1}{24}wa^3 + B\tan\beta = 0.$$

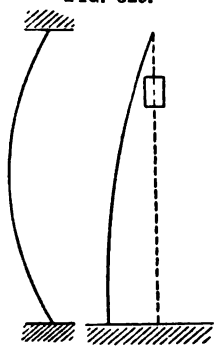
Eliminating F_1, F_2 from these by (143) we obtain

$$a(M_A + 2M_B) + b(M_C + 2M_B) = \tfrac{1}{4}w(a^3 + b^3), \qquad (144)$$

a relation connecting the three moments M_A, M_B, M_C at the three consecutive props. This is known as Clapeyron's Theorem of the Three Moments.

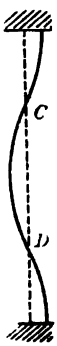
For further particulars on the theory of Elasticity generally and a discussion of some of the most recondite problems of the subject the reader may refer to *Thomson and Tait*, to Professor Pearson's *Elastical Researches of Barré de St. Venant*, and to Professor Love's very valuable treatise. Much has been done by means of graphical methods, especially in the discussion of problems of practical or industrial importance, but for these reference must be made to Treatises on Applied Mechanics.

FIG. 325.



659. **Stability of Long Rod under Thrust. Rod Rounded at Ends.**—The following problem of the *linea elastica* is of practical importance. A long straight rod of given flexural rigidity B, the same in all planes, through the axis, acts as a strut resisting end thrust of given amount W. Clearly if the thrust be too great and the ends be held so that they cannot move laterally the rod will give way at some intermediate point by breaking across (that is by being crushed at one side, and torn asunder at the other), or if one end be free to move laterally it will move into such a position as that shown in the second diagram of Fig. 325. There are two cases which we may consider: (1) that in which the ends are left unconstrained as to direction, by being rounded and therefore made free to turn; (2) that in which the tangents to the rod at the ends are constrained to have the direction of the line of thrust as in Fig. 326. We shall suppose that the ends of the rod lies in the line of thrust in both cases. In the first case the ends are points of zero curvature, that is are points of inflexion on the elastica: in the second there are points of inflexion situated at C, D.

FIG. 326.



Let the rod be deflected *slightly*, while the ends are kept in the line of thrust, as in the first diagram of Fig. 325; then measur-

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ing x along the line of thrust from a fixed end, and denoting the distance of a point of the rod from this line by y we have, putting m for W/B,

$$\frac{d^2y}{dx^2}+m^2y=0.$$

The solution of this equation is

$$y=A\cos(mx+\alpha),$$

subject to the condition that $y=0$ when $x=0$, and when $x=2l$. These conditions are satisfied if

$$\alpha=(2n+1)\frac{\pi}{2},\ 2ml+\alpha=(2n+3)\frac{\pi}{2},$$

so that $2ml=\pi$, or

$$\frac{W}{B}(2l)^2=\pi^2. \tag{145}$$

Thus, if W/B have a greater value than $\pi^2/4l^2$ the rod will be unstable. It is to be remembered that $2l$ is the length of the rod.

660. **Rod or Pillar fixed at both Ends.**—In case (2) a couple of moment M must be applied at each end to keep the tangents at the ends in the line of thrust. In practice this may be done by making the ends of the pillar plane and constraining them to remain perpendicular to the line of thrust. Hence the equation of equilibrium is

$$\frac{d^2y}{dx^2}+m^2\left(y-\frac{M}{W}\right)=0,$$

of which the solution is

$$y=\frac{M}{W}+A\cos(mx+\alpha).$$

The conditions are $y=0$ when $x=0$ and when $x=2l$. Again, $dy/dx=0$, when $x=0$, when $x=l$, and when $x=2l$. These conditions are all satisfied by taking

$$(1),\ \frac{M}{W}=A\ ;\qquad (2),\ \alpha=(2n+1)\pi$$

$$(3),\ ml+\alpha=(2n+2)\pi\ ;\ (4),\ m2l+\alpha=(2n+3)\pi.$$

Hence

$$ml=\pi,$$

or

$$\frac{W}{B}(2l)^2=4\pi^2. \tag{146}$$

Hence, if $W/B>\pi^2/l^2$, the rod will be unstable. The rod has thus four times the strength to resist thrust possessed by the rod (of the same length) considered in § 659.

The value of y at the points of inflexion is M/W, and halfway

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between these points is $2M/W$. Thus the points of inflexion are at distance $\frac{1}{2}l$ from the ends. It will be seen that this gives the maximum distance of the line of the ends from the line of inflexions. It will be noticed that the middle portion, which has half the length l of the whole strut, acts as a strut under the conditions specified in § 659, and that the value of W/B, namely π^2/l^2, corresponds.

There are other cases, but these may all be worked out in the same way. It will be noticed that the theory gives $\pi^2/4l^2$, π^2/l^2, $9\pi^2/16l^2$ for the limiting values of W/B for three struts of the same length $2l$ with their ends in the line of thrust, the first having rounded ends, the second its ends fixed in the direction of thrust, and the third one end rounded the other fixed in the direction of thrust. The results in cases of lateral displacement of the ends can be obtained at once from a consideration of the action of either of the portions of the strut in Fig. 326 between a point of inflexion and the adjacent end, or of either half of the middle part according to the case proposed.

661. **Short Strut. Formula used in Practice.**—It will be obvious that if the strut be very short it will not give way in the manner here contemplated but by yielding to crushing force. An empirical formula is given in Rankine's *Applied Mechanics*, which expresses the results of experiments made on struts of different lengths. The formula was proposed by Professor Lewis Gordon to express the results of experiments made by Mr. Hodgkinson on the strengths of pillars under different circumstances.

If P be the total applied thrust, f and a be constants, L denote the length of the strut, and S the area, and k the radius of gyration of the cross-section, the formula is

$$P = \frac{fS}{1 + 4a\frac{L^2}{k^2}}. \tag{147}$$

The values of f and a are chosen so as to make the values of P accord with experiment. It is found that the theory given above is only in accord with experiment when the strut is very long; for struts of even moderate length the empirical formula must be employed. Mr. Hodgkinson's experiments give the following values of f and a:

	f (lbs. per sq. inch)	a
Wrought iron, solid rectangular section	36000	1/3000
Cast iron, hollow cylinder	80000	1/800
" solid "	80000	1/400

The experiments referred to proved that a pillar or strut, rounded at both ends, so as to be free there as to direction, has the same flexibility as a pillar of the same material and cross-section fixed at both ends and of double the length and has the same strength. This result is expressed in (145) and (146) above.

662. **Imperfection of Elasticity. Permanent Set.**—The subject of the imperfection of elasticity shown by ordinary substances, and

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of the limits of elasticity, is far too vast to be dealt with in this chapter, but some reference to the behaviour of ordinary materials is necessary. In certain circumstances there seems to be practically perfect recovery from even severe strains, for example an ivory or steel ball is very sensibly flattened by being dropped on a marble slab, and perfectly recovers its spherical shape. On the other hand, there are circumstances in which the recovery is only approximate: only a moderate elongation can be given to an iron or steel wire without producing a permanent elongation. Again the power of recovery of bodies seems to depend on the time during which the body is held in a state of strain. An india-rubber band left for a long time stretched is found to have lost its power of unstretching, and the same thing appears to be found in iron hoops. The material in the course of time seems to assume a new molecular arrangement in which the potential energy of the stress does not exist. As the body assumes this new internal state the energy is dissipated *in situ*. The substance is said to have received a permanent set.

The elongation of a wire, even when there is not complete recovery on removal of the stress, is nearly proportional to the stretching force. At first the curve of elongation (abscissal elongations ordinates pulls) is nearly a straight line, afterward the elongation increase more rapidly than the pull, and the curve is concave downward. At still greater elongation the pull reaches what is called the *yield-point*, and a small increase of stretching force produces a large increase of elongation. The specimen has begun to *flow* at certain places.

663. **Effect of Applications and Removal of Stress on Elastic Limits.**—It is observed that when a pull, not too great, is applied and removed a number of times in succession the substance settles down to a steady state in which the elongation produced by application of the stress disappears with removal of the stress. An analogous result is found when by application of stress other changes are produced—for example, changes in the induced magnetism of an iron wire under magnetising force. With the application and removal of stretching force and lateral pressure quasi-elastic changes of magnetisation take place, which assume a constant value only after a considerable number of successive applications and removals of the stress. Thus there seems to be produced a molecular change in the iron by the stress which is related to or of the same nature as the molecular change concerned in magnetisation, and which assumes a constant value after repeated application and removal of the stress.

It has been observed also in experiments on wires that most substances subjected to rapidly succeeding applications and removals of stretching force show a diminution of breaking stress, or tenacity as it is called. Further, if a wire be very gradually stretched by successive small additions to the load up to the breaking point, the breaking stress is very much greater than it would be if the load were rapidly applied. The wire seems to become tempered by long subjection to moderate loads, and to become increased in strength.

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The slow extension of wires under load forms an interesting subject of observation; if the load is fairly large, but still far below the breaking weight, the wire slowly elongates, but more and more slowly as time advances, and the increase of length at last dies out apparently as an exponential function of the time.

There are for every solid substance limits called the *elastic limits* beyond which it cannot be strained by stress of any type without sustaining permanent distortion. These are very narrow for many substances which have nearly perfect elasticity within their elastic limits. A glass ball, if too much distorted by collision, will fly to pieces; yet its elasticity of shape is, within narrow limits, very great.

664. **Solidity. Plasticity.**—It is probable also that every solid substance, if subjected to sufficiently great distorting stress, will continuously alter in shape—that is, *flow* under the stress and suffer continually increasing permanent increase of shape until the stress is in some manner removed. Thus a disk of gold or silver struck by a die is converted into a coin in which the head of the Sovereign, inscription, &c., are perfectly well defined, and remain clear and sharp for ages when the coin is subjected afterwards only to moderate stress. This property of a substance is called *plasticity*.

The limiting distorting stress, S say, at which flow begins is called the solidity of the substance. Some experiments of Tresca's * on lead make its solidity to be about 200,000 grammes per square centimetre, or 2800 pounds per square inch. If a stress S_1 greater than S is applied, the flow proceeds continuously, though not necessarily without limit. It is possible that, as the flow proceeds, the applied forces may give a stress which finally becomes S. For example, a bar subjected to compression by application of thrust may shorten until the stress is reduced to the limiting value in consequence of increase of the cross-section of the bar. Or the substance as it flows, and thus changes its condition, may develop a greater solidity. The whole subject still requires elucidation by experiment. If S_1 be removed at any stage, the substance recovers to an extent depending on its limits of elasticity, and then any force less than the solidity may be applied without again setting up flow.

In a fluid the solidity is zero, of course. A substance such as pitch or sealing-wax has a quasi-solidity, and when a bar of it is bent and the bending couple is at once removed, it unbends to some extent, as if it were really endowed with elasticity of shape. But it is to be remembered that in the resistance of an elastic body to bending both the moduli of an isotropic solid are concerned. The bar of viscous substance offers resistance to the change of shape depending on the rapidity of the change, and a true elastic resistance to the change of volume. The former resistance is very great even for small ratios of distortion. The compressed and extended parts cannot relieve themselves at once from the change of volume, because the change of shape

* Mémoires. Paris, Acad. des Sciences xviii. and xx. 1868 and 1872.

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necessary for this relief cannot at once take place. Hence, when the applied couple is removed, there is an elastic return due to the resilience from compression and dilatation. If, however, the couple is maintained the substance relieves itself by lateral expansion of the compressed region and lateral contraction of the dilated region, and a certain amount of unbending takes place.

It seems probable, however, that there exists a true solidity of pitch when subjected to a rapidly alternating strain of small amount, for a portion of it melted into a bell will sound a musical note of definite pitch, and this could only happen if the return force were proportional to the distortion of the bell from its mean or equilibrium shape. The mass of pitch subjected to a small continuous force, for example, its own weight, slowly yields, and at last flows out into a thin sheet.

No such flow, even under considerable force, takes place in such substances as the metals, which have a large solidity. Gold and silver and bronze ornaments thousands of years old have been found buried in tombs and in the ruins of ancient cities, and display in every detail, as much sharpness of outline and delicacy of workmanship as if they had just been made.

665. **Viscosity of Solids.**—There does exist in solids, however, true viscosity in the sense explained in § 348. The change in shape of a solid involves relative motion of the molecules of the body, and this motion is resisted by frictional forces. This is proved by the mere fact that the vibrations of a tuning-fork die away at a rate considerably greater than that at which energy is communicated to the surrounding air, and also by such experiments as those of torsional oscillations. The rate of diminution of amplitude of a cylindrical vibrator performing torsional oscillations, as described in § 635, is far greater than that due to the resistance of the air to the motion.

666. **Fatigue of Elasticity. Internal Viscosity of Solids.**—It is remarkable that this rate of falling off of amplitude depends, under certain circumstances that have not yet been fully investigated, on the time during which the wire has been kept vibrating. It was found in some experiments made about twenty-five years ago by Lord Kelvin * that the rate of subsidence of torsional oscillation was much greater in a wire that had been kept vibrating for several days than in a precisely similar wire which had been kept quiescent during the same time. Thus, two equal and similar copper wires were loaded with equal lead weights. One wire, called No. 2, was more frequently vibrated than the other (No. 1), and three weeks afterwards the viscosities were compared. Then No. 1 gave subsidence from an initial range 20 to 10 in 97 vibrations, while No. 2 sustained the same diminution of range in 77 vibrations.

It is stated also in the paper referred to that the falling off of amplitude a with time t was for each wire with a given vibrator, according to the law, ae^{kt}, that is the diminution of range per equal

* *Proc. R. S.* May 18, 1865.

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times had throughout the subsidence always the same ratio to the existing range at each instant, in other words, that the retardations were throughout the subsidence proportional to the change of shape; and that further, when two identical wires were caused to perform oscillations with vibrators of the same weight but different moments of inertia, the rates of subsidence were not those corresponding to retardations proportional to the rates of change of shape in the two cases. It is difficult to perceive any physical reason why the same law of retardation should not hold here also.

These results do not agree with some obtained more lately by Streintz * and by Pisati,† who found that a marked diminution of rate of subsidence in wires of steel, copper, silver, brass, and platinum by keeping the wire for a considerable time in continual torsional oscillation.

The viscosities of a large number of wires have been examined at different temperatures by Messrs. Blyth and Dunlop and the author of this work, the result with that in steel, brass, copper (commercial and pure), and soft iron, the rate of subsidence is markedly greater at a temperature of about 90° than at 15°. In one case, that of a specimen of german silver, the rate of subsidence was found to be less at the higher temperature than at the lower.

It was found also that apparently the rate of subsidence is a function of the amplitude as well as of the temperature, a fact which might explain the discordance of results mentioned above as to the effect of continued vibration on rate of subsidence. The experiments just referred to on this subject are still in progress.

667. **Resilience of a Strained Body. Effect of Sudden Application of Stress.**—The *resilience* of a strained body means etymologically the reaction of the body against the applied distorting stress—the tendency to spring back. It is used, however, in the theory of elasticity to designate the work done in straining a body to any given degree.

If the resistance fulfils Hooke's Law, the measure of the resilience is evidently the product of the final applied force into half the final displacement effected. Thus, if a spiral spring be drawn out 60 cms. by a weight of 14 lbs. its resilience in ergs is

$$\tfrac{1}{2} \times 14 \times 453{\cdot}6 \times 60 \times 981 = 186892272,$$

or in foot-pounds, 13·78.

The stress is here supposed to be applied in successive small increments. If the final stress were applied all at once an elongation twice as great as the equilibrium elongation would be produced. For the load, descending let us suppose under the action of gravity, would acquire velocity until the upward force due to the strain was equal to the load, which would continue to descend until the return

* Pogg. Ann. 153 (1874).

† Gazz. Chim. Ital. 1876, 1877, and Sitzungsb. d. Wien. Akad. lxxx. Abth. 2, 1879.

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force due to strain was equal to twice the load. Thus the reader will see how a body may be ruptured by the sudden application of a stress which it would have borne safely if gradually applied.

The amount of work given back by the body when allowed to return towards its former configuration may be perceptibly different from that done in straining it, even though Hooke's Law is fulfilled. In general when set is given it is in great part only subpermanent—that is to say, the strained body, if left to itself, will gradually return towards its original configuration in consequence of what has been called by German experimenters *elastische Nachwirkung*, "elastic after-working." The work spent in straining a body is obviously obtained by plotting a curve of which the abscissæ are the values of the applied force and the ordinates the strains produced. The area between the line of abscissæ, the curve, and the terminal ordinate is the resilience. If, then, when the applied force is gradually applied and then gradually removed, the curve of increasing strain be OCB and the curve of diminishing strain $BC'O$, the whole strain being supposed to come out finally, the work returned is represented by the area $OCBC'O$, and the difference represented by the area $OC'BCO$ has been dissipated, or is represented by unavailable energy of changed internal state of the wire.

668. **Resilience of a Twisted Rod or Shaft.**—The resilience of a twisted shaft is to be found, of course, by multiplying the terminal couple L applied by half the angle θ through which one end has been turned relatively to the other. Thus we have

$$\text{Resilience} = \tfrac{1}{2}L\theta. \tag{148}$$

But the couple is $\frac{1}{2}\pi nr^4\theta/l$ for a right cylindrical rod, and therefore

$$\text{Resilience} = \tfrac{1}{4}\pi nr^4\theta^2/l.$$

For a right cylindrical tube we have

$$\text{Resilience} = \tfrac{1}{2}nAk^2\theta^2/l, \tag{149}$$

where A is the cross-sectional area and k is the radius of gyration of the section about the axis of the tube.

For a solid cylindrical rod twisted until the outermost elements have sustained the utmost torsion they can bear without flowing of the material under the distorting stress we have as above

$$\text{Resilience} = \frac{\pi}{4}nr^4\theta_1^2/l,$$

if θ_1 now denote the utmost twist that can be given without producing flow of the outermost elements. The couple L is then $\frac{1}{2}\pi nr^4\theta_1/l$. The couple resisting turning of the outermost ring of particles is $2\pi nr^3dr\,\theta_1/l$. The area is $2\pi rdr$, and therefore the tangential force per unit area resisting in the cross-section the shearing-strain is $nr\theta_1/l$. If we denote this by η,

$$L = \tfrac{1}{2}\pi\eta r^3. \tag{150}$$

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If now every part of the cross-section be strained up to the point of its solidity, the outer portions will have yielded and flow will have taken place. The couple resisting further turning will then be

$$L' = 2\pi \int_0^r \eta x^2 dx = \tfrac{2}{3}\pi\eta r^3, \tag{151}$$

which is 4/3 of the former couple L.

669. **Torsion of a Shaft Transmitting Power.**—In the case of a shaft transmitting power by torsion the couple L is $\frac{1}{2}\pi n r^4\theta/l$, and the rate at which work is done per minute is $\frac{1}{2}\pi n r^4\theta/l \times 2\pi N$, that is $\pi^2 n N r^4\theta/l$, where N is the number of revolutions per minute. If the rate at which work is delivered by the shaft in horse-power be H, we have, if n be taken in pounds per square foot, and r in feet

$$\pi^2 n N r^4 \frac{\theta}{l} = H \times 33000.$$

The utmost shearing stress that can be applied safely is, we suppose, $S = nr\theta/l$, so that the last equation becomes

$$\pi^2 N S r^3 = H \times 33000,$$

and

$$r^3 = \frac{33000}{\pi^2 S}\,\frac{H}{N}. \tag{152}$$

The radius r of the shaft will be obtained from this equation.

For example, the utmost shearing force regarded as applicable in the case of steel is 13,500 pounds per square inch. This gives for the radius of the shaft in feet

$$r = \cdot 1198 \sqrt[3]{\frac{H}{N}}, \tag{153}$$

or, for the radius in inches

$$r = 1\cdot 438 \sqrt[3]{\frac{H}{N}}. \tag{154}$$

Authorities are not agreed as to what is the proper measure of the tendency of a substance to rupture. It was suggested by Coulomb that it was the greatest shear produced on the substance; Poncelet and later St. Venant held that it was the greatest extension. According to M. Tresca (whose researches on the flow of solids are referred to above) and to Prof. G. H. Darwin it is the maximum difference between the greatest and least principal stresses, a measure which does not differ much from that proposed by Coulomb. In practice the rule adopted is to keep the stress developed in the material under a certain fraction $1/f$ of the stress under which the substance would give way. The number f is called the "factor of safety" for the kind of strain involved.* This factor varies from 6 to 12 in different cases.

* See Love's *Elasticity*, vol. i. p. 106, and Treatises on *Strength of Materials*.

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670. **Impact of Elastic Balls.**—Connected with resilience is the important subject of impact of elastic bodies. This is much too large a subject to be dealt with here and, indeed, the treatment of it from the point of view of the vibrations of elastic solids has not yet been completed. The results so far obtained do not in all cases agree with those derived from general considerations.

Newton found that when two elastic balls impinged directly, that is, when both were moving along the line joining their centres, the velocity of separation after impact bore a ratio to the velocity just before impact depending on the nature of the bodies. Thus if u, u', v, v' were the velocities of the balls before and after impact, m, m' the masses of the balls, the momentum before impact was $mu + m'u'$, and this by the third law of motion must have been the momentum after impact. Hence

$$mu + m'u' = mv + m'v'.$$

The velocity of approach of the two balls was $u - u'$, after impact the velocity of separation was $v' - v$, and

$$v' - v = e(u - u')$$

where e denotes the ratio referred to above. This ratio is often and very improperly called the *elasticity* of the balls. It is better called the *co-efficient of restitution.*

But in all cases of collision the momentum in any direction lost by one of the bodies is equal to the momentum in the same direction gained by the other and this is true at every stage of the collision. Now there must be a stage in the case here considered at which the velocities of the two bodies had just been equalised. If $\mathbf{v}$ was this common velocity (the velocity, it is to be observed, of the centroid of the two balls both before and after the collision),

$$(m + m')\mathbf{v} = mu + m'u' = mv + m'v'.$$

From the equations thus obtained it is easy to prove that

$$\mathbf{v} - v = e(u - \mathbf{v}), \quad v' - \mathbf{v} = e(\mathbf{v} - u'). \tag{155}$$

But $u - \mathbf{v}$ was the velocity of the ball m relatively to the centroid of the two balls before the collision, and $\mathbf{v} - v$ is the velocity of the centroid relatively to the ball m after the collision. Hence this relative velocity is reversed in direction and altered in the ratio of e to 1 by the collision. The other equation has a similar meaning for the other ball.

The action between the two balls being equal and opposite the velocity, $\mathbf{v}$, of the centroid of the two balls during the collision must be throughout equal to the velocity of the surface of contact. If then u' was zero we have by (155) $v' - \mathbf{v} = e\mathbf{v}$, or (supposing, what is very approximately true for two ivory or steel balls, $e = 1$)

$$\mathbf{v} = \tfrac{1}{2}v' = \tfrac{1}{2}u.$$

This is the statement made in § 145 above.

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The kinetic energy of the two balls before impact is in the more general case described above made up of the energy of the two balls supposed both moving with the velocity of the centroid, that is $\frac{1}{2}(m+m')v^2$, and the kinetic energy of the motion of each relatively to the centroid, or $\frac{1}{2}m(v-u)^2+\frac{1}{2}m'(u'-v)^2$. The former remains unchanged. The latter becomes

$$e^2\{\tfrac{1}{2}m(v-u)^2+\tfrac{1}{2}m'(u'-v)^2\},$$

so that there is lost

$$\tfrac{1}{2}(1-e^2)\{m(v-u)^2+m'(u'-v)^2\}.$$

This energy is no doubt in the main converted into energy of vibrational motion of the balls which becomes heat energy as the vibrations subside. A portion also becomes energy of aerial vibrations, as sound was produced by the collision; this also ultimately becomes heat.

671. **End-on Impact of Two Bars.**—Consider two similar prismatic bars of equal length with flat ends, their axes in line, and their cross-sections similarly situated, and moving in the directions of their axes. They collide thus end to end; the two ends remain in contact during the collision, while a wave of compression runs along each from the ends struck. In each unit of time the same change of momentum takes place in each bar, and so the force between them remains constant from beginning to end of the time of contact. After the pulse of compression has passed along each to the free end the bars begin to elongate, and a pulse of elongation runs back from the free ends to the others. When it has reached the latter the bodies separate with the same relative velocity as that with which they approached before the impact, but with their velocities now interchanged.

If v_1, v_2 denote the velocities before impact, the common velocity of the ends during impact is $\frac{1}{2}(v_1+v_2)$. But a portion of the first bar which has not been reached by the wave is moving with the velocity v_1, and consequently the part along which the wave has travelled has shortened by $v_1t-\frac{1}{2}(v_1+v_2)t$, that is by $\frac{1}{2}(v_1-v_2)t$. But the initial length of this portion is at, if a be the velocity of propagation of the pulse. Hence the initial compression is $\frac{1}{2}(v_1-v_2)/a$. Precisely the same compression is propagated along the other bar.

Each part of either bar continues moving with its velocity, v_1 or v_2, until the pulse of compression reaches it, when the velocity of the part changes instantly to $\frac{1}{2}(v_1+v_2)$, and remains at this value until the return pulse of elongation arrives, when the velocity instantly changes to v_2 in the case of the first bar, and to v_1 in the case of the second.

It may be noted that the rebound of a bar, which strikes an unyielding plane end-on, is precisely the same as it would be if it collided, as here described, with an equal bar moving with the same speed in the opposite direction.

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If the bars, otherwise similar, be of unequal lengths, the common velocity of the ends will become instantly $\frac{1}{2}(v_1 + v_2)$. A pulse of compression of amount $\frac{1}{2}(v_1 - v_2)/a$ will run along both bars, and will, when it has reached the free end, be succeeded in each by a wave of elongation running in the opposite direction, as in the case just described. The shorter bar will have its motion altered as if it had struck another of its own length, and will rebound with velocity v_2. The motions of its particles will change as has been described.

The longer bar, however, will be left in a state of longitudinal vibration due to the propagation along it of first a wave of compression travelling in the manner described, and produced by the collision. Upon this will be superimposed, after the short bar has ceased to be in contact, a vibration of a similar nature, due to the sudden cessation of the pressure upon the end which was struck.

672. **Controversy as to Elastic Constants.**—We have not treated in this chapter the subject of the elasticity of æolotropic bodies. In their case the discussion is very much more complicated, and it is unnecessary for the majority of practical purposes. We may note here that each component of stress is necessarily a linear function of each of the six components of strain, so that the full specification of the stresses requires 36 co-efficients or elastic constants. These are reduced to 21 from the fact, proved first by Lord Kelvin,* that the stresses must be the differential co-efficients of a homogeneous quadratic function of the six strains, so that 15 relations must exist among the constants.

A further reduction to 15 is obtained by St. Venant and others by adopting the supposition that the force between any pair of molecules acts along the line joining the molecules, and varies according to a function of the distance of the molecules apart.

This supposition involves a fixed relation between the two principal moduli of an isotropic body, which is expressed by putting Poisson's ratio equal to $\frac{1}{4}$. This result cannot be held to be confirmed by experiments on the approximately isotropic bodies which are met with in nature. Experiments of this kind, however, are difficult to make with accuracy, and there is always the objection brought by the upholders of the rari-constant theory, as it is called, that the substances are not isotropic. The reader will find full information on the subject of this controversy in Prof. Karl Pearson's *History of Elasticity*.

We conclude with a short table of elastic constants of the more important substances employed in the arts.

* *Thermo-elastic Properties of Matter, Math.* and *Physical Papers.* Vol. I.

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APPENDIX TO CHAPTER XV.

Table of Elastic Constants.*

Moduli in 10^6 *grammes per square cm.*

Substance.	Young's Modulus. $9nk/(3k+n)$.	Authority.	Rigidity. n.	Authority.	Bulk-Modulus. k.	Modulus of Simple Extension. $k+\frac{4}{3}n$.	Poisson's Ratio.	Density.
Aluminium . . .	703	*Glasg. Univ. Phys. Lab. Record.*	265	*Glasg. Univ. Phys. Lab. Record.*	675	1028	·326	2·86
5 per cent. copper								
Brass	1001	Rankine	373	Everett	1054	1551	·342	8·47
Copper	1254	Wertheim	456	"	1672	2280	·375	8·9
German silver . .	1380	T. Gray	496	T. Gray	2113	2774	·391	8·8
Glass, flint . . .	614	Everett	243	Everett	432	756	·263	2·94
Iron, cast . . .	1374	"	542	"	985	1708	·267	7·24
" wrought . .	2040	Rankine	785	"	1694	2741	·300	7·68
Steel wire, piano . .	2182	Everett	834	"	1895	3007	·308	7·73
Platinoid . . .	1222	Bottomley	476	Bottomley	941	1576	·294	8·78
Platinum . . .	1704	Wertheim	622	McFarlane	2181	3010	·369	20·81
Quartz fibre . . .	600	Boys	238	Boys	417	734	·26	...
Silver	736	Wertheim	270	McFarlane	895	1255	·363	10·21
Zinc	873	"	360	"	506	986	·212	7·11
Clay, rock . . .	329	T. Gray and Milne	177	T. Gray and Milne	96	332	...	2·68
Slate	686	"	...	...	...	...	...	2·74
Mercury . . .	...	...	...	...	552·5	...	...	13·596
Water	...	...	...	...	22·63	...	..	1·0

* Taken from Maclean's *Exercises in Natural Philosophy*. London: Longmans & Co. 1900.

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CHAPTER XVI.

CAPILLARITY.

673. **Cause of Capillary Phenomena.**—Some calculations of effects of surface-tension of a liquid have been given in chap. iv.; we consider here briefly the physical cause of surface-tension, the phenomena to which it gives rise, and some methods by which it can be measured.

There can be no doubt that there are mutual forces between the molecules of all substances, that is to say between those particles which are ultimate, in the sense that they cannot be divided without a change in the physical properties of the substance. It seems not absolutely impossible that these forces may be merely the gravitational attractions between the particles—attractions made sufficiently intense at small distances by the heterogeneity of the substance which is encountered at distances comparable with the dimensions of the molecular structure—or, it may be, as many seem to hold, that there are special forces of cohesion, &c., entirely distinct from gravitation, which are called into play when the molecules of a substance, or the molecules of different substances, are brought into sufficient proximity with one another. It is certain, however, that when it is possible to estimate the whole circumstances of the actions of molecules upon one another, and to take account of the effects of their relative motions, there will be a considerable reduction in the number of special forces acting between them, and a unity of dynamical action will emerge which will be continuous with that which we find existing between ordinary portions of matter at finite distances apart.

No definite statement of the law of action of molecular forces is yet possible. Of the nature of a molecule we know next to nothing; and if we make the usual statement that the mutual force between two molecules is along the line joining the molecules, and depends upon the distance between them, we assume either that the molecules are mere centres of force, or that they are spherical distributions of matter which act as if the whole mass of each were collected at its centre. It seems probable that the action is to a high degree of approximation the same as that of centres of force. But just as the planets attract one another only approximately as if their masses were collected at their centres, and there are phenomena, for example

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those of precession and nutation, which arise from the fact that the centres of the planetary masses are not accurately centres of the whole external attractions which they exert, and just as the effects of these attractions are to an outside spectator profoundly modified by the motions of the planets, so there are no doubt phenomena which are to be traced to the fact that the molecules are not mere force-centres, and above all to the fact that they are in relative motion.

674. **Forces Insensible at Sensible Distances.**—A very complete theoretical account of capillary phenomena can be framed on the supposition that the members of every pair of molecules act on each other as a pair of force-centres, with forces which are sensible only at distances comparable with the dimensions of a molecule. But the phrase "dimensions of a molecule" requires to be made precise. In any body it is probable that only a small portion of the whole space is really filled by matter; there are always interstices large enough to allow of considerable relative motion of the particles. The phenomena of diffusion of liquids into one another show that there is considerable relative motion of their particles, and the recent experiments of Sir William Roberts-Austen, on the diffusion of gold through lead at ordinary temperatures, show that even in a solid the molecules are not quiescent. Hence the dimensions of a molecule are to be understood as the dimensions of the average volume of the space *occupied* by a molecule, that is the space obtained by dividing the volume of the body by the number of molecules in it. Some account of methods of estimating the number of molecules in a body will be given in vol. ii.

The external attractive force exerted by a molecule is not perceptible beyond a very small distance from the molecule, that is from the centre of force which may be taken to represent the molecule. A sphere of this radius may be called the molecule's *sphere of influence*. It will be clear, then, that if there is a regular arrangement of the molecules in a fluid, and a very large number of them in any finite portion of the space occupied by the fluid, a molecule which is at a greater distance from every part of the boundary than the radius of the sphere of influence will be acted on by the molecules surrounding it so that it experiences no force in any direction. If, however, the molecule be at a smaller distance than this from the boundary the action of the molecules on the side of it towards the fluid will overbalance that of the molecules on the other side, and forces will be brought into play which are not manifested in the interior of the fluid.

675. **Laplace's Theory of Capillarity.***—Laplace founded his theory of capillarity on this idea of forces insensible at sensible distances. The following is a sketch with some slight modifications of his method of investigation. He considered first the attraction of a sphere of uniform density upon a long straight uniform filament of the same density abutting normally against the spherical surface. Let OMN

* *Mécanique Céleste*, t. iv. *Supplement au Xe. Livre.*

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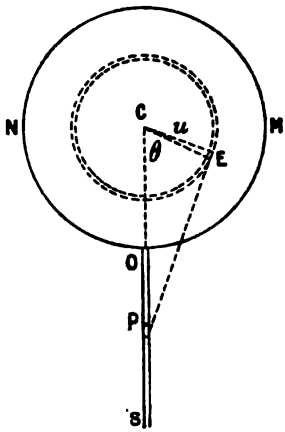
Fig. 327 be the sphere, OS the filament which we shall suppose to be of unit cross-section. Let ρ be the common density of the sphere and rod, and consider the attraction of a shell of radius u and thickness du, on an element dr of the rod at P, distant r from C the centre of the sphere. Take first the attraction on the element dr of a zone generated by revolving an element of breadth $ud\theta$ at E about OS as an axis. The mass of this zone is $2\pi\rho u^2 \sin\theta\, d\theta\, du$, and if a mass m at E exert an attraction $m\phi(f)$ along the line $PE(=f)$ on unit mass at P the attraction along PC on the element dr exerted by the zone is $2\pi\rho^2 u^2 \sin\theta\, d\theta\, du\, dr\, \phi(f)(r - u\cos\theta)/f$, since $(r - u\cos\theta)/f$ is $\cos CPE$. But $f^2 = r^2 + u^2 - 2ru\cos\theta$, so that

FIG. 327.



$$(r - u\cos\theta)/f = df/dr.$$

The attraction of the zone on the element dr is therefore

$$2\pi\rho^2 u^2 \sin\theta\, d\theta\, du\, dr\, \phi(f)\, df/dr.$$

Let $\phi(f)df/dr = -d\{\Pi(f)\}/dr$, then the attraction of the zone on the element dr is

$$-2\pi\rho^2 u^2 \sin\theta\, d\theta\, du\, dr \frac{d\{\Pi(f)\}}{dr}.$$

If this be integrated with respect to r between the limits b and ∞, on the supposition that $\Pi(f)$ is a quantity which diminishes with extreme rapidity as f increases, we get for the attraction of the zone on the filament

$$2\pi\rho^2 u^2 \sin\theta\, d\theta\, du\, \Pi(f)_{r=b}$$

where b is the distance CO.

Now let θ vary while r remains constant and equal to b; then since $\sin\theta\, d\theta = fdf/bu$, the last expression becomes

$$2\pi\rho^2 \frac{u}{b} du\, \Pi(f) f df.$$

This integrated for variations of f corresponding to variations of θ from 0 to π yields

$$-2\pi\rho^2 \frac{u}{b} du\, \{\psi(b+u) - \psi(b-u)\},$$

where ψ is such a function that $\Pi(f)fdf = -\psi'(f)$. The distance f evidently varies from $b-u$ to $b+u$.

If the distance $b+u$ be as we suppose greater than that at which a particle of matter can exert any sensible attraction on the filament

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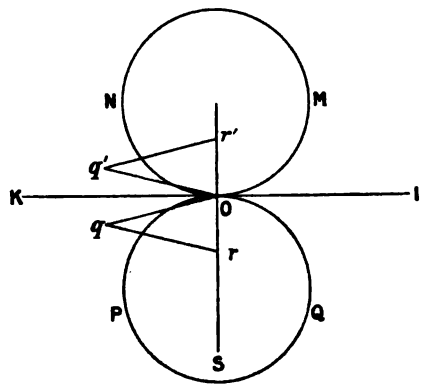
we shall have $\psi(b+u)=0$, and the total attraction of the shell is $2\pi\rho^2 u du\,\psi(b-u)/b$. Denote $b-u$ by z so that $-udu=(b-z)dz$; then the whole attraction of the sphere on the filament is

$$-2\pi\rho^2\int_b^0 \frac{b-z}{b}\psi(z)\,dz = 2\pi\rho^2\int_0^b \psi(z)dz - 2\pi\rho^2\frac{1}{b}\int_0^b \psi(z)dz = K - \frac{H}{b}, \quad (1)$$

if K, H be put for the two integrals. This is Laplace's celebrated expression for the action of a sphere of the fluid upon a filament abutting normally against the sphere. Laplace took $\rho=1$, and the expressions here given differ from his by having ρ^2 as a factor.

FIG. 328.



676. **Intrinsic Pressure in a Fluid.**—The attraction of a sphere of fluid of finite radius b upon an external uniform and infinitely thin filament of fluid normal to the surface, and extending to infinity, is thus $K-H/b$, per unit of cross-section of the filament, so that this must be the pressure exerted on the sphere by the base of the filament and balanced by the reaction of the sphere. The quantities K, H are independent of the radius of the sphere, and as the molecular forces on which these quantities depend are supposed insensible at sensible distances it is clear that $K-H/b$ is the attraction on the filament, per unit of area, of an indefinite mass of fluid bounded by a portion of a spherical surface of radius b to which the filament is normal. It is only necessary, therefore, that every part of the boundary (except the curved surface) of the mass of fluid should be at a finite or sensible distance from the base of the filament, and that the filament should be of finite length, in order to ensure that the pressure at its base should be $K-H/b$.

The quantity H is small in comparison with K. For K is $2\pi\rho^2\int\psi(z)\,dz$, where z is the distance of the base of the filament from the surface of a shell concentric with the sphere of radius b and

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within it, $\psi(z)$ is a function (unknown) of z, and the integral is taken between the limits 0 and b; and, on the other hand,

$$H = 2\pi\rho^2 \int_0^b \frac{z}{b}\psi(z)\,dz.$$

The only values of $\psi(z)dz$ which have a sensible effect in building up either integral are those for which z is very small in comparison with b, because of the supposition that the forces are insensible at sensible distances. Hence H must be small in comparison with K as stated.

It will be observed that if b be made infinite H/b is zero, and therefore the integral K is the pressure which would be produced at the base of the filament if it abutted against the plane boundary of an otherwise infinite mass of fluid, as OS, in Fig. 328, against the plane IK. Clearly then if OMN represent the attracting sphere and OS the filament in contact with the sphere at O, H/b is the amount which must be deducted on account of the want of the meniscus $MIOKN$; that is H/b is the pressure which would be produced by the meniscus acting alone. K has been called the *intrinsic pressure* on the fluid.

677. **Intrinsic Pressure and Pressure due to Curvature.**—Since there is no attraction on a filament of finite length in the interior of an infinite mass of fluid, the attraction upwards exerted on the filament OS by the sphere OMN must be equal and opposite to the attraction in the opposite direction that would be exerted by the mass below the sphere, required to make the attracting matter infinite in all directions. But the latter attraction must be equal to the attraction exerted on the filament by the sphere OPQ, together with the attraction due to the double meniscus $MIQONRP$. The attraction of the single meniscus $MIONR$ is by the result obtained above of amount H/b, and is directed upwards. Hence the downward attraction of all the fluid below the plane IOK upon the infinite filament also below the plane and terminating in it is $K - H/b + H/b$, or K. Thus the attraction of all the matter below the plane IOK on a filament, also below the plane and terminating in it, is equal to the attraction K in the same direction exerted by an infinite mass below the plane on a filament above the plane and terminating in it.

Further, the attraction of the meniscus $PKOIQ$ on the filament below the plane is the same as the attraction of the meniscus $MIONK$ on the same mass. For consider any element of the meniscus $PKOIQ$ at q, Fig. 328, and a corresponding and equal element of $MIOKN$ at q', a point in the normal to IOK through q and as far from the plane IOK as q is. Drawing the isosceles triangles Oqr, $Oq'r'$ (in Fig. 328), we see that the element at q exerts no action along the filament on the part Or, nor q' on Or'. Hence the upward action of q' on the filament terminating at O is the same as that of q on the part of the filament below r. Hence the total action of the

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meniscus $PKOIQ$ on the filament OS is upwards and of the same amount as the attraction exerted on the filament OS by $MIONK$. Hence if S be the downward attraction of the sphere POQ on OS, we have $S - H/b = K$, that is

$$S = K + \frac{H}{b}. \tag{2}$$

Thus the attraction of an infinite mass bounded by a convex spherical surface of radius b on an indefinitely long filament of the mass having its base at O is $K + H/b$ per unit area of cross-section, and this is the pressure at O. On the other hand, if the surface is concave the pressure is $K - H/b$. From this Laplace concludes that the part of this attraction due to a portion of a spherical surface between two planes intersecting in the radius at O, and inclined at a small angle $d\theta$, is $(K - H/b)d\theta$.

Laplace supposes that the density of the fluid is unity; if we suppose that the density is ρ the values of K and H, if these quantities have the meanings assigned to them above, must be multiplied by ρ^2, since both the fluid and the filament have the density ρ. We shall suppose, however, that this has been taken account of in calculating K and H, that is that $K = 2\pi\int\rho^2\psi(z)dz$, and similarly for H.

678. **Secondary Principle of Contractile Film.**—The reader will have perceived that H/b, the pressure due to the curvature of the surface is capable of being regarded as produced by a contractile film on the surface of tension $T = H/2$, where T is defined as in § 204 above. This idea of a contractile film is due to Thomas Young who gave a theory of capillary phenomena in which he used this secondary principle before Laplace took up the subject; and he moreover endeavoured to account for the action by means of molecular forces. A most instructive discussion of the work of Young and Laplace will be found in Lord Rayleigh's papers *On the Theory of Surface Forces*, *Phil. Mag.*, Oct. and Dec. 1890.

679. **Physical Meaning of K.**—The physical meaning of K is explained above, but it may be useful to consider it a little more fully. It is the pull per unit area of cross-section exerted upon the infinitely thin column OS, in consequence of the attraction of an infinite mass in which OS is embedded and which is bounded by a plane surface IOK perpendicular to OS. At any point S, therefore, it must be balanced by a pressure K, applied to the column. The action is similar to that between two portions of a solid held together by mutual attractions. One hemisphere of the earth is pressed against the other across the common plane of contact. The pressures in opposite directions exerted by the fluid beyond on the opposite faces of a central slice are of course balanced by elastic reaction. The pressure would be measured by the force per unit area which would have to be applied to pull the hemispheres apart. That liquids can sustain great tension when freed from air is an ascertained experimental fact.

The pressure K is not experienced by any solid immersed in the

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fluid, or by any material sheet separating two parts of the fluid. For by the insertion of a solid body two other surfaces are brought into existence each of which gives rise to a thrust K, back into the liquid on each column of unit area ending on the surface of the solid. All that any arrangement immersed in the fluid can detect depends on the pressure applied to the surface from without, the pressure due to the action of gravity on the fluid, and the effects of curvature of the surface and motion of the fluid.

If, however, a small spherical hollow of radius r be formed within the fluid the pressure within it becomes $K - H/r$. H/r is prevented from being infinitely great by the discreteness of the substance when r is so small as to be comparable with molecular dimensions, and the pressure is in the limit K. This point cannot be fully discussed here, but it is of great interest to notice that since the law changes when r is of the order of the range of the force of cohesion of the liquid, this range must be of the order of magnitude of K/T. Thus K is the pressure within a spherical hollow in the fluid of very small radius.

680. **Young's Estimate of Molecular Dimensions.** — Young supposed that the attractive force exerted by a molecule was constant over its range and beyond was equal to zero. On this theory he arrived at an estimate a of the range as given by the equation $T = \frac{1}{3}aK$. Taking K as 23,000 atmospheres, and T as 3 grains per lineal inch he found that "the extent of the cohesive force must be limited to about the 250 millionth of an inch," and makes the following remarks on this result: "Within similar limits of uncertainty we may obtain something like a conjectural estimate of the mutual distance of the particles of vapours, and even of the actual magnitude of the elementary atoms of liquids, as supposed to be nearly in contact with each other; for if the distance at which the force of cohesion begins is constant at the same temperature, and if the particles of steam are condensed when they approach within this distance, it follows that at 60° of Fahrenheit the distance of the particles of pure aqueous vapour is about the 250 millionth of an inch; and since the density of this vapour is about one sixty-thousandth of that of water the distance of the particles must be about 40 times as great, consequently the mutual distance of the particles must be about the ten thousand millionth of an inch." After some remarks on the effect of temperature he continues, "on the whole it appears tolerably safe to conclude, that . . . . the diameter or distance of the particles of water is between the two thousand and the ten thousand millionth of an inch."

Further, it will be noticed that if ρ be the density of the fluid and $F\rho$ be the attraction of the infinite mass above IOK (Fig. 328), per unit of length, on any element dx of the filament OS, supposed of unit cross-section, the whole force is $\int F\rho dx$, the integral being taken along the length OS. But this is the whole work which would have to be done in carrying away a quantity of matter ρ from the base O of the filament to an infinite distance from IOK. Thus we

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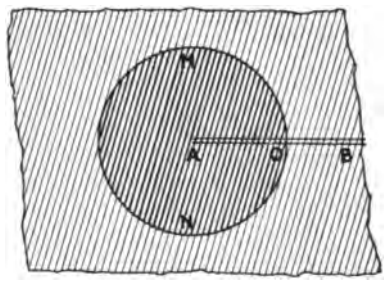
may obtain an estimate of K by measuring the work done in evaporating the liquid. In this way K has been found to be about 25,000 atmospheres for water.

681. **Case of Two Fluids in Contact.**—Lord Rayleigh has extended Laplace's results to the case of two fluids in contact. Let OMN be a sphere of density ρ_2 surrounded on all sides by an infinite fluid of density ρ_1. Let AO, OB be two columns which are theoretically indefinitely long, but which in reality extend from the surface on either side only to a distance which is great in comparison with the range of molecular forces, and which meet at the point O in the surface. If each fluid acts upon the other, as it would act upon itself except for the difference in density, we may suppose that we have here simply the case of an uninterrupted infinite mass of fluid of density ρ_1, with a sphere OMN of density $\rho_2 - \rho_1$ superimposed.

FIG. 329.



The pressure will fall off along the filament AO from A towards O, and the amount of falling-off along any element ds will be the attraction of the sphere of density $\rho_2 - \rho_1$ upon the matter of the element. Hence the whole falling-off of pressure from A to O is

$$\rho_2(\rho_2 - \rho_1)(K + H/b)$$

where b is the radius of the sphere OMN, and K and H are the constants obtained by Laplace's process for a fluid of unit density. Similarly the falling-off of pressure from O to B is

$$\rho_1(\rho_2 - \rho_1)(K - H/b).$$

The pressure at A therefore exceeds the pressure at B by the sum of these, that is by

$$(\rho_2^2 - \rho_1^2)K + (\rho_2 - \rho_1)^2\frac{H}{b} = K' + \frac{H'}{b}.$$

There is thus an excess of pressure K' which is independent of the curvature and proportional to $\rho_2^2 - \rho_1^2$, and a further excess H'/b which is proportional to the curvature and to $(\rho_2 - \rho_1)^2$.

If there be three different media, and H'_{12}, H'_{23}, H'_{31} be values of H' for the first in contact with the second, the second in contact with the third, and the third in contact with the first; then since these are proportional to $(\rho_2 - \rho_1)^2$, $(\rho_3 - \rho_2)^2$, $(\rho_1 - \rho_3)^2$, we have

$$\sqrt{H_{13}} = \sqrt{H_{12}} + \sqrt{H_{23}}.$$

This relation is not confirmed by experiment, and hence the suppo-

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sition made above as to the action of one fluid on another must be incorrect. There is not really any sufficient physical reason for supposing it to be correct.

If the third fluid is air, then approximately $\rho_3 = 0$, and $\sqrt{H_{12}} = \sqrt{H_1} - \sqrt{H_2}$. Since $H_1 > H_2$ this gives $H_{12} < H_1 - H_2$, which *is* confirmed by experiment. See § 685.

The investigation given above may, however, serve to show how capillary pressure may entirely disappear if the transition from one fluid to the other is made sufficiently gradual. Take the case of transition from density 0 to density ρ. Let the change be made in two steps, first from fluid of density zero to fluid of density $\frac{1}{2}\rho$, then from the latter to fluid of density ρ, and let the stratum of fluid of half-density be of thickness great in comparison with the radius of curvature. At each step the difference of capillary pressure is only one quarter of that due to the sudden transition from 0 to ρ, and thus altogether half the effect is lost by the interposition of the layer. If there were three equal steps the effect would be reduced to one-third, and so on. When the number of steps is infinite the capillary pressure disappears altogether.

682. **Surface of Double Curvature.**—In general the curved surface of a liquid is not spherical, but has different curvatures in different planes through the normal at any point. But at any point the surface may be regarded as practically coincident there with a given determinate quadric surface. For, let the origin be taken at any point on the surface, the normal at that point as the axis of z, and the axes x, y at right angles to one another in the tangent plane at the origin. The equation of the surface may be written

$$z = \alpha x^2 + 2\gamma xy + \beta y^2 + \text{ \&c.}, \tag{3}$$

where under the &c. are comprised higher powers and products of x, y. Hence, if z be taken very small and of constant value, c, (3) is the equation of the section of the surface by a plane parallel to the tangent plane and at a distance c from it. If we take x, y as infinitesimals of the first order, and if we neglect infinitesimals of the third and higher orders, *viz.*, x^3, x^2y, xy^2, &c., we have for the equation of the section

$$\alpha x^2 + 2\gamma xy + \beta y^2 = c. \tag{4}$$

This is the equation of what is called the *indicatrix* of the surface, and is an ellipse or a hyperbola or a pair of parallel straight lines according as $\alpha\beta - \gamma^2$ is positive, negative, or zero.

The curvature of the surface at the origin in the plane of zx is d^2z/dx^2, and is therefore 2α, and the curvature in the plane of zy is is similarly 2β. For dz/dx is the tangent of the inclination at O of the tangent to the surface in the plane of xz, and is zero. The rate at which this zero tangent is increasing with x must be the rate at which the tangent line is changing its direction at O per unit length of displacement of the point of contact along the curve. Hence, if 2α

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is positive, the radius of curvature is along the positive direction of z.

Now, let the axes of x, y be turned round through any angle θ, and denote the new x, y by x', y'. We have

$$x = x'\cos\theta - y'\sin\theta, \quad y = x'\sin\theta + y'\cos\theta.$$

The equation of the surface becomes

$$\begin{aligned} z = (\alpha\cos^2\theta + 2\gamma\sin\theta\cos\theta + \beta\sin^2\theta)x'^2 \\ - 2\{(\alpha-\beta)\sin\theta\cos\theta - \gamma(\cos^2\theta - \sin^2\theta)\}x'y' \\ + (\alpha\sin^2\theta - 2\gamma\sin\theta\cos\theta + \beta\cos^2\theta)y'^2. \end{aligned}$$

The curvature in the plane of $x'z'$ is

$$2(\alpha\cos^2\theta + 2\gamma\sin\theta\cos\theta + \beta\sin^2\theta)$$

and that in the plane of $y'z'$ is

$$2(\alpha\sin^2\theta - 2\gamma\sin\theta\cos\theta + \beta\cos^2\theta).$$

The sum of these curvatures is $2(\alpha+\beta)$, so that the sum of the curvatures in two planes at right angles to each other through the normal has always the same value. If, then, the curvature has its maximum value in a plane through the normal it has its minimum value in the plane through the normal at right angles to the former. These are called the principal curvatures, and their sum is $2(\alpha+\beta)$.

Consider now two planes making angles θ and $\theta + d\theta$ with the axis of x. The curvature is $2(\alpha\cos^2\theta + 2\gamma\sin\theta\cos\theta + \beta\sin^2\theta)$, and the portion of the surface between them may be regarded as the fraction $d\theta/\pi$ of the surface of a sphere of radius equal to the reciprocal of this expression. Hence, for the attraction of the whole mass of matter beyond the surface on a filament having its base at O, of unit cross-section, and imbedded in the attracting mass we have

$$\begin{aligned} K \pm \frac{2}{\pi}H\int_0^\pi(\alpha\cos^2\theta + 2\gamma\sin\theta\cos\theta + \beta\sin^2\theta)d\theta \\ = K \pm \tfrac{1}{2}H(\alpha+\beta) = K \pm \tfrac{1}{2}H\left(\frac{1}{R} + \frac{1}{R'}\right), \end{aligned} \tag{5}$$

the upper or lower sign being taken according as the surface is concave or convex. R and R' denote the two principal radii of curvature.

683. **Energy Theory of Capillarity. Surface Tension Identical with Energy per Unit of Surface.**—Some twenty-five years later a new theory of capillarity was given by Gauss, in which the idea of the energy of the substance was employed. The molecules of the fluid being supposed to act on one another, and on the molecules of a solid with which the fluid is in contact, with forces which are functions of the distance between the particles, an expression can be written down for the potential energy of the fluid, and from the

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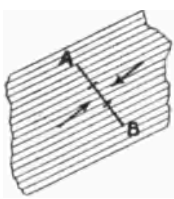
principle that the potential energy must always tend to become less, the form of the surface in different cases can be found. In this way Gauss discussed the subject, and pointed out the restrictions as to the law of force between the particles required to bring the conclusions of theory into harmony with observed phenomena. One point left unexplained by Laplace was fully covered by Gauss's theory, the observed fact that when a given fluid and solid are in contact the fluid surface always meets the solid at the same angle. We shall not deal with this theory at present, but will assume the secondary principle of superficial tension, and the constancy of the angle of contact of a fluid with a solid.

The theory of energy leads not less directly to the notion of superficial tension than does the theory of Laplace. For, if a portion of a fluid has its surface increased, the increased number of molecules which now lie in or near the surface have been withdrawn from the neighbourhood of other molecules against molecular attractions, and so work must have been spent on the substance from outside to effect the extension. Consequently, apart from any change of energy due to thermal action, or to the action of gravity, the potential energy of the substance has in this case been increased from E_0 to E_1, where $E_1 - E_0$ is the work which has been done by external forces in extending the surface. If the surface be increased from S_0 to S_1, the work done per unit area in stretching the film is $(E_1 - E_0)/(S_1 - S_0)$. And it is clear that the molecular forces will tend to diminish the area of the surface so as to diminish E_1 again towards E_0, that is there seems to observation, which can take no account of the molecular forces, to be a contractile film connected by cohesion with the fluid beyond, and in many cases with the solid with which the fluid is in contact, and by this film the phenomena appear to be caused. We shall speak as if this film really existed, and in a sense it does so exist, but the reader must not be led into any too gross conception of its reality.

FIG. 330.



Let AB be a line drawn on a surface film, then along the surface across any element ds of AB the film tends to contract. Let F be the force across ds; F/ds is called the surface tension of the film, and is usually denoted by T. It will be clear that if the film be in equilibrium in a horizontal position the surface tension must be the same at all points. If the film be stretched from area S_0 to S_1 without alteration of surface-tension we must have $T(S_1 - S_0) =$ work spent from external sources in increasing this area. For let the boundary be displaced along the tangent plane to the film, through a small distance u in a direction inclined at an angle θ to the normal drawn outward in that plane at the point. The work done against the surface tension (which acts perpendicular to the boundary) across an element ds of the boundary is $Tu\cos\theta ds$. But $u\cos\theta ds$ is the area dS described by the element ds, and therefore the whole work done in stretching

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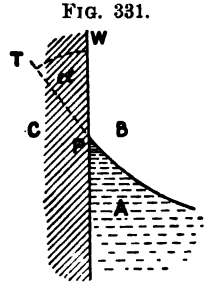
the film is $\int TdS$, the integral being taken round the boundary. Of course at places where θ is greater than $\pi/2$ the element of area added is negative, that is, there is there contraction of the film. If T be constant the work done is $T\int dS$ or $T(S_1 - S_0)$. Clearly on the supposition of constant tension the whole work done in creating a surface of area S is TS. T may therefore be interpreted as the energy per unit of area of the surface.

As an elementary example a simple arrangement, by which it is possible to obtain a rough determination of surface tension, may be considered. A rectangular frame is placed in a vertical position and is filled by a soap-film the lower edge of which adheres to a horizontal wire which is movable upwards or downwards. If l be the distance between the vertical sides of the frame the whole upward force with which the film pulls up the wire is Tl. If the wire is pulled down a distance h the work done is Thl, that is TS, if S denote the increase lh of surface. The surface-tension T can be approximately determined by measuring the force Tl.

684. **Thermal Change in Stretching a Liquid Surface.**—When a film of liquid is extended quickly its temperature changes, falling as a rule; and heat must be supplied to the film in order that its temperature may remain constant. In the case of a water-film the amount of heat which must be supplied in order to keep the temperature constant as the film is extended, is dynamically equivalent to about half as much as energy as is spent in stretching it against the constant surface-tension. This arises from the fact that the surface-tension is a function of the temperature; the theory of the action will be explained in vol. ii. in the chapter on *Thermodynamics*.

FIG. 331.



It will be observed that the dimensional formula of T is that of force per unit length, that is $[MT^{-2}]$.

Let Fig. 331 be a section of a portion of the fluid bounded by a vertical wall, straight, or only so slightly curved that it may be regarded as straight. The section is supposed made by a plane at right angles to the line of junction of the media. In general the fluid surface, whether the wall be vertical or not, makes a definite angle with the wall-surface. If PT be the intersection with the plane of the diagram of a tangent-plane to the surface at P, the angle WPT is called the angle of contact of the fluid with the surface at P. In most cases the three media A, B, C in contact at P are the liquid, air, and the material of the solid wall. For mercury in contact with glass in presence of air as a third medium the angle WPT is about 48°. Frequently the supplement of the angle WPT is taken as the angle of contact. When the medium B is air the angle so taken is often called the air-angle of contact of the liquid with the solid.

685. "**Triangle of Tensions.**"—In some cases the three media are three liquids, or two liquids and air, and theoretically it might

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appear that there should be equilibrum with the three surfaces, inclined to one another, so that each of the different tensions T_1, T_2, T_3 in a plane cutting the meeting edge at right angles should be proportional to the sine of the angle between the other two, as in the triangle of forces. But, as has been pointed out by Lord Rayleigh,* the triangle of tensions is never possible, for one of three fluids will always flow over the separating surface of the other two. All experimenters have arrived at this result. For example, Marangoni (*Pogg. Ann.* 143, 1871) states that "the interface between two fluids has a smaller surface-tension than the difference of the surface-tensions of the substances themselves (with the exception of quicksilver)." Thus, if we have two fluids of which the surface-tensions in contact with air are T_1, T_2, the surface-tension T_{12} of the two in contact is less than $T_1 - T_2$. Hence $T_1 > T_2 + T_{12}$ and the fluid T_2 is spread over the surface separating the fluid of tension T_1 from air. A drop of oil cannot stand upon water unless the water is already covered with a film of oil. Nor is mercury really an exception. As has been proved by Quincke, a drop of water will spread over the surface of clean mercury; in ordinary circumstances a drop of water only stands on a mercury surface because of the existence over the latter of a greasy film.

686. **Film of Oil on Surface of Water. Effect of Oil on Waves.**—A drop of oil placed on a water surface immediately spreads in a thin film over the surface, and thus the effective superficial tension of the water-surface is altered. After the film has become thin, still further thinning increases the surface-tension. Thus, if the water changes the form of its surface, as in wave-motion, where there is contraction of the area there is thickening of the oil film and consequent diminution of tension, and where there is extension of area there is thinning of the film and increase of tension. Forces are thus called into play which tend to restore the former configuration, and the film of oil opposes the periodic changes of area of surface involved in wave-propagation. The ripples and small superimposed waves are by this action smoothed out, and the effect which these would have in causing the crests of the large waves to advance and break are avoided. This explanation of the action of oil upon waves is due to Professor Osborne Reynolds.†

687. **Effect of Alteration of Thickness on Surface-Tensions.**—Lord Rayleigh has pointed out that the increase and diminution of surface-tension with thinning and thickening of the film is shown also by films of soap and water. Let a soap-film be caught in a ring of wire by dipping the ring into a flat dish containing a soap-solution. If the plane of the ring be kept horizontal every part of the film must have the same tension. But if the plane be held in the vertical position, and the film remain in equilibrium, it is clear that the tension at the top must be greater than that at the bottom,

* "On The Theory of Surface Forces," *Phil. Mag.*, Dec. 1890; see also Lord Rayleigh's note in Maxwell's *Theory of Heat*, 10th edition, p. 287.

† *Brit. Assn. Rep.*, 1880.

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by the amount necessary for the support of the intermediate portion of the film. The liquid slowly drains from the top to the bottom and the film thins above. This goes on until the upper part reaches a certain limiting thickness when the surface-tension begins to diminish and the film breaks. This takes places when the thickness becomes comparable with molecular dimensions, and the measurement of the limiting thickness enables an estimate of these dimensions to be arrived at. Experiments on this subject have been made by Professors Reinold and Rücker.* The results will be described in vol. ii.

688. **Diminution of Surface-Tension of Water produced by Dissolved Substances.**—The solubility of a substance in water leads to a gradation of surface-tension from the value for pure water to that for as strong a solution of the substance as may be made. For example, a fragment of camphor placed on water dissolves, and the tension of the water-surface is diminished, and to a greater extent the greater the amount of camphor present in the water at the surface. If this be, as it usually is, unequal round the piece of camphor, the substance is pulled towards the side on which there is greater tension; and the curious motions of a piece of camphor on water are due to this cause. The camphor dissolves, the part dissolved is drawn out in consequence of the superior tension of the surrounding water-surface. If the surface be touched by the finger the motion may be stopped by the film of grease communicated to and rapidly spread over the clean surface of the water. Lord Rayleigh has determined the thickness of olive oil necessary to stop the motion of small fragments of camphor, by determining the weight of oil added and the area of the vessel. He found that ·8 milligramme upon a circular surface 84 cms. in diameter sufficed. This gives a film $1{\cdot}6 \times 10^{-7}$ cm. thick; and Lord Rayleigh estimates that a thickness of 2 millionths of a millimetre would have been enough if the surface had been perfectly clean to begin with.

The diminution of surface-tension produced by adding alcohol is very clearly shown by flooding an inclined plate of glass with water coloured by aniline, and then drawing horizontally along the middle of the adhering coating of water a camel's-hair brush which has been dipped in alcohol. The liquid will be drawn on each side from the alcohol, and will run upwards on the upper side and downwards on the lower side, leaving bare the middle part of the plate. The running up will continue until the liquid has become so deep on the plate that it flows back by the action of gravity. If a brush dipped in ether be held close to the surface of the layer of water, the water will be drawn away from that place all round.

689. **The "Tears of Strong Wine."**—In the same way, as was pointed out by Professor James Thomson, is to be explained the phenomenon of "tears of strong wine." Wine partially filling a bottle or glass gives a film standing up round the side, which is

* *Phil. Trans. R. S.* 1881.

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continuous with a thin film wetting the walls of the vessel. From this the alcohol evaporates more quickly than the water, and so the tension increases. The tension being stronger above, the liquid is pulled up the sides until it runs down in drops. If the wine is contained in a partially filled stoppered bottle the motion does not go on, as the evaporation ceases when the vapour is in equilibrium with the liquid. The motion can be made to begin, however, by opening the bottle and drawing off the vapour by sucking it up through a tube inserted so that its lower end is above the surface of the liquid. If the alcoholic solution is too strong or too weak the "tears" will not be shown. A solution of about 25 per cent. strength gives the best effect.

690. **Elevation and Depression of Liquid in Tubes and between Plates.**—We can now explain the elevation or depression of a liquid

FIG. 332.

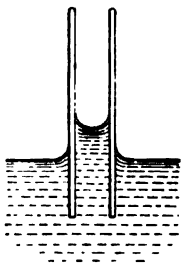


FIG. 333.

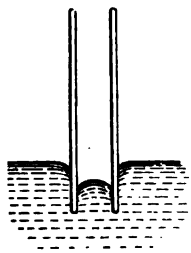
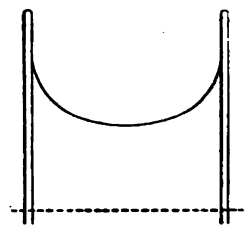


FIG. 334.



in a tube or between two plates. Fig. 332 shows the elevation of water in a glass tube above the level of the horizontal surface in a wide outer vessel, Fig. 333 the depression of a fluid like mercury in a similar tube. In the case of water the general result of experiments gives zero for the angle of contact, that is the air-angle is 180°, in the case of mercury the angle is finite and about 48°. Consider two points, one P on the free level surface in the outer vessel, the other Q on the surface of the meniscus in the tube, at some distance from the wall of the tube. The pressure in the liquid at the surface is K, at the same level under the tube it is $K - \frac{1}{2}H(1/r_1 + 1/r_2)$, if r_1, r_2 be the principal radii of curvature at the point Q, and z be its height above P. Since these two pressures must be the same we have

$$g\rho z = \tfrac{1}{2}H\left(\frac{1}{r_1} + \frac{1}{r_2}\right),$$

or writing $2T$ for H,

$$g\rho z = T\left(\frac{1}{r_1} + \frac{1}{r_2}\right). \qquad (6)$$

Figs. 334 and 335 show more exactly the form of the surface in wide tubes. The dotted line in each case is the undisturbed level.

If the tube be of small radius a, it is easy to show that r_1 must

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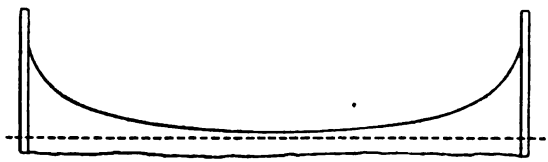
be very nearly equal to r_2. At the bottom of the meniscus r_1 is equal to r_2: calling each r we have

$$g\rho z = 2T\frac{1}{r}. \tag{7}$$

If the liquid meet the tube at a finite angle α, the curvature in a meridian section at the edge is, we suppose, $1/r'$, that in the other normal plane is $\cos\alpha/a$. For the surface at the line of contact with the tube is inclined at the angle α to the generating lines of the tubular surface. Hence a narrow ring of the surface at that line must coincide with a spherical surface of radius $a/\cos\alpha$. Hence if z

FIG. 335.



be the height of the edge of the meniscus above the free surface level we have

$$g\rho z = T\left(\frac{1}{r'} + \frac{\cos\alpha}{a}\right). \tag{8}$$

But the total upward pull exerted by the film upon the fluid is $2\pi aT\cos\alpha$, and if h be the mean height to which the liquid is raised, we have

$$\pi a^2 g\rho h = 2\pi aT\cos\alpha,$$

or

$$g\rho h = 2T\frac{\cos\alpha}{a}. \tag{9}$$

Now, since the tube is very narrow, h is very nearly equal to z, and therefore

$$g\rho h = T\left(\frac{1}{r'} + \frac{\cos\alpha}{a}\right), \tag{10}$$

that is $r' = a/\cos\alpha$, and the two radii of curvature are to a high degree of approximation the same at the edge of the meniscus. But, at the lowest point $g\rho z = 2T/r$, if r be the radius of curvature there: hence we have there also $1/r = \cos\alpha/a$, and each of the curvatures is the same both at the lowest point of the meniscus and also at the edge, and they must be the same approximately at all points. The meniscus is therefore part of a spherical surface of radius $a/\cos\alpha$.

If the angle of contact is zero, the radius of curvature is a. This is the case for water and for every liquid which wets a solid with which it is in contact.

The height to which a liquid rises in a tube is inversely as the

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radius of the tube. The same law holds for the rise of a liquid between two near parallel plates; but here the formula is slightly different. There is only one curvature, which is $\cos\alpha/a$, if a be the half distance between the plates and α the angle of contact. Hence, in this case, if h be the mean height we have

$$g\rho h = T\frac{\cos\alpha}{a}. \tag{11}$$

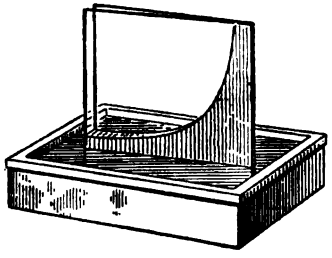
The height h is thus only half that for a tube of diameter equal to the distance between the plates.

In this case if the radius of curvature at any point of the curved surface be r, the height z to which the liquid is raised is given by

$$g\rho z = \frac{T}{r}. \tag{12}$$

Of course, if the surface be convex, the depth h to which any point at which the radii of curvature are r_1, r_2 is sunk below the free surface is given by (6), and the depression between the plates by (12). This is the case in which the liquid does not wet the walls of the tube, as, for example, water within a tube the walls of which have been covered with a film of oil.

FIG. 336.



An interesting experiment illustrating the law of rise of a liquid between two plates is made by arranging two rectangular plates of glass so that they touch along a vertical edge, as shown in Fig. 336, and are slightly separated by a thin wedge at the other two vertical edges, and placing them with their lower horizontal edges in a trough of water coloured with aniline. If a be the distance to which the wedge separates these latter edges, and l the horizontal length of the plates, we have for their distance apart at distance x from the vertical edge at which they are in contact xa/l. Approximately, therefore, the height z to which the water rises at this point is given by $g\rho z = Tl/xa$. Hence

$$xz = T\frac{l}{g\rho a}, \tag{13}$$

that is xz is a constant. Hence the curve of contact of the water with the plate is a rectangular hyperbola, of which the vertical edge of contact of the plates and a line along either plate coincident with the surface are the asymptotes.

691. **Form of Surface near Plane Plate dipping into Liquid.**—Consider now the form of the surface on either side of a plane plate which dips into a liquid. Fig. 337 shows the form of the surface, at

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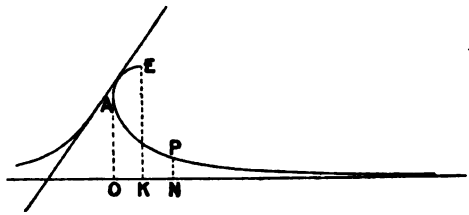
any rate, at all points which are not infinitely close to the wall. The form will be given for any angle of contact by drawing the plate in position so as to intersect the film at the proper angle. A section of the surface by a plane perpendicular to the horizontal edge in which the liquid meets the plate, must be the *linea elastica*, since the curvature at any point P of this section of the surface

FIG. 337.



is proportional to the vertical distance of the point from the undisturbed level. Hence if c be a constant the equation of the section is as we have seen in § 655

$$x = -(4a^2 - y^2)^{\frac{1}{2}} + a\log\frac{2a + (4a^2 - y^2)^{\frac{1}{2}}}{y} + c. \qquad (14)$$

If the value $a\sqrt{2} - a\log(1 + \sqrt{2})$ be chosen for c, O will be taken at the foot of the vertical tangent and is measured horizontally and y vertically. OA, the height of the point of contact above O, is $a\sqrt{2}$.

If p denote dy/dx, the curvature at any point P is $(dp/dx)/(1 + p^2)^{\frac{3}{2}}$, and this, as we have seen, is connected with the ordinate y of the point by the equation.

$$T\frac{\frac{dp}{dx}}{(1 + p^2)^{\frac{3}{2}}} = g\rho y. \qquad (15)$$

If we put a^2 for $T/g\rho$ the equation becomes

$$a^2\frac{\frac{dp}{dx}}{(1 + p^2)^{\frac{3}{2}}} = y. \qquad (16)$$

A first integral of this equation is obtained by multiplying both sides by p and integrating. This gives

$$-\frac{a^2}{(1 + p^2)^{\frac{1}{2}}} = \tfrac{1}{2}y^2 + C = \tfrac{1}{2}y^2 - a^2, \qquad (17)$$

since $y = 0$ where $p = 0$. Thus, if θ be the inclination of the tangent at any point to the axis of x, this equation may be written, since $p = \tan\theta$, in the form $a^2\cos\theta = a^2 - \frac{1}{2}y^2$, or $a^2(1 - \cos\theta) = \frac{1}{2}y^2$, that is

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$$\sin \tfrac{1}{2}\theta = \frac{y}{2a}. \tag{18}$$

692. **Theory of Large Drop on Horizontal Plate or Bubble of Air below Plate.**—We can make various applications of this last result. Drawing a horizontal line through E, the point at which the curve is horizontal, we obtain approximately the meridianal section of a very large circular air bubble in water below a glass plate well wetted by the liquid, that is a section made by a plane through the axis of figure.

The height of the bubble is KE, and this is given by (17) with the negative value of the radical $(1+p^2)^{\frac{1}{2}}$. Hence, $y = 2a$ for this point. The height of the point A above the axis of abscissæ is $a\sqrt{2}$, since $y = 2a \sin \frac{1}{2}\theta$, and $\theta = 90°$.

If the height of the bubble or the height of the point A be measured, the surface tension can be calculated subject to a correction required for the neglect of the curvature in the other normal plane.

693. **Determination of Surface Tension from Bubbles and Drops.**—In a case in which the air angle of contact is α, let a horizontal line be drawn through the point C', at which the curve is inclined at that angle to the horizontal. Then we shall have for the ordinate, h say, of that point

$$h = 2a \sin \tfrac{1}{2}\alpha. \tag{19}$$

Let this, which is the height of the bubble, be measured. Let also the height OA, that is $a\sqrt{2}$, be measured. Then a is known, and all that is required is a determination of ρ to enable the surface tension to be calculated. Also since a is known the angle of contact can be calculated by (19).

This method may be applied to large drops of liquid resting on a horizontal plate which they do not wet; for example, a drop of mercury resting on a glass plate, as in Fig. 338. If the origin be taken at the top of the drop and y be measured downwards, the formulæ given above apply without change. The density ρ is, however, the density of the liquid forming the drop.

FIG. 338.

Measurement of drops for the determination of surface tension was recommended by Gauss in his memoir on Capillarity, and it has been carried out with much care by Quincke. It is to be noticed, however, that the theory is imperfect, as the complete differential equation is not (16), but

$$a^2 \left\{ \frac{\dfrac{dp}{dx}}{(1+p^2)^{\frac{3}{2}}} + \frac{\cos\theta}{r} \right\} = y, \tag{20}$$

where r is the radius of the horizontal section of the drop at

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which the curvature is taken, and θ is the angle which the normal at the point makes with the plane of the section. If R be the radius of the greatest section, and x the abscissa of the point $r = R - x$, and θ is the complement of $\tan^{-1}p$. Hence, $\cos\theta = p/\sqrt{1+p^2}$, and the equation is

$$a^2\left\{\frac{\frac{dp}{dx}}{(1+p^2)^{\frac{3}{2}}} + \frac{p}{(R-x)\sqrt{1+p^2}}\right\} = y. \tag{21}$$

This equation has been fully treated by Laplace in a second supplement to the Tenth Book of the *Mécanique Céleste*, in which he discusses further the subject of capillarity. He compares with theory the height, measured by Gay-Lussac, of a drop of mercury a decimetre in diameter taking the angle of contact as 48°, and finds 3·397 millimetres for the height, as against the observed height 3·378 millimetres. The height of a drop of the same size had been previously found by Segner to be 3·4067 millimetres.

The correction of the surface tension of mercury required for this term has been calculated by Worthington, *Phil. Mag.* Jan. 1885.

694. **Plateau's Experiments.**—Some remarkable experiments were made by Plateau with drops of olive oil immersed in a mixture of alcohol and water, adjusted to have exactly the specific gravity of the oil, so that the figure of a mass of it could be studied without any distortion except that due to motion imposed upon the mass. It was seen that the figure of a drop of the oil was spherical when at rest under no force, a result that of course must hold for all fluids. The surface-film contracts so as to take the smallest area consistent with a given volume, and that involves, as is well known, the spherical figure. The spherical figure is the only one of finite extent which, when there is no constraint as to boundary, will satisfy the condition that

$$\frac{1}{R_1} + \frac{1}{R_2} = \frac{1}{a}, \tag{22}$$

where a is a constant, which must hold at all points.

Also, when one of these drops was pierced through its centre by a vertical wire oiled so as to adhere to the drop, and the wire was turned round its own axis the drop bulged out horizontally under centrifugal force until ultimately, under some little disturbance, an equatorial ring parted from the main body and remained rotating round the equator of the central mass, like the ring of Saturn about the equator of that planet.

Many other remarkable experiments were carried out by Plateau, and are described in his *Statique . . . des Liquides soumis aux seules Forces Moléculaire*, to which the reader may refer for further information.

695. **Films having Two Surfaces.**—Now let us consider films which have two surfaces separated by a thin stratum of the fluid. Such are soap films, which, when proper precautions are taken in

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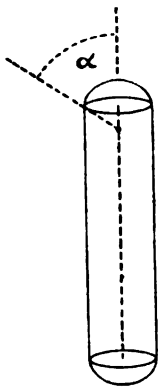
preparing the solution, can be kept in existence for days and studied in various positions, and can be attached to boundaries of different shapes.* In these cases the tension of the film may be taken as twice the tension of either surface.

Let the bulb be closed, p be the pressure of the air within it, and P be the external atmospheric pressure. If R_1, R_2 be the principal radii of curvature at any point, we have

$$\frac{1}{R_1}+\frac{1}{R_2}=\frac{p-P}{2T}. \tag{23}$$

696. **Cylindrical Film.**—As an illustration, take a film in the form of a cylinder bounded by two spherical caps. This is very easily formed by suspending two circles of wire in horizontal planes at a suitable vertical distance apart, and so that their centres are in one vertical. This can be done by supporting the lower circle by three diverging legs of wire, and the upper one by three thin diverging threads attached at points in the ring, then taking up a portion of soap solution at the mouth of a small thistle-funnel or tube, placing the end of the tube between the rings and blowing the bubble until it comes in contact with both rings. If the tube is then gently withdrawn while the finger closes the open end of the tube, the bubble can be left resting whole between the rings. If, then, the upper ring be vertically raised, the film between them will be stretched, and will, as its length increases, approach the cylindrical form. For by the raising of the upper ring the curvature in a vertical section through the axis is diminished, while the curvature in a horizontal section is increased. When the curvature in the vertical section has become zero, the curvature of the other is at each point $1/a$, where a is the radius of either ring. The caps closing the ends will be spherical. If a be the radius of either ring, the difference of pressure between the inside and outside being $p-P$ as before, we have for the cylindrical body of the film

FIG. 339.



$$2T\frac{1}{a}=p-P. \tag{24}$$

Also if r be the radius of a spherical cap, we have

$$4T\frac{1}{r}=p-P. \tag{25}$$

* Take 1 part Marseilles soap cut in small pieces, dissolve in 40 parts by weight of distilled water moderately warm; filter and then mix 15 parts by volume with 11 of pure glycerine, and allow to stand for a week. On the eighth day cool to 3°C. for 6 hours, then filter, placing in the filter a closed test-tube containing ice to prevent deposit from re-dissolving. Repeat filtering until solution is clear, then put away in stoppered bottles at ordinary temperature.

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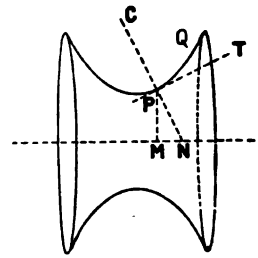
Hence $2/r = 1/a$, or $2a = r$. The radius of the ends is therefore equal to the diameter of the tube. Hence we have (Fig. 339) $r \sin \alpha = \frac{1}{2}r$, or $\sin \alpha = \frac{1}{2}$, that is $\alpha = 30°$.

697. **The Catenoid.**—If the spherical caps be punctured, the pressure outside and inside the cylindrical film will at once be equalised, and the cylindrical form will disappear; and the surface will become one of revolution about the line of centres of the two circles, having double curvature at each point given by the equation

$$\frac{1}{R_1} + \frac{1}{R_2} = 0. \qquad (26)$$

FIG. 340.



But if $1/R_1$ be the curvature in a plane through the axis, a meridianal plane as we shall call it, the curvature $1/R_2$ in the plane perpendicular to this, and also containing the normal at the point P, is, by Meunier's theorem,* connected with the distance of P from the axis by the equation $1/R_2 = \cos\theta/r$, when r is the distance of the point P from the axis, and θ is the angle MPN. Hence

$$\frac{1}{R_1} = -\frac{\cos\theta}{r}$$

or

$$r = -R_1 \cos\theta. \qquad (27)$$

Hence the curvature in the meridianal plane is opposite to that in the other normal plane, and the radius of curvature in the former plane is PN, the distance of the point P from intersection of the normal with the axis of the surface. The section of the surface by an axial plane is thus (§ 324 above), a catenary of which the axis is the directrix. Hence the surface is called a *catenoid.*

If the wires be brought closer to one another, or carried farther open than corresponds to the cylindrical shape of the film between them, the film will in the former case bulge out convex in a meridian-plane, and in the latter will be pressed in by the outer air. If we

* This theorem connects the curvature of the surface at any point P in any normal plane with the curvature in a section intersecting the former plane in a tangent line to the surface at the point. If R be the radius of curvature in the normal plane, and α the angle between the two planes, the radius of curvature in the other plane is $R\cos\alpha$. For the intersections of the two planes at P infinitely nearly coincide over a small arc ds, which has its centre at P. The chords of these two nearly coincident arcs are in the limit of the same length. But if d be the distance of the chord from P in the normal section, the distance of the other chord from P is $d/\cos\alpha$. If 2θ be the angle subtended by ds at the centre of curvature in the normal section, we have $R(1-\cos\theta) = d$, or since θ is small $\frac{1}{2}R\theta^2 = d$, that is $\frac{1}{2}ds^2/R = d$. Similarly if r be the radius of curvature in the other section $\frac{1}{2}ds^2/r = d/\cos\alpha$. Hence $r = R\cos\alpha$, which is Meunier's theorem.

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suppose (Fig. 340) s to increase from P to Q, we shall have for the curvature at P, $-d\theta/ds$.

Also, if y be the distance of P from the axis of the curve, we have for the curvature in the other plane $\cos\theta/y$. Hence,

$$-\frac{d\theta}{ds}+\frac{\cos\theta}{y}=\frac{1}{a}, \tag{28}$$

if $2a$ denote $2T/(p-P)$. But $\cos\theta=y/\rho$, where ρ is PN, the radius of curvature in the second normal plane, and $\sin\theta=dy/ds$. Hence, $ds=dy/\sin\theta$, and the equation just written becomes

$$-y\sin\theta\, d\theta+\cos\theta\, dy=\frac{1}{a}y\, dy,$$

of which the integral is

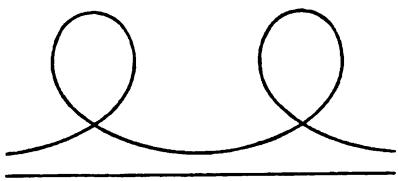
$$y\cos\theta=\frac{y^2}{2a}+C, \tag{29}$$

where C is a constant. Now $\cos\theta=y/\rho$, and therefore

$$y^2\left(\frac{1}{\rho}-\frac{1}{2a}\right)=C. \tag{30}$$

698. **Capillary Surfaces of Revolution, Generation of Unduloid, Catenoid, and Nodoid.**—Now let the curve PQ be described by the rolling of another curve along the axis, touching that line at N. PM is the perpendicular let fall from the point P on the tangent to the rolling curve. The relation between the radius vector ρ and y given by (30) is exactly the relation between the length of the radius vector ρ drawn from the focus of a conic section to a point on the curve and the perpendicular let fall from the focus on the tangent to the conic section at P. This the reader may make out at once for the ellipse by comparing with the equations of § 79 above. One semi-axis of the conic is a, the other is $\sqrt{2aC}$. Hence the curve is that traced out by the focus of a conic section as it rolls along the axis of the curve.

FIG. 341.



When the rolling curve is an ellipse the curve traced by the focus is called the unduloid; when the curve is a parabola the curve is a nodoid, consisting of a series of loops as shown in Fig. 341.

699. **Stability of Cylinder and Catenoid.**—We have no space to deal with the stability of the cylinder and catenoid. For the investigations the reader may refer to Clerk Maxwell's article on Capillarity in the *Encyclopædia Britannica*, 19th edition. The results are that a cylindrical film between the edges of two parallel circular end

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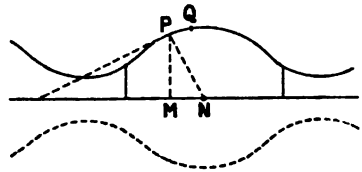
plates, and acted on by no forces, is stable only when the distance between the plates is greater than half the circumference.

A catenoid bounded at the ends by two open rings is stable when the tangents at the extremities of the generating catenary intersect above the directrix of the curve, that is before reaching the axis of the catenoid.

The former case is of importance in its bearing on the stability of a jet of fluid. Cylindrical portions longer than the limit stated will break up into detached pieces, each of which will tend to become spherical, and if they continue to move freely under the action of gravity will end by becoming spherical after a sufficient number of vibrations about the spherical figure.

FIG. 342.



700. **Influence of Surface Tension on Propagation of Waves.**—The influence of capillary action on the velocity of propagation has been investigated by Lord Kelvin.* The velocity of propagation of waves in water, which are short in comparison with the depth of the water, is given by the equation

$$v = \sqrt{g\frac{\lambda}{2\pi}}, \tag{31}$$

where λ is the wave length, and g is such that $g\rho z$ is the pressure at a depth z below the surface at any point. In ordinary cases g is simply the acceleration due to gravity. But the action of surface tension is to increase the pressure at any depth z if the surface be convex, and to diminish it if the surface be concave. Let the wave profile along the direction of propagation be given by

$$y = a\sin\frac{2\pi}{\lambda}x,$$

where y is the height of the surface above the mean surface-level. We have for the curvature

$$-\frac{d^2y}{dx^2} = \frac{4\pi^2}{\lambda^2}y;$$

and the pressure at the surface due to the curvature is therefore

$$-T\frac{d^2y}{dx^2} = \frac{4\pi^2}{\lambda^2}Ty. \tag{32}$$

The pressure at the mean surface-level is this together with $g\rho y$, so that the whole pressure there is $g(1+4\pi^2T/g\rho\lambda^2)\rho y$. Hence,

* "Hydrokinetic Solutions and Observations," *Phil. Mag.* Oct. 1871.

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the wave propagation takes place as if gravity were increased from g to $g(1+4\pi^2T/g\rho\lambda^2)$. Hence, the velocity of propagation is

$$v=\sqrt{\frac{g\lambda}{2\pi}+\frac{2\pi T}{\lambda\rho}}. \tag{33}$$

This will have different values for different values of ρ. It will have its least value when $g\lambda/2\pi=2\pi T/\lambda\rho$, that is, when $\lambda^2=4\pi^2T/g\rho$, and then

$$v=\left(4g\frac{T}{\rho}\right)^{\frac{1}{4}}. \tag{34}$$

701. **Waves and Ripples.**—Values for which λ is greater than $2\pi\sqrt{T/g\rho}$ are controlled more by gravity than by surface tension; waves for which λ is less than this limit are controlled more by surface tension than by gravity. The latter Lord Kelvin calls *ripples*. Ripples are very clearly seen when a fine fishing line is held stretched by a plummet sunk in water from a boat, and the boat is rowed forward. A pattern of waves extending to some distance in front of the line is produced. The waves travel out from the line in front in the water (which just at the line is moving with the same speed) until a point is reached where, in consequence of the slower motion of the water, the velocity of propagation of the waves is just equal to the rate of advance of the boat. This marks the front of the pattern of stationary waves which move forward with the boat. The stationary wave nearest the line is the wave of minimum velocity, then come the two sets of more rapidly travelling waves, the gravitation waves and the ripples. The former increase, the latter diminish in wave length, and both diminish in amplitude as the distance of the same increases. If w be the velocity of propagation of the waves, and V the velocity of the source, the outside of the wave pattern is at a distance from the front of the pattern inclined at the angle $\sin^{-1}w/V$ to the direction of motion.

702. **Determination of Surface Tension by Ripples.**—This theoretical result has been tested by Lord Rayleigh * and by Matthiessen,† and has been employed by Lord Rayleigh for the experimental determination of surface tension.‡ The method is preferable to any other, as it is possible to take precautions ensure that the surface of the liquid is kept clean, which it is impossible to employ in other methods.

A shallow porcelain tray 10″ by 12″ contained the water; the waves were produced by a straight edge of glass attached to one prong of a tuning-fork maintained in vibration by an electro-magnet,

* "On the Crispations of Fluid resting upon a Vibrating Support." *Phil. Mag.* July 1883.

† *Wied. Ann.* 38, 1889.

‡ "On the Tension of Water Surfaces, Clean and Contaminated, investigated by the Method of Ripples." *Phil. Mag.* Nov. 1890.

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which had a frequency, determined by comparison with standard forks, of 42·12. The waves were rendered visible by means of a gas flame placed in the focal plane of a large lens of about 34 inches focus. The lens was only a few inches above the surface of the water, and the flame was placed just a little on one side of the axis of symmetry, so that the focus after reflexion and a second passage through the lens might not coincide with the source.

A fork kept vibrating in unison with the exciter screened off the light of the flame at each vibration by means of a piece of tin plate. The surface was thus rendered instantaneously visible at each vibration always in the same phase of generation of the waves.

The eye was placed close to the focus to which the reflected light was brought, and the arrangement was so sensitive that waves were seen which were perfectly invisible when the surface was viewed in the ordinary way.

In some of the experiments a somewhat graver fork was used (frequency 40·9), and the results were found to be quite concordant with those obtained by the other fork.

To keep the surface of the water clean a hoop of thin brass, of breadth greater than the depth of the water, was coiled up close round the dipper which generated the waves, and then allowed to expand out towards the edge of the dish. One or two operations cleaned the surface very thoroughly, and they were repeated from time to time as impurities settled on the surface from the air.

The number of ripples between two points close to the surface of the liquid was counted, and this gave the wave length. The known period then enabled the surface tension to be calculated by (33), $g\lambda/2\pi$ being supposed negligible.

The mean result gave T for water as 74 dynes per lineal centimetre at temperature 65° Fahr. This is markedly below the value (81 dynes per linear centimetre) obtained by Quincke from observations on air-bubbles below glass plates. It agrees, however, with other observations of Lord Rayleigh's on the transverse oscillations of water jets,* for which Quincke's results gave too low a wave length.†

703. **Method with Microscope or Telescope.**—The following method has been worked out by Mr. J. A. Craw in experiments on the superficial tension of mercury in contact with various solutions, in which he is at present engaged as a research student in the Physical Laboratory of the University of Glasgow. Liquid is placed in a shallow vessel supported on the resonance box of a monochord. When the monochord is excited, a well-defined pattern of standing ripples is produced by interference of waves proceeding from the sides of the vessel. The waves comprise different component series, and any one of these can be isolated by means of glass rods properly placed on the surface.

Fig. 343 shows the arrangement. T is a rectangular trough

* "Capillary Phenomena of Jets." *Proc. R. S.* No. 196 (1979).

† Worthington, *Phil. Mag.* vol. xx. 1885.

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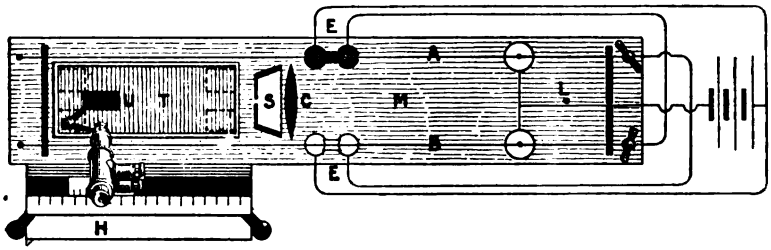
containing the liquid firmly attached to the resonance box. Light from a source L, rendered parallel to the condenser C, passes through a slit S, is projected by the mirror U (which is adjustable in any position) on the surface of the liquid, and is thence reflected into the reading microscope, or better, the telescope of a horizontal cathetometer H. The mirror is rigidly attached to the telescope.

The monochord is maintained in vibration by an electrical arrangement. Two wires, A and B, are tuned to the pitch of a

FIG. 343.



standard fork, and are excited by electromagnets EE, connected in parallel, one above and the other below its wire. The mercury contact for each wire is used as an interrupter for the electromagnet of the other, so that uniformity of pitch and phase of the two wires is secured. The slit-image and the spider line of the eye-piece are adjusted parallel to and exactly in coincidence with the image of the crest of a ripple, and the microscope (or telescope) is moved over m crests. The difference of the scale readings d is taken and gives $\lambda = 2d/m$ the length of a single ripple. Measurements made in this way have been found to agree within the experimental error (about ·10 per cent. on the surface tension) with those obtained by a simple series of plane waves of small amplitude, illuminated instantaneously by a spark and photographed on a quick plate or measured by direct observation with the microscope.

704. **Apparent Attraction or Repulsion between Two Plates. Motion of a Drop in a Tapering Tube.**—When a liquid rises between two close plates the plates appear to attract each other; but they are really pressed together by the excess of pressure on the outer sides. There is no difficulty in calculating the force in any given case. Let water rise between two plates placed vertical and parallel at a short distance apart in a large vessel, to a mean height h above the level in the vessel. The pressure at a height x is $P - g\rho x$, if P be atmospheric pressure. But the pressure on the outside is P. Hence the resultant thrust on either plate on a horizontal strip of length l and breadth dx is $g\rho lxdx$. The whole push on a length l of the plates is $g\rho l\int_0^h xdx = \frac{1}{2}g\rho lh^2$. If the liquid has an angle of contact α with the plates a term $Tl\sin\alpha$ must be added.

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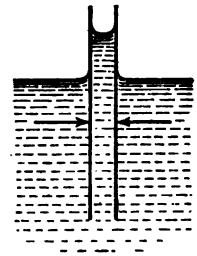
If the surface of the liquid between the plates be below the outside undisturbed level and the plates be close, the plates appear to repel each other with a force which can be calculated in the same way.

FIG. 344.



Consider a circular layer of liquid between two close parallel horizontal plates Let the angle of contact be a. The surface will be concave in any section at right angles to the plates and through their centre, and convex in any section parallel to the plates.

If d be the distance of the plates apart, the radius of curvature the in first section is approximately $\frac{1}{2}d/\cos a$, so that the curvature is $2\cos a/d$. If R be the radius of the plate, the other curvature is $1/R$. Hence the diminution of pressure is $T(2\cos a/d - 1/R)$. The area is πR^2, and therefore the thrust on each plate towards the other due to difference of pressure is

$$\pi R^2 T(2\cos a/d - 1/R) = 2\pi R^2 T\cos a/d - \pi RT.$$

To this must be added the pull $2\pi RT\sin a$ given by the film, so that the whole force is $2\pi R^2 T\cos a/d - \pi RT(1 - 2\sin a)$.

Thus two blocks of ice placed together with a film of water between them, are drawn together and adhere. Part of the ice brought into contact melts in consequence of the lowering of the freezing point brought about by the pressure, and the pressure being in part thus relieved, the film freezes, welding the two blocks together.

Also a drop of liquid placed in a tapering tube will move toward the narrower or the wider parts of the tube according as the liquid does or does not wet the tube. The curvature is greater, and a greater force is applied by the film to the liquid, at the narrower than at the wider end of the drop. This force is towards the narrower end of the tube, if the surface is concave, and in the opposite direction if the surface is convex.

705. **Measurement of Surface Tension by Capillary Tubes.**—Many determinations of the surface tension of water have been made by measuring the elevation or depression of liquids in tubes. One of the first of these was made by Gay-Lussac, and is described by Laplace in his supplement already cited. The tube was of white glass, and had an internal diameter of 1·29441 mm. The elevation of the lowest point of the surface was 23·1634 mms. at a temperature of 8·5° C. The water wetted the tube, and the angle of contact was zero. Adding $\frac{1}{6}$ of the diameter of the tube, we get for mean height 22·3791 mms. From this we get for the surface tension of water 8·5° C., the value 74·2 degrees per lineal centimetre, almost exactly the result obtained by Lord Rayleigh for the temp. 65° F.

An experiment made with a wide tube gave an almost exactly concordant result.

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In the carrying out of such experiments it is essential that the tubes should be most carefully cleaned by the most efficient chemical methods, and that the liquids used should be as free from impurity as possible. The slightest film of grease on the surface of water in a tube would lower the height of the column, and give an inexact result. The height of the column should be carefully read from a distance by a cathetometer telescope.

706. **Form of Drops.**—Attempts have been made to determine the surface-tension of liquids by weighing drops let fall from a hori-

FIG. 345.

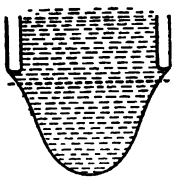
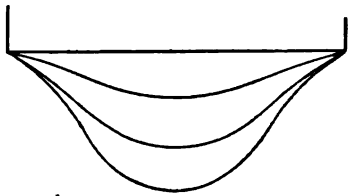


FIG. 346.



zontal circular mouthpiece communicating with a vessel containing the liquid experimented on. It has, however, been erroneously concluded in such experiments that the weight of the drop was $2\pi rT$ where r is the radius of the drop at the level of rupture. If the drop is cylindrical at that level the pressure in the interior must exceed atmospheric by T/r, which would give a downward force on the falling drop of πrT. Thus the weight of the drop could only equal πrT, and the tension obtained from the weight by the erroneous formula would be twice the proper value. The method is not reliable for other reasons. The detachment of the drop is influenced by the bore of the tube. Even with an outside diameter, the same in both cases and attachment at the outer edge, the weight of the drop detached is sensibly different according as the bore is very wide or very narrow. The reader may refer to a paper by Lord Rayleigh in the *Phil. Mag.* for Oct. 1899.

Fig. 345 show the theoretical form of a pendent drop. The horizontal line in Fig. 346 shows level of atmospheric pressure in the drop. The reader will perceive that there the curvatures are opposed, and they, in fact, just balance each other, so that $1/R_1 + 1/R_2 = 0$, and the pressure-difference is zero. Fig. 346 is taken from an interesting paper on Pendent Drops by Mr. A. M. Worthington (*Proc. R. S.* June 16, 1881). The drop was formed by turpentine falling away from a vertical cylindrical tube, the lower end of which was ground flat with a sharp edge. Its magnified image was formed on a screen by a lens, and traced in different stages of the development of the drop.

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CHAPTER XVII.

MEASUREMENTS AND INSTRUMENTS.

707. **Measurement of Mass. The Balance.**—Most of what we have to say regarding units of measurement will be found in chap. i. and in connection with the definitions of derived units as these occur. But there are one or two matters connected with the measurement of mass, the measurement of length in certain particular cases, and the measurement of angle, which we may conveniently discuss here.

First of all the standard instrument for the measurement of mass is the balance. In its elements the balance consists of a double lever, called a *beam*, which is movable about a horizontal axis at right angles to its length, and carries attached at its extremities two scale-pans in which can be placed the masses to be compared. In general the arms are made of exactly equal length, that is to say, when the line joining the points of attachment of the scales to the beam is horizontal that line is at right angles to the axis, and these points are at equal distances from the vertical plane containing the axis. Further, the arrangement is so constructed and adjusted that when the scale-pans are in position and carry no weights, the beam rests stably in the horizontal position. Any deviation from horizontality of the beam can be corrected by a small pivoted arm carried by the beam above its centre. As this is turned round the centroid shifts slowly in one direction or the other along the beam, and also, of course, sideways. Thus the centroid of the beam and scales is below the axis about which the beam turns.

708. **Arrangement of a Precision Balance.**—To avoid errors from friction the beam is supported on two horizontal "knife-edges" of hardened steel most carefully made and ground so as to have a straight smooth edge. These knife-edges face downwards, and rest on smooth horizontal plates of agate placed one on each side of the beam. At each end of the beam is a horizontal knife-edge turned upwards. On each of these knife-edges rests by an agate plate a piece to which the scale-pan at each end is attached. The distance between the knife-edges is the length of the beam. The various parts will be seen in the engraving (Fig. 347) of a balance of precision made by Brunée of Göttingen. An arrangement of levers enables the beam to be placed on its knife-edges when the balance is to be used, and to be otherwise supported

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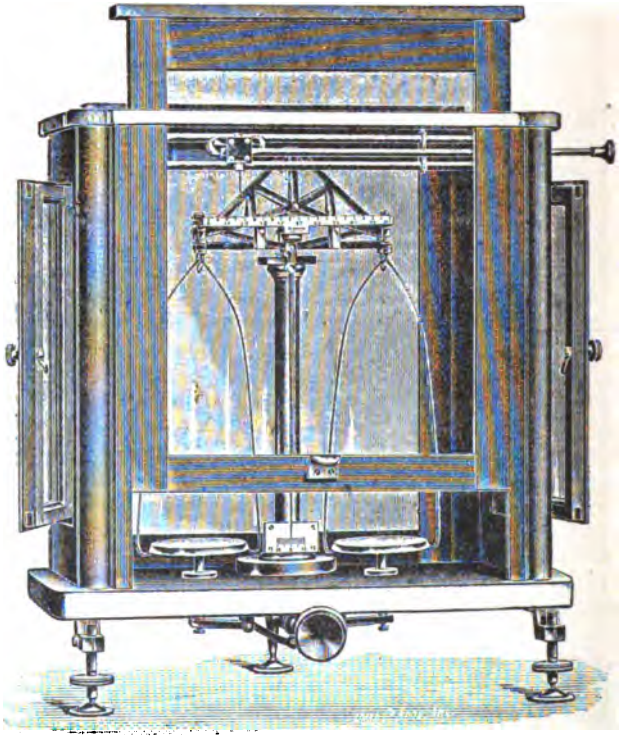
when not in use. The point of a long index depending from the beam and turning with it ranges along a graduated scale near the base of the supporting pillar, and enables a small deflection of the beam from the horizontal to be measured.

The balance if at all delicate is enclosed in a well-made case so that the beam, &c., may be protected from currents of air, and

FIG. 347.



arrangements are adopted in the finest work for putting on and removing weights without opening the case. The case may with advantage be covered with thin sheet copper inside where there is no glass, to equalise the internal temperature.

An arm projecting from one end of the case enables small riders to be placed at different points of the beam to slightly vary the weights by known amounts. For this a graduated scale along the beam is provided as will be seen in the figure. The effect of a rider of given weight can be seen from the graduations on the scale.

From Fig. 347 can be made out the arrangement of the "arrestment" by which the beam can be brought back to its normal position

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when required, and the central knife-edge brought always into a definite position on its bearing. The precautions adopted to ensure that the terminal knife-edges bear also with perfect definiteness on agate plates will also be seen. The pans should be connected to the terminal piece above by a loose link or two to obviate the effect of placing a weight at different positions on the scale-pan. A complete acquaintance with the arrangement, use, and adjustment of the balance can only be obtained by long careful use of the instrument, and we shall not attempt to enter into details here.

709. **Theory of the Balance. Sensibility.**—Let us consider then a balance the central and terminal knife-edges of which are parallel and horizontal and lie nearly in one plane. Let the weights, including pans, &c., attached to the terminal pieces be w and $w+\delta w$, and the lengths of the corresponding arms l and $l+\delta l$. Let the centroid ("centre of gravity") of the beam when it is horizontal be situated at a distance h below the plane of the terminal knife-edges, and the central knife-edge be at a height h' above the plane through the terminal knife-edges; let the weight of the beam be w', and the beam be turned through any angle θ from the horizontal. The moment tending to restore equilibrium is as the reader will see by making a diagrammatic sketch

$$w(l\cos\theta+h'\sin\theta)+w'(h+h')\sin\theta-(w+\delta w)\{(l+\delta l)\cos\theta-h'\sin\theta\}.$$

If this couple is zero, that is if there be equilibrium in the deflected position,

$$\tan\theta=\frac{w\delta l+\delta w(l+\delta l)}{(2w+\delta w)h'+w'(h+h')}.$$

If the arms be of equal length $\delta l=0$, and

$$\frac{\tan\theta}{\delta w}=\frac{l}{(2w+\delta w)h'+w'(h+h')}. \qquad (1)$$

The ratio $\tan\theta/\delta w$ is taken as the measure of the sensibility of the instrument. A scale, divided so as to show tangents of deflections, would, if placed horizontally at the base of the supporting pillar, and ranged over by a pointer carried by the beam, give values of $\tan\theta$ for different values of δw. But in practice as the deflections are small it is sufficient to use a circular scale if uniformly divided. As a rule the balance is so constructed that the knife-edges are all in one plane, and therefore h' has only a small variable value due to flexure of the beam under the weights attached to its ends. The sensibility is therefore greater the greater l and the smaller the weight w' of the beam. Since h' is small the sensibility is nearly independent of the load $2w$, through a considerable range of its values. As a rule h' is zero for a certain mean load, and it has therefore opposite values according as the load is smaller or greater than this beam. The sensibility, therefore, first increases then diminishes with increase of load. It can be altered by screwing up or down a small weight

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carried by the beam and so varying the value of h. When its value, s, is known the value of δw can be obtained by reading the deflection θ, and is approximately θ/s.

710. **Stability of the Balance.**—A balance should be not only sensitive, but as quick in action as possible. To find on what the latter characteristic depends we must find the time of a small oscillation. Thus let the beam be deflected through an angle θ, with $\delta w = 0$, and $\delta l = 0$. The return couple is

$$wg(l\cos\theta + h'\sin\theta) + w'g(h+h')\sin\theta - wg(l\cos\theta - h'\sin\theta),$$

or

$$2wgh'\sin\theta + w'g(h+h')\sin\theta.$$

The distances of the verticals in which the weights w are situated from the vertical through the axis are $l\cos\theta + h\sin\theta$, $l\cos\theta - h\sin\theta$. Hence if w be the total mass suspended at each end of the beam, and we suppose that the weights w move at each instant in the vertical through the point of support, the moment of momentum of the weights about the axis is

$$w\dot{\theta}(l\cos\theta + h\sin\theta)^2 + w\dot{\theta}(l\cos\theta - h\sin\theta)^2 = 2w\dot{\theta}(l^2\cos^2\theta + h^2\sin^2\theta).$$

Also if k be the radius of gyration of the beam about the axis the moment of momentum of the beam is $w'k^2\dot{\theta}$. Equating the time-rate of variation of the momentum of the system (neglecting terms multiplied by $\dot{\theta}^2$) to the return couple, we obtain for the period T of a small oscillation

$$T^2 = 4\pi^2 \frac{2wl^2 + w'k^2}{g\{2wh' + w'(h+h')\}}.$$

If as is usual h' is zero or nearly, we have

$$T = 2\pi\sqrt{\frac{2wl^2 + w'k^2}{gw'h}}. \qquad (2)$$

If we denote by s the sensibility $\tan\theta/\delta w$, we have by (1)

$$T = C\sqrt{s}$$

where C is a constant depending on the load in the pans. Hence for a given load the period varies directly as the square root of the sensibility.

A short period enables readings of the deflection on each side to be quickly read, and an estimate to be formed as to whether equilibrium has been nearly attained. But clearly this desirable quickness of return is incompatible with sensibility and therefore it is only possible to make a compromise between the two, which will depend upon the degree of accuracy to be attained in the weighings. The length of the beam should always be great enough to swamp any

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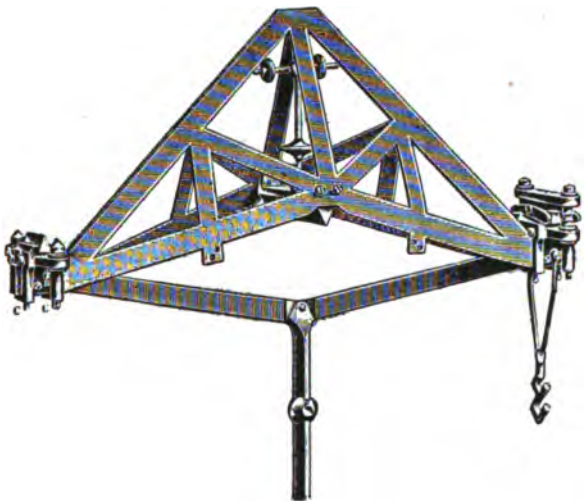
uncertainty as to the effective length of the arm due to shifting of the positions of the end pieces on their knife-edges. It is always desirable to make the beam as light as possible, and this is effected by making it in the form of a light girder, and of as light material as possible. Balances are now made of aluminium, and Fig. 348 shows a very light and stiff form of beam made of this material and used in the balance shown in Fig. 347.

711. **Adjustment of the Balance.**—When the balance is to be used it must be carefully levelled by means of the foot-screws, and

FIG. 348.



adjusted by the movable weight to the sensibility required. The increase of sensibility may be traced by observing the increase of period: it has been shown above that the sensibility is proportional to the square of the period. The horizontality of the beam when the pans are unloaded is adjusted by the turning arm or other arrangement provided for the purpose.

If the balance is working properly it will be subject to little frictional resistance. This may be tested by causing the beam to vibrate about the middle position and noticing the divisions on the horizontal scale which mark the extent of the successive deflections of the pointer. These should fall off slowly in amplitude. Further, the pointer ought to come back always to the same position on the scale, when the beam is taken off and replaced on its knife-edges a number of times in succession.

It should be observed whether the effect of a weight is independent of the position on the scale-pan at which it is placed. There will be little variation of this kind if as stated above the scale-pan

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is connected by a perfectly loose link or two with the hook attached to the terminal piece.

712. **Weighing by Oscillations.**—To save time it is desirable when weighing to take successive readings of the pointer on the scale, for from these readings the position at which the pointer would come to rest can be obtained without waiting for complete subsidence of the oscillations. It is convenient for taking these readings that the zero of the scale should be at the right hand end, so that the centre point should be numbered 10, or 20, or some other number according to the divisions. Let successive readings at which the pointer comes to rest, beginning with one on the left, be n_1, n_2, n_3, &c., and let the position at which the pointer would finally rest be n. The first deflection from the position of rest is $n_1 - n$, the second is $n - n_2$, the third is $n_3 - n$. If we suppose that there is the same falling off in the three successive deflections we have $n_1 - n - (n - n_2) = n - n_2 - (n_3 - n)$, or $n = (n_1 + n_3 + 2n_2)/4$.

If the division at which the balance pointer rests with equal loads on the pans be m, $n - m$ is the deflection due to the inequality of loads. If the amount w of this inequality be known (and it can be made of any small value required by putting equal known weights W, W on the pans and adding to them a known small weight w) the sensibility is proportional to $(n - m)/w$.

This, of course, will be the sensibility for the load $2W$, and different determinations should be made for different loads, and a curve, with sensibilities as ordinates and loads as abscissæ, laid down on squared paper to exhibit the results, in a form available for the determination of the difference of weights.

713. **Determination of Ratio of Arms of Balance.**—The ratio of the lengths of the arms of the balance may be determined in the following manner, which does not assume that the weights used have their nominal values. Let the nominal value of the two weights in the left and right pans respectively which produce equilibrium of the beam be w and w'. Then let *these same weights* be interchanged, and w altered to $w + \delta w$ in order to find equilibrium. [If δw must be negative for equilibrium the weight representing it must be placed in the scale-pan with w'.] Then, if l, l' be the lengths of the left and right arms respectively, we have $wl = w'l'$, $w'l = (w + \delta w)l'$, and therefore approximately

$$\frac{l}{l'} = 1 + \tfrac{1}{2}\frac{\delta w}{w}. \tag{3}$$

As an example we may take the following, given in Kohlrausch's Physical Measurements. A single weight nominally 50 grammes placed in the left hand scale was equilibrated by a group of weights making a nominal aggregate of 50·00083 grammes. The weights were then interchanged, and it was found that when ·00256 gramme was added to the 50-gramme weight and ·00083 was removed from the group, equilibrium was again obtained. Thus δw was clearly ·00256 + ·00083 (that is ·00339) of a gramme. Hence

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$$\frac{l}{l'} = 1 + \frac{\cdot 00339}{100} = 1\cdot 0000339.$$

The ratio of the sum of the weights of the group making up nominally 50·00083 grammes to the real weight of the piece marked 50 grammes was thus l'/l or 1·0000339. Thus, taking the 50-gramme weight as correct the group of weights making up the 50 had an excess weight of ·000865 gramme. This will illustrate the method of comparing and testing a set of weights.

714. **Double Weighing.**—Error from inequality of l and l' may be avoided by the process of double weighing. The body is placed in one pan, say the left, and balanced by weights w_1 in the right. The body is then transferred to the other pan and balanced by a weight w_2 in the left. Then, if W be the true weight of the body, $Wl = w_1 l'$, and $w_2 l = Wl'$. Hence, $W = \sqrt{w_1 w_2}$. Since w_1, w_2 differ little from one another, the arithmetic mean $\frac{1}{2}(w_1 + w_2)$ may be taken as the value of W.

This gives also a comparison of l, l', but has the disadvantage of assuming that w_1, w_2, the nominal values of the weights, are the correct values. Thus, $l/l' = \sqrt{w_1/w_2}$.

715. **Reduction of Weights to Vacuo.**—Weighings are usually made in air, and require correction for the buoyancy of the air, as explained in § 423 above. Let ρ be the density of the body, and ρ_a that of air. Then, if W be the true weight of the body, the weight of air displaced by it is $W\rho_a/\rho$. Thus, the weight measured by the process of weighing in air is $W(1 - \rho_a/\rho)$. On the other hand, if the nominal value of the weights which balance it be W' and ρ_w be their density, or the average density if they are made of different materials their effective weight is $W'(1 - \rho_a/\rho_w)$. Hence,

$$W(1 - \rho_a/\rho) = W'(1 - \rho_a/\rho_w),$$

and

$$W = W' \frac{1 - \frac{\rho_a}{\rho_w}}{1 - \frac{\rho_a}{\rho}} = W'\left(1 + \frac{\rho_a}{\rho} - \frac{\rho_a}{\rho_w}\right), \tag{4}$$

very nearly, since ρ_a is small in comparison with ρ.

An approximate value of ρ obtained by the process of weighing in air without correction for buoyancy is exact enough for use in this equation. That of ρ_w is generally known exactly enough. For brass weights ρ_w may be taken as 8·4, for platinum weights 21·5. If the nominal value of the brass weights be W_1', and of the platinum weights W_2',

$$\rho_w = \frac{21\cdot 5 \times 8\cdot 4 W'}{21\cdot 5 W_1' + 8\cdot 4 W_2'}. \tag{5}$$

The value of ρ_a for ordinary atmospheric air is generally taken as ·0012 gramme per cubic centimetre.

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Thus, if a quantity of copper of density 8·9 be weighed with brass weights, and W' be the apparent weight in air in grammes, the correction is $\cdot 0012(1/8\cdot 9 - 1/8\cdot 4)$ of a gramme per gramme of W', that is, ·008 of a milligramme per gramme of W', and the true weight is

$$W'(1 - \cdot 000008).$$

716. **Correction to Vacuo of Specific Gravity. Determinations.**—In determinations of specific gravity the substance is weighed in air and in water. The correction for the weighing in air is to be made as just described. Let W be the true weight just found. The density of the water depends on the temperature. Let it be ρ'. The apparent loss of weight of the body when immersed in water is $W\rho'/\rho$. If the weight on the other pan giving equilibrium be W'', the true weight in water is $W''(1 - \rho_a/\rho_w)$, where as before ρ_w is the density of the weights. We have therefore $W(1 - \rho'/\rho) = W''(1 - \rho_a/\rho_w)$. Hence, $W\rho'/\rho = W - W''(1 - \rho_a/\rho_w)$, and

$$\left.\rho = \frac{W\rho'}{W - W''\left(1 - \dfrac{\rho_a}{\rho_w}\right)}.\right\} \qquad (6)$$

But if W' be as before the nominal weight of the body in air $W = W'(1 - \rho_a/\rho_w + \rho_a/\rho)$, and from this we obtain, putting for ρ in the term in brackets the approximate value $W'/(W' - W'')$, and neglecting small terms of the second order,

$$\rho = \frac{W'}{W' - W''}(1 - \rho_a) + \rho' + \rho_a,$$

or

$$\rho = \rho_1(\rho' - \rho_a) + \rho_a, \qquad (7)$$

where ρ_1 is the uncorrected value of ρ.

717. **Comparison of Lengths. The Comparator.**—For the exact measurement of lengths various appliances are used. One of these is the *Comparator*, which is used for determining by means of a graduated measuring rod the distance between two marks on a rod or scale. On the standard measuring rod is fixed a sliding piece carrying a reading microscope. The instrument is arranged so that the line the length of which is to be determined is parallel to the scale, and underneath the object glass of the microscope. One of the marks is then brought under and focussed in the microscope in coincidence with a "cross wire" carried by the eyepiece. This wire is in the focal plane of the objective when the object is in focus. A reading is then taken of the position of an index which traverses the scale when the sliding piece is moved. The sliding piece is then displaced along the scale until the other mark is brought into focus and made coincident with the cross wire in the field of view. The reading of the mark is again taken and the displacement along the rod gives the distance required.

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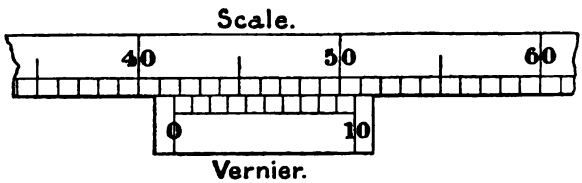
718. **The Vernier.**—For the more exact determination of the distance a *vernier* may be used. This contrivance is represented in Fig. 349. On an edge of the sliding-piece arranged to move along the graduated edge of the scale is divided a scale of the 10 equal divisions in a space equal to 9 divisions of the scale, and these marks (not the spaces) are numbered 0, 1, 2, 3, ... 10, let us say in the direction of increasing numbers on the scale. The mark 0 is called the zero of the vernier and lies between two of the graduation-marks on the scale. The distance of the zero in front of the mark behind it on the scale is obtained as follows. Let the mark numbered n of

FIG. 349.



the vernier coincide with a division of the scale, then the distance just specified is n-tenths of one space or division of the scale. To see this suppose the zero of the vernier put back through the distance in question so as to coincide with the mark formerly immediately behind. The division mark 1 on the vernier will be behind the mark on the scale in front of it by just the difference between a vernier-space and a scale-space, that is 1/10 of a scale-space, the mark 2 will be behind the scale-mark next in front of it just 2/10 of a scale-space, and so on, so that the vernier-mark n is behind the scale-mark next in front a distance just equal to n-tenths of a scale-space.

Hence, if the distance through which the zero has been put back be n-tenths of a scale-space the mark n on the vernier was coincident with a scale division when the zero of the vernier was the sighted position.

If, as may happen, no mark on the vernier coincide with a scale-mark two consecutive marks on the vernier, say the $n-1$th and the nth, will be between two scale-marks. Then an imaginary mark $n+f$ vernier-spaces from the zero (where f is a fraction) will coincide with a scale-mark, and the distance to be measured is $n+f$ tenths of a scale-division from the zero. The fraction f can be quite exactly enough estimated in most cases. The result can be checked by estimating the fraction f' of a vernier-space which the mark $n-1$ is in front of the scale-division behind it. Then the distance to be measured is also $(n-1+f')$ tenths of a scale-space.

Sometimes the numbering of the vernier graduation runs in the opposite direction to the numbering on the scale. In this case 10 spaces of the vernier must include 11 spaces of the scale. Also some verniers are divided to 50 spaces for 49 of the scale, or 50 spaces for 51 of the scale, according as the vernier numbering runs the same

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way as, or in the reverse way to, the scale numbering. The vernier then enables a fraction of a scale-space to be reckoned in 50ths of a scale-space. In some cases the vernier numbers are doubled so that the vernier-reading gives the result in hundredths of a scale-space.

719. **Measurement of Distances by Optical Bench.**—The arrangement of a sliding-piece moving along a graduated scale can be used for determining the distance parallel to the scale between two parallel surfaces directed across the scale. In most cases these stand at right angles to the scale, as in the case of screens and lenses mounted on what is called an *Optical Bench.* (See vol. ii.) In this case the sliding-piece carries one of the surfaces the distance between which is to be measured. The reading of its vernier zero having been taken with the surfaces in position, the sliding-piece is moved until the surface carried by it is brought into contact with the nearer end of a gauge-rod, the further end of which is in contact with the surface. Its vernier is read, and the displacement obtained from the two readings. This added to the known length of the gauge-rod gives the distance required.

720. **The Screw-Gauge and Vernier-Callipers.**—Screw-gauges are used for the measurement of small distances, such as the thickness of wires. In all a point moves through a certain distance for each turn of a screw provided with a micrometer head on which fractions of a turn can be read. On the stem of the screw itself is a scale which tells the distance through which the screw has been moved from its zero position, which is generally that in which its point is in contact with the projecting piece in front of it. When the screw is turned back, a gap is formed between its points and this piece, and in this gap the body to be measured is properly inserted. The screw is then moved forward until the body is held tightly between the screw point and the projecting piece. The number of whole divisions on the scale, and the fraction of a division indicated by the reading on the micrometer head give the dimension to be measured. Thus, if the scale be in half-millimetres, the head be divided into 20 parts, and the screw make one turn for each division of the scale, and if the scale reading in any measurement be 4, and the circle reading be 13, the distance is $4\frac{13}{20}$ half-millimetres or about 2·325 millimetres.

The construction of the ordinary screw-gauge is faulty in that the screw-head is hardly ever made large enough to enable exact graduations to be obtained; the screw-thread should also be very carefully cut to run without "backlash." The projecting piece should always be capable of adjustment to enable any error of zero for the instrument to be corrected.

Another form of this instrument is the *Vernier-Callipers.* Here what are called the *jaws* of the callipers are two projecting pieces movable along a graduated metal rule. One of these is furnished with a vernier moved by an adjusting screw, by which the position of the sliding-piece can be read off on the scale. The body to be

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measured is placed in the gap, and the sliding-piece moved up against it. The vernier is then read and the distance deduced.

721. **Spherometer.**—A form of the screw-gauge is arranged for the measurement of the thickness of bodies as well as of the curvature of surfaces—for example, those of lenses. Three parallel steel legs ending in rounded points, adjusted to lie in one plane at the vertices of an equilateral triangle, are held in a frame, which also supports a nut by which a fourth steel point can be moved at right angles to the plane of the triangle just referred to. This point, when it is in the plane of the triangle, is situated exactly at the centroid of the triangle, that is, at a distance from each of the three vertices of $s/\sqrt{3}$, where s is the side of the triangle. A large micrometer head, the edge of which traverses a scale parallel to the line of motion of the fourth point, is carried by the nut. One turn of the head corresponds to one division of the scale, which is generally graduated to half-millimetres, and thus if the head, as is usual, is divided into 250 spaces, a turning of the head through one space gives a motion of the point of $\frac{1}{500}$ of a millimetre, or $\frac{1}{5000}$ of a centimetre.

When the instrument is to be used to measure the thickness of a glass plate, it is set down on a horizontal plate of glass, the upper face of which is accurately plane, and the fourth point is screwed back a sufficient distance. The plate is then laid within the triangle formed by the three points, and the fourth point is gently screwed down into contact. Contact will be perceived at once by the tendency of the instrument to rock or swivel about the fourth point thus brought into bearing, and the perception of this by an experimenter accustomed to the instrument is instant and delicate. The readings on the scale and micrometer head give the distance through which the fourth point has been drawn back from co-planarity with the three points, when the micrometer head is at the zero of the scale.

When this arrangement is used for the measurement of the curvature of a spherical surface, the instrument is set down upon the surface, and the fourth point adjusted so as just to give contact. The readings are then taken, and give the distance d of a point on the surface from the plane of the three feet. Consider the plane through the centre of the sphere and one of the feet F, and bisecting the line joining the other two at a point P. The distance FP is $s\sqrt{3}/2$, and at a point on the line $\frac{2}{3}FP$, that is $s/\sqrt{3}$, from F, the radius is perpendicular to FP. Hence if r be the radius of the surface $(r-d)^2+s^2/3=r^2$, or

$$r=\frac{s^2}{6d}+\tfrac{1}{2}d. \qquad (8)$$

722. **The Bifilar Balance. Adjustment.**—The measurement of couples by the bifilar suspension has already been referred to (§ 302) and the theory of the arrangement discussed. The return couple exerted by the bifilar is given by (19) and (20) of chap. vi., and we add here a few remarks about the adjustment of the arrangement and

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its advantages. The suspension is generally used in an instrument which can be slightly changed in position by levelling screws. Thus the line of the points to which the fibres are attached can be slightly inclined to one side or the other. Supposing the points of attachment of the threads at the top to be on the same level, as also those at the bottom, and the threads to be of equal length, so that the bifilar is properly adjusted, a slight raising of one of the sides would alter the quantity c of § 302 without appreciably affecting f. For rather more weight would be thrown on one fibre than on the other, and the sensibility as measured by the ratio of deflection to opposing couple will be increased. If, therefore, the adjustment is only nearly made, f will be very small, and any increase in c will increase the sensibility. But if the effect of inclining the instrument is to equalise more nearly the weights on the fibres the sensibility will be diminished. Any change in the sensibility will be shown at once by observation of the period of vibration, which is proportional to the square root of the sensibility. If raising one fibre a little by inclining the instrument increases the period of swing while raising the other diminishes the period, more weight is borne by the former fibre than by the latter, and the inequality is to be removed by shortening the fibre which bears the smaller weight and lengthening the other, until a state of adjustment is obtained such that deviation from it by an elevation of either fibre increases the period.

723. **Correction of Bifilar for Rigidity, Torsional and Flexural.**—Each fibre acts as a unifilar suspension by its torsion. If τ be the torsional rigidity of each wire, the total couple given by the bifilar is shown in § 302, with the notation there adopted, to be for a small angle

$$L = Mg\frac{ab}{l}\sin\theta + 2\tau\frac{\theta}{l}. \qquad (9)$$

If the wires of the bifilar have flexural rigidity each is bent by the deflection into the curve shown in Fig. 326 above by the half of the bar between either extremity and the middle. The two ends we suppose are held vertical by the manner of attachment, and a couple is brought into play by the bending of the wires. If the vertical stretching force in each fibre be F, the flexural rigidity (§ 651) of the fibre be B, and the fibre be long, the couple derived by the equation just given is to be increased in the ratio 1 to $1 - 2\sqrt{B/l}\sqrt{F}$ (as the reader may prove from the principles set forth in § 651), that is, the couple is the same as that which would be calculated by the equation from a suspension shorter than l by the amount $2\sqrt{B/F}$. The amount of this shortening is quite sensible, and for wires of $\frac{1}{100}$ cm. in diameter each stretched by a weight of 50 grammes it is ·22 cm. for copper, ·17 cm. for silver, ·18 cm. for gold, and ·26 cm. for platinum. These corrections are to be halved if only one end of the wire is fixed in the vertical direction.

724. **Comparison of Bifilar and Unifilar Balances.**—A comparison of the bifilar with a unifilar suspension giving a return

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couple by torsion shows that for silver, gold, and copper wires (the double wire in the bifilar and the single wire in the unifilar) just strong enough to support the weight, the bifilar and the unifilar have the same sensibility when the ratio of the distance apart of the wires to the diameter is about 5, and for platinum when the ratio is 6.

The bifilar has the advantage over the unifilar that it is only very slightly affected by temperature. The alteration of length of the fibres has far less effect upon it than the alteration of torsional rigidity. It might be made self-compensating by attaching wires to a bar which by its expansion would carry the ends just far enough apart to compensate the increase of length of the fibre.

725. **Measurement of Small Couples by Pendulums.**—A very simple method of measuring small couples is to apply the couple in a horizontal plane by the deflections in opposite directions of two simple pendulums, of the same length and having equal bobs. Let the pendulums be hung from two points on the same level, and let a horizontal cross bar to which the couple is to be applied be placed above the bobs and against the threads, so that the threads are deflected in opposite directions when the bar is turned round, while they remain in parallel planes. This latter condition can be fulfilled by making the vertical parts of the threads just above the bobs move outwards along two horizontal rules placed at right angles to the plane in which the threads hang when undisturbed.

If M be the mass of each bob the couple is $2Mgp\sin\theta$, if p be the distance of the points of suspension of the pendulums apart.

726. **Hole, Slot, and Plane Arrangement for Supporting an Instrument.**—It is often very convenient to be able to remove an instrument from one place to another and afterwards replace it exactly in the position which it formerly occupied. This can be done by supporting the instrument on three feet, for example the lower rounded ends of three levelling screws, and using the hole, slot, and plane device suggested by Lord Kelvin. On the supporting table is cut a trihedral hollow, a hollow in the shape of an inverted triangular pyramid, and in line with that, with its centre at a distance from that of the hollow equal to that between two of the feet of the instrument, a V-shaped notch or groove, both wide enough to admit a foot of the instrument. If then the instrument be set with one foot in the hollow, another in the groove and the third on the plane surface, and there levelled, it can only stand in one position with the same feet on the hollow and the groove. Hence if the levelling screws are not altered it can be removed and replaced with absolute certainty, if no change has taken place on the supporting table, that it is made to occupy precisely the same position as before.

727. **Dynamometers or Ergometers.**—Ergometers, or, as they are commonly called, *Dynamometers*, are work measurers. They are of two kinds—*friction-dynamometers* and *transmission-dynamometers*. The former waste all the work done, but enable the whole work given out in any time by a prime mover to be estimated by the

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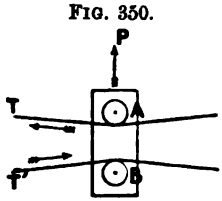
frictional resistance overcome and the speed at which it is overcome. Round a fly-wheel or vertical pulley a rope or band is wrapped, and weights are attached at the two ends to make it bear upon the rim of the wheel. The pulley is now driven in the direction to pull upon the heavier weight by the friction applied by the rim to the band, and if the frictional resistance is great enough the weight will be lifted. If this balance can be maintained and the speed measured, the rate at which work is being done can be calculated.

Professors Ayrton and Perry have found that when a grooved pulley is used with a rope brake, a knot on the rope, which is slightly jammed into the groove by the pull on the rope, is sufficient to give the necessary self-adjusting power to the arrangement to enable it to run stably under alterations of the frictional co-efficient. The rubbing surfaces are kept cool by soapy water trickling over them. When large friction dynamometers are to be used, for example those in which a metal band faced with blocks of wood surrounds the wheel-rim, and is prevented from turning by a spring or weight pulling it on the proper side, the rim is sometimes made hollow, and a large flow of water is maintained round it to prevent accident by overheating. The whole work is spent in producing heat, which is therefore generated in the tests of a large engine at a very great rate.

The calculation of the rate of working is very simple. Let C be the circumference of the wheel in feet, W the difference of weights in pounds at the two ends of the band (or the pull in the spring on the arrangement just referred to) and N the number of revolutions per minute. The total resisting couple on the wheel is $WC/2\pi$. The angle turned through in one turn is 2π, and therefore the work done in one turn is WC. The work done per minute is thus WCN, in foot pounds, and the activity in horse-power is $WCN/33{,}000$.

728. **Transmission-Dynamometers.**—There are several forms of transmission-dynamometers. We shall mention only two, that called the Hefner-Alteneck and the spring dynamometer of Ayrton and Perry. In the former a vertical piece carries two pulleys A, B; on these bear the two parts of the driving-belt or cord as shown in the cut. One side, the driving side of the belt, is tight; the other comparatively slack. The driving side passes over the under pulley B, and the slack side under the upper pulley A. The piece is thus pulled down with a force P depending on the difference of stretching forces, and this force is measured by a spring, when the piece is balanced in the position in which the parts of the cord on the two sides of each pulley all make the same angle with the horizontal. Let this angle be θ. Then if T', T be the stretching forces in the tight and slack parts of theb elt respectively, we have $2(T' - T)\sin\theta = P$, or $T' - T = P/2\sin\theta = P/2\theta$, if θ be small. Let P be taken in pounds, and let the belt be travelling at v feet

FIG. 350.



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per minute. The rate at which work is being transmitted is $Pv/2\theta$, in foot pounds per minute. The activity in horse-power is $Pv/66{,}000\theta$.

There are other dynamometers for use when the work is transmitted directly from one shaft to another without the intervention of belting. One of these is the invention of Professors Ayrton and Perry. The connection between the driving and driven shafts is made by springs, the deflection of which by the driving couple causes a silvered bead to describe a circle the diameter of which depends on the amount of the couple. An observation of the diameter of this circle and the speed enables the activity to be found.

The principle of the latter dynamometer is somewhat similar to that of one invented long ago by the late Professor James Thomson, and used by Dr. Joule in the last determinations which he made of the dynamical equivalent of heat. Water in a cylindrical vessel was stirred by a paddle driven round within the vessel by cords unwound from a vertical spindle by falling weights. The friction between the water and the vessel tended to carry the vessel round with the paddle, and the torque or turning motive exerted on the paddle was found by measuring the couple required to keep the vessel at rest. The product of the couple into the angular velocity gave the activity.

The same principle has been used to measure the work spent in driving the armature of a dynamo in the magnetic field of the magnets, and thus to determine exactly the activity spent in generating currents in the circuit. The field-magnets and armature are supported by separate frames, and that carrying the magnets is capable of turning round an axis in line with that of the armature. The electro-magnetic action tends on the one hand to stop the armature, and on the other to carry the magnets round in the same direction as that in which the armature is rotating. The couple required to keep the magnets at rest is determined and the activity found as before.

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